BOLLETTINO
UNIONE MATEMATICA ITALIANA

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 605–611.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_605_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 605-611

A Note on Countable Butler Groups.

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Sunto. – *Nel 1986 fu pubblicato un esempio di un gruppo di Butler numerabile che non è un sottogruppo puro di un gruppo completamente decomponibile. Recentemente, si è scoperto un errore nella dimostrazione. In questa nota la dimostrazione viene corretta, e si prova anche che questo gruppo non ha sottogruppi puri precobilanciati di rango uno.*

Summary. – *An example of a countable Butler group that is not a pure subgroup of a completely decomposable group was published in 1986. Recently, an error in the proof was discovered. This note includes a corrected argument, as well as a proof that this group has no pure precobalanced subgroups of rank one.*

The main purpose of this note is to correct the proof, published in [1] and repeated verbatim in [2], that there is a countable Butler group that is not a pure subgroup of a completely decomposable group.

We begin with some definitions and a context for this example, e.g. see [2]. A subgroup H of a torsion-free abelian group G is a *pure subgroup* of G if $nG \cap H = nH$ for each nonzero integer n. A *type* is an isomorphism class $\tau = [X]$ of a torsion-free abelian group X of rank 1. The set of types is partially ordered by setting $\tau = [X] \leq \sigma = [Y]$ if $\mathrm{Hom}(X, Y) \neq 0$. A type $\tau = [X]$ may be represented by an equivalence class $[(n_p)_{p \in \Pi}]$ of the sequence $(n_p)_{p \in \Pi}$, where Π is the set of primes and each n_p is either a non-negative integer or ∞. In this case, X is isomorphic to the subgroup of the rationals $\mathbb{Q}$ generated by $\{1/p^{i_p} : 0 \leq i_p \leq n_p, p \in \Pi\}$.

Let G be a torsion-free abelian group. The type of a non-zero element x of G is defined to be $[X]$, where X is the pure rank-1 subgroup of G generated by x. For any type τ, $G(\tau) = \{x \in G : \mathrm{type}(x) \geq \tau\}$, $G^*(\tau)$ is the subgroup of G generated by $\{x \in G : \mathrm{type}(x) > \tau\}$, and $G^{\#}(\tau)$ is the pure subgroup of G generated by $G^*(\tau)$.

An exact sequence $0 \to A \to B \to C \to 0$ of abelian groups is *balanced* if every rank 1 torsion-free abelian group is projective with respect to this exact sequence. In this case we say B is a balanced extension of A by C. The inequivalent balanced extensions of A by C form a subgroup $\mathrm{Bext}(C, A)$ of the group $\mathrm{Ext}(C, A)$ of all extension of A by C.

A torsion-free abelian group G is *completely decomposable* if G is a direct sum of groups of rank-1. Pure subgroups of finite rank completely decomposable groups, under the name Butler groups, admit several interesting characterizations indicated in the following theorem, see [4], [3]:

THEOREM 1. – *The following statements are equivalent for a finite rank torsion-free abelian group G:*

(i) G is a pure subgroup of a finite rank completely decomposable group;

(ii) G is a homomorphic image of a finite rank completely decomposable group;

(iii) $Bext(G, T) = 0$ for any torsion abelian group T;

(iv) The set $T(G)$ of types of elements of G is finite and for each $\tau \in T(G)$, $G(\tau) = G^{\#}(\tau) \oplus G_\tau$ where G_τ is the direct sum of finitely many copies of a rank 1 group X of type τ and the factor group $G^{\#}(\tau)/G^{*}(\tau)$ is finite.

What happens if we replace finite rank by countable rank in the above theorem? Clearly $(i) \Rightarrow (ii)$ since any countable group is the homomorphic image of a countable rank free group which also implies that $(ii) \Rightarrow (i)$. Statements (i) and (iv) are not equivalent, [9]. It is shown in [3] that $(i) \Rightarrow (iii)$ for countable groups and that (iii) is equivalent to the condition that G is a countable group in which any finite rank pure subgroup is a Butler group. Such groups are called *countable Butler groups* (or *countable finitely Butler groups*).

