Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 613–621.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_613_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 613-621

Finite Simple Groups Admitting Minimally Irreducible Characters of Prime Power Degree.

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Sunto. – *In questo lavoro si classificano i gruppi semplici finiti che ammettono un carattere complesso irriducibile avente grado la potenza di un primo e la cui restrizione ad ogni sottogruppo proprio è riducibile.*

Summary. – *In this paper we classify the finite simple groups that admit an irreducible complex character of prime power degree which is reducible over any proper subgroup.*

1. – Introduction.

A complex character χ of a group G is said to be minimally irreducible, if it is irreducible and if the restriction of χ to any proper subgroup of G is reducible. The study of the minimally irreducible characters is related to computational problems and, more importantly, to irreducible cross-characteristic embeddings of finite groups of Lie type over a field of characteristic $p > 0$.

In this paper we classify all finite simple groups possessing a minimally irreducible character χ of prime power degree $\chi(1) = s^d$. The case where $\chi(1) = s$ is a prime was analyzed in [5]. Moreover, in [6] the authors classified the finite non-simple groups with non-soluble socle having minimally irreducible characters of degree $\chi(1) = pq$, where p, q are two distinct primes. Our main result can be summarized as follows:

THEOREM 1. – *Let G be a finite simple group. Let χ be an irreducible complex character of G, such that $\chi(1) = s^d$, where s is a prime and $d > 1$. Then χ is minimally irreducible, if and only if one of the following holds:*

1. *G is a finite simple group of Lie type of characteristic s and $\chi = St$ is the Steinberg character of G;*
2. *$G \cong PSL(2, q)$ and $\chi(1) = \dfrac{q+1}{2}$, where q is odd;*

3. *$G \cong PSL(2,q)$ and $\chi(1) = q+1$;*
4. *$G \cong PSL(2, 2^d+1)$ and $\chi(1) = 2^d$;*
5. *$G \cong PSL(n,q)$, $q > 2$, n is an odd prime, $(n, q-1) = 1$ and $\chi(1) = \dfrac{q^n-1}{q-1}$;*
6. *$G \cong PSU(n,q)$, n is an odd prime, $(n, q+1) = 1$ and $\chi(1) = \dfrac{q^n+1}{q+1}$;*
7. *$G \cong PSU(3,3)$ and $\chi(1) = 2^5$.*

The proof of Theorem 1 will be established in the subsequent sections of this paper using the classification of quasi-simple groups possessing irreducible characters of prime power degree. Alternating groups admitting an irreducible character of prime power degree were investigate in [1]. The complete classification follows from the work of G. Malle and A.E. Zalesskiĭ ([13]). From these results we conclude the following:

THEOREM 2 (cf. [1, 13]). – *Let G be a finite simple group, and let χ be an irreducible complex character of G of prime power degree s^d, $d > 1$. Then one of following holds:*

1. *G is a finite simple group of Lie type of characteristic s and χ is the Steinberg character of G;*
2. *$G = PSL(2,q)$ and $\chi(1) \in \{q \pm 1\}$, or q is odd and $\chi \in \{(q\pm 1)/2\}$;*
3. *$G = PSL(n,q)$, $q > 2$, n is an odd prime, $(n, q-1) = 1$, $\chi(1) = (q^n-1)/(q-1)$;*
4. *$G = PSU(n,q)$, n is an odd prime, $(n, q+1) = 1$, $\chi(1) = (q^n+1)/(q+1)$;*
5. *$G = PSp(2n,q)$, $n > 1$, $q = r^k$ where r is an odd prime, kn is a 2-power and $\chi(1) = (q^n+1)/2$;*
6. *$G = PSp(2n,3)$, $n > 1$ is a prime, $\chi(1) = (3^n-1)/2$;*
7. *$G = \mathbb{A}_{s^d+1}$, $\chi(1) = s^d$;*
8. *$s^d = 16$, $G \in \{M_{11}, M_{12}, PSL(3,3)\}$;*
9. *$s^d = 27$, $G \in \{\mathbb{A}_9,\ PSp(6,2),\ {}^2F_4(2)'\}$;*
10. *$s^d = 32$, $G = PSU(3,3)$;*
11. *$s^d = 64$, $G = G_2(3)$.*

In case that the maximal subgroups of the simple group G together with their associated permutation characters are known, the following lemma turns out to be useful (cf. [14, Lemma 4.1]):

LEMMA 1. – *Let K be a finite group, $\chi \in Irr(K)$ and $H \leq K$, such that $1_H^K = 1_K + \sum_i a_i\phi_i$, where $1_K \neq \phi_i \in Irr(K)$. Then $\chi_{|_H}$ is an irreducible character of H, if and only if $(\chi, \chi\cdot\phi_i)_K = 0$, for all i.*

2. – Proof of Theorem 1.

