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The Banach-Lie Group of Lie Automorphisms of an H^*-Algebra.

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Sunto. – *Studiamo il gruppo di Banach-Lie* $\mathrm{Aut}\,(A^-)$ *degli automorfismi di Lie di una* H^**-algebra associativa complessa. Vengono anche ottenute alcune conseguenze riguardanti la sua algebra di Lie, cioè l'algebra delle derivazioni di Lie di A. Per una A topologicamente semplice, nel caso di dimensione infinita si ha* $\mathrm{Aut}\,(A^-)_0 = \mathrm{Aut}\,(A)$, *il che implica che* $\mathrm{Der}\,(A) = \mathrm{Der}\,(A^-)$. *Nel caso di dimensione finita,* $\mathrm{Aut}\,(A^-)_0$ *è il prodotto diretto di* $\mathrm{Aut}\,(A)$ *e di un certo sottogruppo di derivazioni di Lie* δ *da A al suo centro, che annullano i commutatori.*

Summary. – *We study the Banach-Lie group* $\mathrm{Aut}\,(A^-)$ *of Lie automorphisms of a complex associative* H^**-algebra A. Also some consequences about its Lie algebra, the algebra of Lie derivations of A, are obtained. For a topologically simple A, in the infinite-dimensional case we have* $\mathrm{Aut}\,(A^-)_0 = \mathrm{Aut}\,(A)$ *implying* $\mathrm{Der}\,(A) = \mathrm{Der}\,(A^-)$. *In the finite dimensional case* $\mathrm{Aut}\,(A^-)_0$ *is a direct product of* $\mathrm{Aut}\,(A)$ *and a certain subgroup of Lie derivations* δ *from A to its center, annihilating commutators.*

1. – Preliminary results and definitions.

We recall that an H^*–*algebra* A over $\mathbb{C}$ is a, non–necessarily associative, $\mathbb{C}$–algebra whose underlying vector space is a complex Hilbert space, endowed with a conjugate–linear map $* : A \to A$ ($x \mapsto x^*$), such that $(x^*)^* = x$, $(xy)^* = y^*x^*$ for any $x, y \in A$ and the following hold

$$(xy|z) = (x|zy^*) = (y|x^*z)$$

for all $x, y, z \in A$. The map $*$ will be called the *involution* of the H^*–algebra. The continuity of the product of A is proved in [7]. We call the H^*–algebra A, *topologically simple* if $A^2 \neq 0$ and A has no nontrivial proper closed ideals. H^*–al-

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gebras were introduced and studied by W. Ambrose [1] in the associative case, and have been also considered in the case of the most familiar classes of non-associative contexts [3, 6, 7, 12] and even in the general nonassociative contexts [7]. Given an associative H^*-algebra A, for any $x, y \in A$, we shall denote by $[x, y]$ the usual bracket $[x, y] := xy - yx$. In this context, a *Lie derivation* of A is a linear map $D : A \to A$ such that

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in A$. If A and A' are associative H^*-algebras, a *Lie isomorphism* $f : A \to A'$ is a linear isomorphism, such that

$$f([x, y]) = [f(x), f(y)]$$

for all $x, y \in A$. In [7] it is proved that any H^*–algebra A with continuous involution splits into the orthogonal direct sum $A = Ann(A) \perp \overline{\mathcal{L}(A^2)}$, where $Ann(A) := \{x \in A : xA = Ax = 0\}$ is the *Annihilator* of A, and $\overline{\mathcal{L}(A^2)}$ is the closure of the vector span of A^2, which turns out to be an H^*–algebra with zero annihilator. Moreover, each H^*–algebra A with zero annihilator satisfies $A = \overline{\perp I_a}$ where $\{I_a\}_a$ denotes the family of minimal closed ideals of A, each of them being a topologically simple H^*–algebra. This focuses the interest on H^*-algebras to the topologically simple case. If A is an associative complex topologically simple H^*-algebra, W. Ambrose proved in [1] that up to a positive factor of the inner product, A is isometrically $*$-isomorphic to the algebra of Hilbert-Schmidt operators $\mathcal{HS}(H)$ on a complex Hilbert space H. Under this $*$-isomorphism, the H^*-involution of A is identified with the map $\mathcal{HS}(H) \to \mathcal{HS}(H)$, $T \mapsto T^\sharp$ where $T^\sharp$ is the adjoint of T relative to the inner product $(\cdot | \cdot)$ in H. The inner product in A is identified with the usual inner product in $\mathcal{HS}(H)$ given by

$$\langle T, S \rangle := \sum_a (T(e_a)|S(e_a))$$

where $\{e_a\}$ is a Hilbert basis of H.