Finally, (iii) does not imply (i), since an example of a countable Butler group which is not a pure subgroup of a completely decomposable group is constructed in [2]. However, there is an error in this proof. The error occurred in the first two paragraphs of the proof. It was claimed that if S is the set of words on the alphabet $\{0, 1\}$, then there is an infinite tree of proper subgroups $\{X_s : s \in S\}$ of $\mathbb{Q}$ such that:

(i) if $t = s1$ and $u = s0$, then $X_s = X_t \cap X_u$ and

(ii) if τ is a type and for each positive integer n, there is a word $s(n) \in S$ of length n, with $\tau \geq [X_{s(n)}]$, then $\tau = [\mathbb{Q}]$.

Such a construction is not possible. For example, choose a prime q and $s \in S$ with $qX_s \neq X_s$. By (i), either $qX_{s0} \neq X_{s0}$ or $qX_{s1} \neq X_{s1}$. Repeating this argument results in an infinite chain of words $s(n)$ of increasing length n and a type $\tau = [(n_p)_{p \in \Pi}]$ with $n_q \neq \infty$ and $\tau \geq [X_{s(n)}]$ for each n. In particular, $\tau \neq [\mathbb{Q}]$, contradicting (ii).

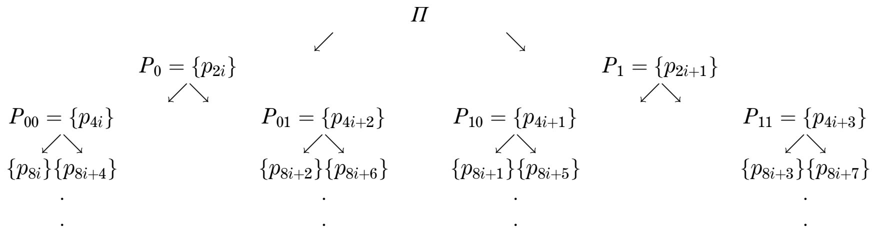
To correct the proof, we begin with the following proposition.

PROPOSITION 1. – *If S is the set of words on the alphabet $\{0, 1\}$, then there is an infinite tree of proper subgroups $\{X_s : s \in S\}$ of $\mathbb{Q}$ such that if $t = s1$ and $u = s0$, then $X_s = X_t \cap X_u$ and $X_t + X_u = \mathbb{Q}$.*

PROOF. – We first construct a sequence of partitions of $\Pi = (p_i)_{i\in\mathbb{N}}$, the set of prime numbers. The X_s's will be defined in terms of these partitions.

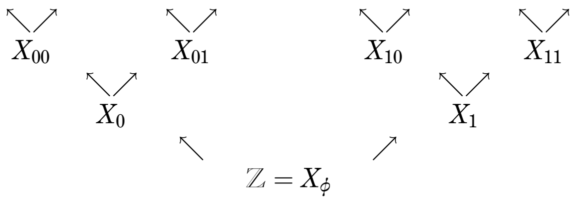
The first partition of Π is $\pi_1 = \{P_0, P_1\}$ with $P_0 = \{p_{2i} : i \geq 1\}$ and $P_1 = \{p_{2i+1} : i \geq 0\}$ corresponding to elements of S with length 1. The second partition is $\pi_2 = \{P_{00}, P_{01}, P_{10}, P_{11}\}$, where $P_{00} = \{p_{4i} : i \geq 1\}$, $P_{01} = \{p_{4i+2} : i \geq 0\}$, $P_{10} = \{p_{4i+1} : i \geq 0\}$, and $P_{11} = \{p_{4i+3} : i \geq 0\}$ corresponding to elements of S with length 2. In general, $\pi_i = \{P_s : s$ is a word of length $i\}$ and each P_s consists of primes indexed by a modulo 2^i equivalence class.