2.1 – *The Steinberg character.*

The first character we analyze is the Steinberg character St. We briefly recall the definition of this character. For more details, see [2] and [4].

Let G be a finite group with a (B,N)-pair of rank l and let W be the Weyl group of G. Then $W = \langle w_1, \ldots, w_l \rangle$, where w_j is a fundamental reflection, with $j \in I = \{1, \ldots, l\}$. Let W_J be the standard parabolic subgroup of W generated by the set $\{w_j : j \in J\}$ and let P_J be the standard parabolic subgroup of G corresponding to W_J. The Steinberg character of G is the (virtual) character

$$St = \sum_{J \subseteq I} (-1)^{|J|} 1_{P_J}^G.$$

It turns out (e.g., see [4, 67.10]) that this virtual character is actually an irreducible character of G. Moreover, if G is a finite group with a split (B,N)-pair of rank l and characteristic s, then $St(1) = |G|_s$, the order of a Sylow s-subgroup of G.

This character has the following property:

PROPOSITION 1. – *Let G be a finite simple group of Lie type of rank l and characteristic s and let St be the Steinberg character of G. Then St is minimally irreducible.*

PROOF. – The statement was proved in [14] within the framework of the theory of unipotent characters of groups of Lie type. For the convenience of the reader we give a short self-contained proof.

Let $s^d = |G|_s$ and suppose that St is not minimally irreducible. Then there exists a maximal subgroup H of G such that $St_{|_H}$ is irreducible. Hence $St(1) = s^d \mid |H|$ and H contains a Sylow s-subgroup U of G of order s^d. Then, by a result of J. Tits (e.g. see [15, 1.6]), H is G-conjugate to a standard parabolic subgroup P of G. In particular, $P = U \rtimes L$, where L denotes the Levi complement of P and $U = O_p(P) \neq 1$. As $(St, 1_B^G)_G = 1$ (e.g., see [4, 67.10]),

$$1 = (St, 1_B^G)_G = (St_{|_B}, 1_B)_B, \quad \text{by Frobenius reciprocity.}$$

Hence

$$(St_{|_U}, 1_U)_U > 0.$$

Since $U \triangleleft P$, Clifford's theorem implies that $U \subseteq Ker(St)$, a contradiction, and the claim follows.

Observe that the restriction of the Steinberg character of ${}^2F_4(2)$ to the group $G = {}^2F_4(2)'$ splits into the sum of two distinct irreducible characters of degree 2^{11}. Using [3], one can prove that both these characters are minimally irreducible.

2.2 – *The groups* $PSL(2, q)$.

Let $G = PSL(2, q)$, where $q = p^a$, p a prime. The subgroups of G were classified about one hundred years ago by L. E. Dickson (e.g., cf. [10] for a modern account). The character table of G is also well-known (e.g., see [7]). In particular, the non-trivial irreducible characters of G have the following degrees:

1) q, $q \pm 1$, $(q+1)/2$ if $q \equiv 1 \pmod 4$;
2) q, $q \pm 1$, $(q-1)/2$ if $q \equiv -1 \pmod 4$;
3) q, $q \pm 1$, if $p = 2$.

From this information one concludes the following:

PROPOSITION 2. – *Let $G = PSL(2, q)$ and let χ be an irreducible complex character of G of degree s^d, where s is a prime and $d > 1$. Then χ is minimally irreducible if and only if:*

- $s^d = q$;
- $s^d = \dfrac{q+1}{2}$ *and p is odd;*
- $s^d = q - 1$ *and* $q = 2^d + 1$;
- $s^d = q + 1$ *and* $q \neq 3$.