We recall that any derivation on arbitrary H^*–algebras with zero annihilator is continuous [15]. Also, isomorphisms of H^*–algebras with zero annihilator are continuous [5, Corolario 1-2-37, p. 21].

The aim of the present paper is to study the Banach-Lie group of an (associative) H^*-algebra with zero annihilator. As a consequence, we will describe its Lie algebra, the algebra of Lie derivations of A. We finally note that the description of the Lie derivations of A maybe could be done from the structure theory of H^*-algebras and some classical results on the subject in [2, 9, 10]. However, in order to make the exposition as self-contained as possible and to show an application of the nice relation between a Banach-Lie group and its associate Lie algebra, we opt for developing the study of the Lie derivation as it is given in Section 3.

2. – Automorphisms and derivations of associative *H**-algebras.

Let $A = \mathcal{HS}(H)$ be the complex H^*-algebra of all Hilbert-Schmidt operators in the Hilbert space H with inner product $(\cdot \mid \cdot)$. We consider now H as a left complex vector space, and also as a right vector space H', with the action $x\lambda := \overline{\lambda}x$ for all $x \in H$ and $\lambda \in \mathbb{C}$. Then the couple (H, H') is a pair of dual vector spaces in the sense of [9, Definition 1, p. 69], relative to $(\cdot \mid \cdot)$. The H'-topology of H is defined in [9, Definition 2, p. 70]. A linear map $f : H \to H$ turns out to be continuous for the H'-topology of H, if and only if it has an adjoint (see [9, THEOREM 1, p. 72]). The complex algebra of continuous linear maps $H \to H$ (relative to the H'-topology of H), will be denoted by $\mathfrak{L}_{H'}(H)$ (see [9, p. 73]). This algebra agrees with that of continuous linear maps relative to the norm topology of H. We shall denote by $\mathfrak{F}_{H'}(H)$ the ideal of finite rank elements in $\mathfrak{L}_{H'}(H)$. Of course we have $\mathfrak{F}_{H'}(H) \subset \mathcal{HS}(H) \lhd \mathfrak{L}_{H'}(H)$ and $\mathfrak{F}_{H'}(H)$ is also an ideal in $\mathfrak{L}_{H'}(H)$ hence in $\mathcal{HS}(H)$. So the algebra $\mathcal{HS}(H)$ is an example of a prime algebra with nonzero socle. In the context of nonzero socle, primeness is equivalent to primitiveness so we can also say that $\mathcal{HS}(H)$ is a primitive algebra.

Consider now any $f \in \mathrm{Aut}\,(A)$. Applying the ISOMORPHISM THEOREM in [9, p. 79], we have the existence of a $\mathbb{C}$-linear homeomorphism $S : H \to H$ such that $f(T) = STS^{-1}$ for any $T \in \mathcal{HS}(H)$. On the other hand, if $D \in \mathrm{Der}\,(A)$, by applying [9, Theorem 3, p. 87], there is a continuous linear map $G : H \to H$ such that $D(T) = [G, T]$ for each $T \in \mathcal{HS}(H)$. The group $\mathrm{Aut}\,(A)$ is algebraic of degree ≤ 2 (see [14, Definition 7.13, p. 117 and example 7.15, p. 119]), hence it is a Banach Lie group in the operator norm topology. Its Lie algebra is then $\mathrm{Der}\,(A)$ (see for instance [14, Theorem 7.14, p. 118]).

In case H is finite-dimensional, the polar decomposition provides a retraction from $\mathrm{GL}\,(n, \mathbb{C})$ onto $\mathrm{U}\,(n, \mathbb{C})$ (the unitary group) which becomes a strong deformation retract via the map $(X, s) \mapsto X(\overline{X}^t X)^{-s/2}$. As $\mathrm{U}\,(n, \mathbb{C})$ is connected, we conclude that $\mathrm{GL}\,(n, \mathbb{C})$ is also connected and therefore $\mathrm{Aut}\,(A)$ is it. Suppose now that H is infinite-dimensional. Since any operator in $\mathrm{L}(H)$, the Banach algebra of bounded linear operators on the Hilbert space H, allows polar decomposition, the same retraction and homotopy as before, prove that the general linear group $\mathrm{GL}\,(H)$ of invertible operators in $\mathrm{L}(H)$ is connected (see [13]). Thus $\mathrm{Aut}\,(A)$ is a connected Banach Lie group in any case.

3. – Lie Automorphisms and derivations.