These partitions are represented by the following diagram:



For $s \in S$, we define $X_s = \cap\{\mathbb{Z}_{(p)} : p \in P_s\}$, a subgroup of $\mathbb{Q}$, where $\mathbb{Z}_{(p)}$ is the localization of the integers $\mathbb{Z}$ at a prime p.

The result is an infinite lattice of proper subgroups of $\mathbb{Q}$, indexed by elements of S and with unique minimum element $\mathbb{Z} = X_\phi = \bigcap_{p\in\Pi} \mathbb{Z}_p$, represented by the following diagram.



For example,

$[X_0] = [\cap\{\mathbb{Z}_{(p)} : p = p_{2i}\}] = [(\infty, 0, \infty, 0, ...)]$,
$[X_1] = [\cap\{\mathbb{Z}_{(p)} : p = p_{2i+1}\}] = [(0, \infty, 0, \infty, 0, ...)]$,
$[X_{00}] = [(\infty, \infty, \infty, 0, \infty, \infty, \infty, 0, ...)]$, $[X_{01}] = [(\infty, 0, \infty, \infty, \infty, 0, \infty, \infty, ...)]$,
$[X_{10}] = [(0, \infty, \infty, \infty, 0, \infty, \infty, \infty, ...)]$, and $[X_{11}] = [(\infty, \infty, 0, \infty, \infty, \infty, 0, \infty, ...)]$.

It is now clear from the construction of the groups X_s that if $t = s1$ and $u = s0$, then $X_s = X_t \cap X_u$ and $X_t + X_u = \mathbb{Q}$. □

EXAMPLE 1. – There is a countable Butler group G which is not a pure subgroup of a completely decomposable abelian group.

PROOF. – Let $\{X_s : s \in S\}$ be the infinite tree of proper subgroups of $\mathbb{Q}$ constructed in Proposition 1. Given a nonnegative integer n, define

$$G_n = \oplus\{X_s : s \in S \text{ with length } s = n\}.$$

There is a monomorphism $f_n : G_n \to G_{n+1}$ defined by the diagonal map $f_n(x_s) = (x_s, x_s) \in X_{s0} \oplus X_{s1}$ for each $x_s \in X_s$ and $s \in S$ of length n.

For each $n, f_n(G_n)$ is a pure subgroup of G_{n+1} because $X_s = X_{s0} \cap X_{s1}$ for each word s of length n. Since G_{n+1} is a finite rank completely decomposable group, $f_n(G_n)$ is a finite rank Butler group for each n.

Define G to be the direct limit of $\{G_n, f_n : n \geq 0\}$. Then G is the union of an ascending chain $0 \subseteq B_1 \subseteq ... \subseteq B_n \subseteq B_{n+1} \subseteq ...$ of pure subgroups such that each B_i is a finite rank Butler group. Hence, G is a reduced countable group with infinite rank and a Butler group by [3].

Assume, by way of contradiction, that G is a pure subgroup of a completely decomposable group $C = C_1 \oplus C_2 \oplus ... \oplus C_i \oplus ...$ with rank $C_i = 1$ for each i. Since G is reduced, G is a pure subgroup of $C/d(C)$, where $d(C)$ is the divisible subgroup of C, so that it is sufficient to assume that each C_i is reduced. For each i, let $\pi_i : G \to C_i$ denote a projection homomorphism.

We now identify each X_s with its image in G. The group $X_\phi = \mathbb{Z}$ is a pure subgroup of $C_1 \oplus ... \oplus C_m$ for some fixed m, whence $\pi_i(X_\phi) = 0$ for each $i > m$. It is now sufficient to prove that $\pi_i(X_s) = 0$ for each word $s \in S$, and $i > m$. In this case, the infinite rank group G is a subgroup of $C_1 \oplus ... \oplus C_m$, a contradiction.

Assume, by way of induction on the length of a word, that $s \in S$ has length n and $\pi_i(X_s) = 0$ for each $i > m$. As a consequence of Proposition 1, there is an exact sequence

$$0 \to X_s \to X_{s0} \oplus X_{s1} \to \mathbb{Q} \to 0$$

because $X_s = X_{s0} \cap X_{s1}$ and $X_{s0} + X_{s1} = \mathbb{Q}$.