PROOF. – Set $q = p^a$. According to Dickson's theorem (cf. [10]), a subgroup H of G must satisfy one of the following properties:

(a) H is abelian;
(b) H is dihedral;
(c) H is isomorphic to $\mathbb{A}_4$ or $\mathbb{S}_4$;
(d) H is isomorphic to $\mathbb{A}_5$ and either $p = 5$ or $p^{2a} \equiv 1 \pmod 5$;
(e) H is isomorphic either to $PSL(2, p^k)$, where $k \mid a$, or to $PGL(2, p^k)$, where $2k \mid a$;
(f) H is the semidirect product of an elementary abelian p-group P of order p^k with a cyclic group of order t, where $t \left| \left(\dfrac{p^k - 1}{d}, p^a - 1 \right) \right.$ and $d = (p^a - 1, 2)$.

Subgroups satisfying (a) or (b) have irreducible characters of degree ≤ 2 and hence they are obviously ruled out. Similarly, we may rule out the subgroups in (c). As for case (d), the group $\mathbb{A}_5$ only has non-trivial irreducible characters of degree 3, 4 and 5.

1. Let $s^d = q$. Then χ is the Steinberg character of G. As $q \geq 4$, G is simple. Hence this character is minimally irreducible, by Proposition 1.

2. Let χ be an irreducible character of degree $s^d = \dfrac{q-1}{2}$. Denote by $\tilde{\chi}$ the irreducible character of $SL(2, q)$ to which χ lifts, and let T be the subgroup of

$SL(2,q)$ consisting of the lower triangular matrices. A direct computation shows that $(\tilde{\chi}_{|_T}, \tilde{\chi}_{|_T})_T = 1$, and hence $\tilde{\chi}_{|T}$ is irreducible. It follows that χ is not minimally irreducible.

3. Let χ be an irreducible character of degree $s^d = \frac{q+1}{2}$. Suppose that χ is not minimally irreducible. Then there exists a proper subgroup H of G such that the restriction $\chi_{|H}$ is irreducible, and $\chi(1) = \frac{q+1}{2}$ divides $|H|$. Scrutiny of the list of subgroups of G shows that this cannot occur. Indeed, H cannot be as in (d), since $4 = \frac{q+1}{2}$ implies $q = 7$, contradicting the restrictions on p. In case (e), $\chi(1)^2 > |H|$, which contradicts the character degrees formula. Finally, case (f) is ruled out by observing that s and p are coprime and $\frac{p^a+1}{2} \nmid p^a - 1$. Thus we have excluded every candidate for H provided by Dickson's list.

4. Let χ be an irreducible character of degree $s^d = q-1$. If $p = 2$, as in 2, it is enough to consider $\chi_{|_T}$, where T is the subgroup of G consisting of the lower triangular matrices. Direct computation shows that $\chi_{|_T}$ is irreducible. Next, suppose that p is odd. Then $s = 2$. Suppose that χ is not minimally irreducible. Then there exists a proper subgroup H of G, such that the restriction $\chi_{|H}$ is irreducible, and hence $\chi(1) = 2^d$ divides $|H|$. All items in Dickson's list from (a) to (e) are easily ruled out (in particular, in case (d) one obtains $\chi(1) = 2^2$ and hence $H = G = PSL(2,5)$, a contradiction; whereas in case (e) one obtains $|H| < \chi(1)^2$). In case (f), $|H| = p^k \cdot t$. By Ito's theorems, $\chi(1)$ divides $|H : P| = t$. This implies that $\chi(1) = p^a - 1$ divides $\frac{p^k-1}{2}$, which is impossible.

5. Let χ be an irreducible character of degree $s^d = q+1$. First of all, observe that G has an irreducible character of this degree only for $q \geq 4$. Next, if s is odd, then $p = 2$ and hence $\chi(1) = 3^2$ and $q = 8$. Using [3] we obtain that χ is minimally irreducible. Thus, we may assume $s = 2$. It follows that q is a Mersenne prime, and hence $q = p$ and d is a prime. Suppose that χ is not minimally irreducible. Let H be a proper subgroup of G such that $\chi_{|_H}$ is irreducible. Then $q+1 \mid |H|$ and inspection of Dickson's list shows that all possibilities from (a) to (f) can be ruled out. In particular: if H belongs to (f), then $|H| = p \cdot t$, where $t \mid p-1$. As $2^d = p+1$ and p are coprime, 2^d should divide t, and hence $p-1$, which is impossible.