For any associative H^*-algebra A, we denote by A^- the antisymmetrized H^*-algebra of A. Both algebras have the same underlying Hilbert space, involution and inner product. The only difference is the product

$$[\cdot, \cdot] : A^- \times A^- \to A^-$$

of A^- which can be written in terms of the product of A by the formula $[x,y] := xy - yx$ for all $x, y \in A$. Using this notion, the group of Lie automorphism of A is just $\mathrm{Aut}\,(A^-)$ while the algebra of Lie derivations of A is just the Lie algebra $\mathrm{Der}\,(A^-)$. We have proved in [4], the following fact: let $f : A \to A'$ be a Lie isomorphism of associative H^*-algebras with zero annihilator, and $A = \overline{\perp_{a\in\mathcal{A}} I_a}$ the decomposition of A as the closure of the orthogonal direct sum of its minimal closed ideals I_a, then $A = P \perp Q$ for some closed ideals $P, Q \triangleleft A$ with $P = \overline{\perp_{a\in\mathcal{A}_1} I_a}$, $Q = \overline{\perp_{a\in\mathcal{A}_2} I_a}$, $\mathcal{A} = \mathcal{A}_1 \dot{\cup} \mathcal{A}_2$, and there exists a $\mathbb{C}$-linear bijective map $f' : A \to A$ such that (1) the restriction $f'|_P$ is an isomorphism, (2) the restriction $f'|_Q$ is the negative of an anti-isomorphism, (3) $f'|_{I_a} = f|_{I_a}$ for each infinite-dimensional I_a, and (4) $\delta_a := f'|_{I_a} - f|_{I_a}$ is a linear map from I_a to the center of A', mapping commutator to zero, for each finite-dimensional I_a. In particular, if A and A' are topologically simple, we conclude some of the following excluding possibilities:

1. If A is infinite-dimensional then $f : A \to A'$ is an isomorphism or the negative of an anti-isomorphism.
2. If A is finite-dimensional, there is an map $f' : A \to A'$ which is an isomorphism or the negative of an anti-isomorphism such that $f = f' + \delta$ where $\delta : A \to \mathrm{Z}\,(A')$ (the center of A') and δ maps commutator to zero.

Let now $A = \mathcal{HS}(H)$ be an associative topologically simple H^*-algebra. As in the previous section, we have again the structure of a Banach-Lie group on $\mathrm{Aut}\,(A^-)$. Trivially $\mathrm{Aut}\,(A) \subset \mathrm{Aut}\,(A^-)$ and indeed $\mathrm{Aut}\,(A)$ is a Banach-Lie subgroup of $\mathrm{Aut}\,(A^-)$. In the same way $\mathrm{Der}\,(A)$ is a subalgebra of $\mathrm{Der}\,(A^-)$. According to our previous results, if A is infinite-dimensional, then for any $f \in \mathrm{Aut}\,(A^-)$, we have $f \in \mathrm{Aut}\,(A)$ or $f = -g$ for some anti-automorphism $g : A \to A$. Denoting by $\mathrm{Antiaut}\,(A)$ the set of anti-automorphisms of A, and writing $-\mathrm{Antiaut}\,(A) = \{-f : f \in \mathrm{Antiaut}\,(A)\}$ we have $\mathrm{Aut}\,(A^-) = \mathrm{Aut}\,(A) \cup (-\mathrm{Antiaut}\,(A))$. Moreover, it is not difficult to see that $\mathrm{Aut}\,(A) \cap (-\mathrm{Antiaut}\,(A)) = \emptyset$. We also know that $\mathrm{Aut}\,(A)$ is a connected Banach-Lie group, and taking into account the adjoint map $\sharp : A \to A$ (an involutive antiautomorphism of A), the map $\mathrm{Aut}\,(A) \to -\mathrm{Antiaut}\,(A)$ such that $f \mapsto -\sharp \circ f$, is an homeomorphism. This proves that $\mathrm{Aut}\,(A)$ and $-\mathrm{Antiaut}\,(A)$ are the connected components of $\mathrm{Aut}\,(A^-)$, and the identity component of $\mathrm{Aut}\,(A^-)$ is $\mathrm{Aut}\,(A^-)_0 = \mathrm{Aut}\,(A)$.