If $\pi_i(X_{s0})$ or $\pi_i(X_{s1})$ is non-zero, then $\pi_i(X_{s0} \oplus X_{s1}) \neq 0$. Since C_i is reduced and $(X_{s0} \oplus X_{s1})/X_s = \mathbb{Q}$, it follows that $\pi_i(X_s) \neq 0$ and so $i \leq m$. Hence, $\pi_i(X_{s0}) = \pi_i(X_{s1}) = 0$ for each $i > m$. By induction, $\pi_i(X_s) = 0$ for each $i > m$ and each $s \in S$. □

The next theorem provides several characterizations of those countable Butler groups that are pure subgroups of completely decomposable groups.

Following [7], an exact sequence $0 \to B \to G \to C \to 0$ of abelian groups is *cobalanced* if for each rank-1 torsion-free abelian group X, the induced sequence

$$0 \to \text{Hom}(C, X) \to \text{Hom}(G, X) \to \text{Hom}(B, X) \to 0$$

is exact. A subgroup B of G is a *cobalanced subgroup of* G if the exact sequence

$$0 \to B \to G \to G/B \to 0$$

is cobalanced.

A subgroup B of a torsion-free abelian group G is said to be *precobalanced* if for each subgroup K with B/K torsion-free of rank 1, there are finitely many subgroups $K_1, ..., K_n$ of G with each G/K_i torsion-free of rank 1 such that inclusion of B in G induces a pure embedding $B/K \to G/K_1 \oplus ... \oplus G/K_n$ [8]. Clearly, a cobalanced subgroup is precobalanced.

THEOREM 2 [6]. – *Let G be a countable torsion-free abelian group. The following statements are equivalent:*

(a) *Every pure rank-*1 *subgroup of G is precobalanced;*

(b) *Every pure finite rank subgroup of G is a Butler group and is precobalanced in G;*

(c) *G is a pure subgroup of a completely decomposable group;*

(d) *G is the union of an ascending chain* $0 \subseteq B_1 \subseteq ... \subseteq B_n \subseteq B_{n+1} \subseteq ...$ *of pure subgroups such that each* B_i *is a finite rank Butler group and is precobalanced in G.*

An example of a countable Butler group G such that *no* pure rank-1 subgroup of G is precobalanced in G is given in [6]. Since the arguments for this example used the flawed proof in [1], we give a corrected example and proof.

Let $\{X_s : s \in S\}$ be the infinite tree of subgroups of $\mathbb{Q}$ constructed in Proposition 1. We identify precisely those types τ such that for each positive integer n, there is a word $s(n) \in S$ of length n with $\tau \geq [X_{s(n)}]$.

LEMMA 1. – *Let* $\{X_s : s \in S\}$ *be the infinite tree of subgroups of* $\mathbb{Q}$ *constructed in Proposition* 1. *If* τ *is a type and for each positive integer n, there is a word* $s(n) \in S$ *of length n, with* $\tau \geq [X_{s(n)}]$, *then either* $\tau = [\mathbb{Q}]$ *or* $\tau = [\mathbb{Z}_{(p)}]$ *for some prime p.*

PROOF. – In view of the construction of the X_s's, $[X_s] = [(h_p]$, where $h_p = 0$ if $p \in P_s$ and $h_p = \infty$ if $p \notin P_s$. Given a type $\tau \geq [X_{s(n)}]$ for each n, then $\tau \geq [(n_p)]$, where $n_p = 0$ if $p \in \cap_n P_{s(n)}$ and $n_p = \infty$ otherwise.