2.3 – *The groups* $PSL(n,q)$, $n \geq 3$.

In this and the subsequent section we investigate properties of the Zsigmondy primes. Recall (cf. [17]) that, if a and b are integers such that $a \geq 2$, $b \geq 3$ and $(a,b) \neq (2,6)$, then there exists a prime $\zeta_b(a)$ which divides $a^b - 1$, but does not divide $a^c - 1$ for all $c = 1, \ldots, b-1$. Such a prime is called a primitive prime divisor or a Zsigmondy prime for the pair (a, b). For these prime divisors one has the following (e.g., cf. [11, Prop. 5.2.15]):

LEMMA 2. – *Assume $a \geq 2$, $b \geq 3$ and $(a, b) \neq (2, 6)$. Let $\zeta_b(a)$ be a Zsigmondy prime for the pair (a, b).*

(a) *If $\zeta_b(a) \mid a^c - 1$, then $b \mid c$;*
(b) $\zeta_b(a) \equiv 1 \pmod{b}$.

PROPOSITION 3. – *Let $G = PSL(n, q)$, where n is an odd prime, $q > 2$ and $(n, q-1) = 1$. Let χ be an irreducible character of G of degree $\chi(1) = \dfrac{q^n - 1}{q - 1}$. Suppose that $\chi(1) = s^d$, where s is a prime and $d \geq 2$. Then χ is minimally irreducible.*

PROOF. – First, we observe that s is the unique Zsigmondy prime for the pair (q, n). It follows that s is odd and $s \equiv 1 \pmod{n}$. Furthermore, as n is a prime, $s > n+1$ and $s \geq 7$. Suppose that χ is not minimally irreducible. Then there exists a maximal subgroup H of G such that the restriction $\chi_{|H}$ is irreducible. The maximal subgroups of G fall into 8 'natural' classes $\mathcal{C}_i$ $(1 \leq i \leq 8)$, the so-called Aschbacher classes, plus a class $\mathcal{S}$ of 'small' subgroups, which are almost simple and act projectively and absolutely irreducible on the natural G-module $V(n, q)$. For an accurate description of the order and structure of subgroups belonging to the Aschbacher classes, the reader is referred to [11, Chapter 4].

Since s^d divides $|H|$, H must contain a Sylow s-subgroup of G, which is cyclic of order s^d (a 'Coxeter' torus of G). By [12, Theorem 1.1] we know the maximal subgroups H of G containing such a subgroup: either they belong to the class $\mathcal{C}_3$ or $(H, G) \in \{(PSL(3, 2), PSL(3, 4)), (\mathbb{A}_7, PSL(4, 2))\}$.

Under our assumptions on the degree of χ, the only possibility are the groups $H \cong \mathbb{Z}_{s^d}.\mathbb{Z}_n$, belonging to the class $\mathcal{C}_3$ (the class of 'field extension stabilizers'). Hence, by Ito's theorem, s^d divides n. But this contradicts the assumption that n is a prime.

2.4 – *The groups $PSU(n, q)$, $n \geq 3$.*

PROPOSITION 4. – *Let $G = PSU(n, q)$, where n is an odd prime and $(n, q+1) = 1$. Let χ be an irreducible character of G of degree $\chi(1) = \dfrac{q^n + 1}{q + 1}$. Suppose that $\chi(1) = s^d$, where s is a prime and $d \geq 2$. Then χ is minimally irreducible.*

PROOF. – The proof is similar to that provided in 2.3. First of all, as in 2.3, we observe that s is the unique Zsigmondy prime for both the pairs $(q, 2n)$ and (q^2, n). It follows that s is odd, $s \equiv 1 \pmod{2n}$ and $s \nmid q^i - 1$ for all $1 \leq i \leq 2n - 1$. Furthermore, $s \geq 7$.

Suppose that χ is not minimally irreducible. Then there exists a maximal subgroup H of G such that the restriction $\chi_{|H}$ is irreducible. We have to consider two cases: either H belongs to one of 7 Aschbacher classes $\mathcal{C}_i$ $(1 \leq i \leq 7)$, or H belongs to the class $\mathcal{S}$. Since s^d divides $|H|$, H must contain a Sylow s-subgroup of G, which is cyclic of order s^d. By [12, Theorem 1.1] we know the maximal subgroups H of G containing such a subgroup: either they belong to the class $\mathcal{C}_3$ or $G \in \{PSU(3,3), PSU(3,5), PSU(4,3), PSU(5,2), PSU(6,2)\}$. Once again, under our assumptions, the only possibility are the groups $H \cong \mathbb{Z}_{s^d}.\mathbb{Z}_n$, belonging to the class $\mathcal{C}_3$ (the class of 'field extension stabilizers'). Hence, by Ito's theorem, s^d divides n. But this contradicts the assumption that n is a prime.