If A is finite-dimensional, and $f \in \mathrm{Aut}\,(A^-)$ then, there exists $g \in \mathrm{Aut}\,(A) \cup (-\mathrm{Antiaut}\,(A))$ such that $f - g = \delta : A \to \mathrm{Z}\,(A)$ where δ is a linear map annihilating commutators: $\delta([A,A]) = 0$. We can consider the following sets: S_1 is the one formed by all the linear maps $f + \delta : A \to A$ such that $f \in \mathrm{Aut}\,(A)$, and $\delta : A \to Z(A)$ is linear, $\delta([A,A]) = 0$ and $\delta(1) \neq -1$; on the other hand S_2 is defined as the set of all $f + \delta$ such that $f \in -\mathrm{Antiaut}\,(A)$, and $\delta : A \to Z(A)$ is linear,

$\delta([A,A]) = 0$, $\delta(1) \neq 1$. It is straightforward that $S_i \subset \mathrm{Aut}\,(A^-)$ for $i = 1, 2$. Also $\mathrm{Aut}\,(A^-) = S_1 \cup S_2$ and it is not difficult to prove that $S_1 \cap S_2 = \emptyset$. Moreover S_1 is connected: consider an element $f + \delta \in S_1$, with $f \in \mathrm{Aut}\,(A)$; we have $f(T) = PTP^{-1}$ for any $T \in A$ and $P \in A$ being an invertible element. Thus $P \in \mathrm{GL}\,(n, \mathbb{C})$ (we can identify A with $\mathcal{M}_n(\mathbb{C})$ and $A^\times$ with $\mathrm{GL}\,(n, \mathbb{C})$). Next, since $\mathrm{GL}\,(n, \mathbb{C})$ is connected, consider a continuous path $Q : [0, 1] \to \mathrm{GL}\,(n, \mathbb{C})$ such that $Q(0) = id$ and $Q(1) = P$. Now define $f : [0, 1] \to S_1$ by $f(t)(M) = Q(t)MQ(t)^{-1} + \delta(M)$, $M \in A$. This is a continuous path in S_1, joining $id + \delta$ with $f + \delta$. So S_1 is connected and in a similar way S_2 is it. Of course these are the connected components of the Banach-Lie group $\mathrm{Aut}\,(A^-)$. Summarizing the previous paragraphs we can claim:

PROPOSITION 3.1. – *Let A be a topologically simple complex H^*-algebra. Then the group $\mathrm{Aut}\,(A^-)$ of Lie automorphisms of A is a Banach-Lie group with two connected components $\mathrm{Aut}\,(A^-) = \mathrm{Aut}\,(A^-)_0 \cup \mathrm{Aut}\,(A^-)_\sharp$. The connected component $\mathrm{Aut}\,(A^-)_0$ agrees with the set of Lie automorphisms of the form $f + \delta$ where $f \in \mathrm{Aut}\,(A)$ and $\delta \in \hom\,(A, \mathrm{Z}\,(A))$, with $\delta(\overline{[A, A]}) = 0$, $\delta(1) \neq -1$. The other component is formed by the Lie automorphisms of the form $f + \delta$ where $f \in -\mathrm{Antiaut}\,(A)$ and $\delta \in \hom\,(A, \mathrm{Z}\,(A))$, with $\delta(\overline{[A, A]}) = 0$, $\delta(1) \neq 1$. If A is infinite-dimensional $\overline{[A, A]} = A$ and therefore $\delta = 0$, hence $\mathrm{Aut}\,(A^-)_0 = \mathrm{Aut}\,(A)$ while $\mathrm{Aut}\,(A^-)_\sharp = -\mathrm{Antiaut}\,(A)$.*

Since the Lie algebra of the Banach-Lie group $\mathrm{Aut}\,(A^-)$ is $\mathrm{Der}\,(A^-)$, and the Lie algebra of $\mathrm{Aut}\,(A)$ is just $\mathrm{Der}\,(A)$, taking into account that in the infinite dimensional case, both groups have the same identity component $\mathrm{Aut}\,(A^-)_0 = \mathrm{Aut}\,(A) = \mathrm{Aut}\,(A)_0$, we conclude

$$\mathrm{Der}\,(A^-) = \mathrm{Der}\,(A). \tag{1}$$

In order to refine our knowledge of $\mathrm{Aut}\,(A^-)_0$ we can prove the following:

PROPOSITION 3.2. – *Any element $g \in \mathrm{Aut}\,(A^-)_0$ can be written as $g = f + \delta$ for a unique $f \in \mathrm{Aut}\,(A)$ and a unique $\delta : A \to \mathrm{Z}\,(A)$, such that $\delta(\overline{[A, A]}) = 0$ and $\delta(1) \neq -1$.*