Assume that $\tau \neq [\mathbb{Q}]$. Then $\cap_n P_{s(n)}$ is non-empty. It follows that

$$P_{s(0)} \supset P_{s(1)} \supset ... \supset P_{s(n)} \supset ....$$

because $\{P_s : s \text{ has length } n\}$ is a partition of Π. Hence, for each n, $s(n+1) = s(n)0$ or $s(n)1$ and

$$X_{s(0)} \subset X_{s(1)} \subset ... \subset X_{s(n)} \subset X_{s(n+1)} \subset ...$$

is a chain of subgroups of $\mathbb{Q}$.

Given n, choose i minimal with $p_i \in P_{s(n)}$. Then the least j with $p_j \in P_{s(n+1)}$ is either i or $i + 2^n$ since $s(n+1) = s(n)0$ or $s(n)1$. For example, let $s(2) = 10$. Then $p_1 \in P_{s(2)}, p_1 \in P_{s(2)0} = P_{100}$, and the least j with $p_j \in P_{s(2)1} = P_{101}$ is $j = 5 = 1 + 2^2$.

As $\cap_n P_{s(n)}$ is non-empty, there is a minimal j with $p_j \in \cap_n P_{s(n)}$. Hence, there is a sufficiently large n with such that j is minimal with $p_j \in P_{s(m)}$ for each $m > n$.

Consequently, if $m > n$, then for each $p_t \in P_{s(m)}$ with $j < t$, there is some $0 < r(t)$ such that $P_{s(m+1)} = \{p_j, p_{t+r(t)} : p_t \in P_{s(m)}, j < t\}$. In other words, passing from $P_{s(m)}$ to $P_{s(m+1)}$ shifts all subscripts t (except j) of primes in $P_{s(m)}$ by a positive $r(t)$. For example, assume that $j = 1$ is minimal with $p_j \in \cap_n P_{s(n)}$ and $s(1) = 1$. If $s(2) = 10$ and $p_t \in P_{s(2)} = P_{10}$ with $t > 1$, then $t = 4i + 1 = (2i + 1) + 2i$ for some $i \neq 0$ and $p_{2i+1} \in P_1 \backslash \{p_1\}$. On the other hand, if $s(2) = 11$ and $p_t \in P_{s(2)} = P_{11}$ with $t > 1$, then $t = 4i + 3 = (2i + 1) + 2(i + 1)$ for some $i \neq 0$ and $p_{2i+1} \in P_1 \backslash \{p_1\}$.

It now follows that $\cap_n P_{s(n)} = \{p_j\}$ has cardinality 1, because j is the only unshifted index in this intersection, and so $\tau = [\mathbb{Z}_{(p_j)}]$. □

EXAMPLE 2. – Let G be the group constructed in Example 1. If B is a pure rank-1 subgroup of G, then B is not precobalanced in G.

PROOF. – There is some least n with B a pure subgroup of the image of G_n in G.

Assume B is precobalanced in G. Then there are finitely many subgroups $K_1, ..., K_m$ of G with each G/K_i torsion-free of rank 1 such that inclusion of B in G induces a pure embedding $\alpha : B \to G/K_1 \oplus ... \oplus G/K_m$.

Fix an i. For each $j > n$, there is a word $s(j)$ of length j with $(X_{s(j)} + K_i)/K_i \neq 0$ because a diagonal embedding of G_n in G_j induces an inclusion of the image of G_n into the image of G_j in G. Hence, $\tau_i = [G/K_i] \geq [X_{s(j)}]$ for each $j \geq n$. By Lemma 1, $\tau_i = [\mathbb{Q}]$ or $\tau = [\mathbb{Z}_{(p)}]$ for some prime p. As α is a pure embedding, $[B] = \inf\{\tau_1, ..., \tau_m\}$ is p-reduced for at most finitely many primes p. This is a contradiction to the assumption that B is isomorphic to a pure subgroup of a finite direct sum

$$G_n = \bigoplus \{X_s : \text{length } s = n\}$$

and the fact that each X_s is p-reduced for infinitely many primes p. □

Acknowledgments. The authors would like to thank Dr. Lutz Struengmann and Daniel Herden for pointing out the error in the original proof of the example.

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Pervenuta in Redazione
il 21 febbraio 2006