2.5 – *The groups $PSp(2n, q)$.*

It is well-known that the group $G = Sp(2n, q)$, q odd, has exactly two irreducible complex characters η, η^* of degree $(q^n - 1)/2$ and exactly two irreducible complex characters ξ, ξ^* of degree $(q^n + 1)/2$. These are the so-called Weil characters of G (e.g., see [8, 9, 16]). In [5] it was shown (regardless of whether the degree is a prime power or not) that these characters are never minimally irreducible.

2.6 – *The alternating groups.*

The irreducible characters of prime power degree of $\mathbb{A}_n$ were described in [1].

PROPOSITION 5. – *Let p be a prime, with $d > 1$ and suppose $p^d \geq 4$. Then the irreducible character of $\mathbb{A}_{p^d+1}$ having degree p^d is not minimally irreducible.*

PROOF. – The proof follows the same lines as that given in [5] for the case where $d = 1$. Nevertheless we offer it here, for the sake of completeness.

It is well-known (e.g., cf. [1]) that if $n \geq 5$ then $G = \mathbb{A}_{p^d+1}$ has a unique irreducible character χ of degree p^d. Let Ω be the canonical G-set of order $p^d + 1$. Then, $\chi_{|_H}$ is irreducible for a subgroup H of G, if and only if Ω is a 2-transitive H-set. Since $PSL(2, p^d) \leq G$ acts 2-fold transitively on Ω, this yields the claim.

2.7 – *Other groups and characters appearing in Theorem* 2.

In this section we prove that the characters listed in [13] as items 8, 9 and 11 are not minimally irreducible, whereas the two characters of degree 32 of $PSU(3,3)$, listed as item 10, are indeed minimally irreducible.

$G = M_{11}$, $s^d = 16$. G has two irreducible characters of degree 16. Using Lemma 1 and [3], one sees that the restrictions of these characters to a maximal subgroup $H \cong M_{10}$ are irreducible.

$G = M_{12}$, $s^d = 16$. G has two irreducible characters of degree 16. Using Lemma 1 and [3], one sees that the restrictions of these characters to a maximal subgroup $H \cong M_{11}$ are irreducible.

$G = PSL(3,3)$, $s^d = 16$. G has four irreducible characters of degree 16. Using Lemma 1 and [3] one sees that the restriction of these characters to a maximal subgroup H isomorphic to $3^2 : 2\mathbb{S}_4$ is irreducible.

$G = \mathbb{A}_9$, $s^d = 27$. G has a unique irreducible character of degree 27. Using Lemma 1 and [3], one sees that the restriction of this character to a maximal subgroup H isomorphic to $PSL(2,8) : 3$ is irreducible.

$G = PSp(6,2)$, $s^d = 27$. G has a unique irreducible character of degree 27. Using Lemma 1 and [3], one sees that the restriction of this character to a maximal subgroup H isomorphic to $PSU(3,3) : 2$ is irreducible.

$G = {}^2F_4(2)'$, $s^d = 27$. G has two irreducible characters of degree 27. Using Lemma 1 and [3], one sees that the restrictions of these characters to a maximal subgroup H isomorphic to $PSL(3,3) : 2$ are irreducible.

$G = G_2(3)$, $s^d = 64$. G has two irreducible characters of degree 64. Using Lemma 1 and [3], one sees that the restrictions of these characters to a maximal subgroup H isomorphic to $PSU(3,3) : 2$ are irreducible.

$G = PSU(3,3)$, $s^d = 32$. G has two irreducible characters of degree 32. The maximal subgroups of $PSU(3,3)$ have orders 96, 168 or 216 (see [3]). Since $1024 = 32^2 > 216$, these characters are minimally irreducible.

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Pervenuta in Redazione
il 2 marzo 2006 e in forma rivista il 5 luglio 2007