PROOF. – The uniqueness property is the only thing we have to prove. Suppose $f + \delta = f' + \delta'$ with $f, f' \in \mathrm{Aut}\,(A)$, $\delta, \delta' : A \to \mathrm{Z}\,(A)$, $\delta(\overline{[A,A]}) = \delta'(\overline{[A,A]}) = 0$, $\delta(1), \delta'(1) \neq -1$. In the infinite-dimensional case there is nothing to prove: necessarily $\delta = \delta' = 0$ and $f = f'$. If A is finite dimensional the maps δ and δ' are completely determined by $\delta(1)$ and $\delta'(1)$ respectively. Then, $f(1) + \delta(1) = f'(1) + \delta'(1)$ and as $f(1) = f'(1) = 1$ we conclude $\delta(1) = \delta'(1)$, hence $\delta = \delta'$ implying $f = f'$. □

In the finite dimensional case, the linear maps $\delta : A \to \mathrm{Z}(A)$ such that $\delta([A,A]) = 0$, are completely determined by the element $\delta(1) \in \mathrm{Z}(A)$. In fact, $\mathrm{Z}(A) = \mathbb{C}1$ and $A = [A,A] + \mathrm{Z}(A)$ imply that assertion. So we can define a map $\theta : \hom(A/[A,A], \mathrm{Z}(A)) \to \mathbb{C}$ such that $\delta \mapsto \delta(1 + [A,A])$ for any $\delta \in \hom(A/[A,A], \mathrm{Z}(A))$. It is not difficult to see that θ is a bijective linear map. We can restrict θ (removing the null element in each set) so as to have a group isomorphism $\theta : \hom(A/[A,A], \mathrm{Z}(A))^* \to \mathbb{C}^*$ for the unique possible operation in $\hom(A/[A,A], \mathrm{Z}(A))^*$, making θ a group isomorphism. Consider now the following sequence of group homomorphisms

$$1 \to \mathrm{Aut}(A) \xrightarrow{i} \mathrm{Aut}(A^-)_0 \xrightarrow{p} \mathbb{C}^* \to 1, \tag{2}$$

where i is the inclusion map and for any $g \in \mathrm{Aut}(A^-)_0$, we define $p(g) = g(1)$, (the fact that p is a group epimorphism is an easy consequence of its definition). It can be checked that $pi = 1$ and that $\ker(p) = \mathrm{Aut}(A)$. Thus the sequence (2) is in fact a short exact sequence of groups. Moreover the maps i and p are homomorphisms of Lie groups (recall that a continuous homomorphism $f : G_1 \to G_2$ between the topological groups underlying, the (finite-dimensional) Lie groups G_i $(i = 1, 2)$, is necessarily a Lie groups homomorphism). So, finally (2) is a short exact sequence of (finite-dimensional) Lie groups. Furthermore, we can assert:

THEOREM 3.1. – *The short exact sequence* (2) *is split: there is a monomorphism of Lie groups* $j : \mathbb{C}^* \to \mathrm{Aut}(A^-)_0$ *such that* $pj = 1$. *The subgroups* $\mathrm{Aut}(A)$ *and* $j(\mathbb{C}^*)$ *of* $\mathrm{Aut}(A^-)_0$ *satisfy* $f\delta = \delta f$ *for any* $f \in \mathrm{Aut}(A)$ *and* $\delta \in j(\mathbb{C}^*)$. *Thus we have an isomorphism of Lie groups*

$$\mathrm{Aut}(A^-)_0 \cong \mathrm{Aut}(A) \times \mathbb{C}^*, \tag{3}$$

such that $g \mapsto (f, g(1))$ *(the unique* f *provided by Proposition* 3.2*).*

PROOF. – Let us denote by δ_μ the linear map $\delta_\mu : A \to \mathrm{Z}(A)$ annihilating all commutators and making $\delta_\mu(1) = \mu 1$. Define now $j : \mathbb{C}^* \to \mathrm{Aut}(A^-)_0$ by $j(\lambda) = 1 + \delta_{\lambda - 1}$. This is obviously an element in $\mathrm{Aut}(A^-)_0$ and a routine computations reveals that it is in fact a monomorphism of groups. The continuity of j is also easy to prove hence j is a monomorphism of Lie groups and trivially it verifies $pj = 1$. So the group $\mathrm{Aut}(A^-)_0$ is a semidirect product of $\mathrm{Aut}(A)$ and $j(\mathbb{C}^*)$. But if we take $f \in \mathrm{Aut}(A)$ and $\delta = \delta_\mu \in j(\mathbb{C}^*)$ we have $f\delta = \delta f$, since for any $x = c + a1$ with $c \in [A,A]$ and $a \in \mathbb{C}$, we can write $f\delta(x) = f\delta(a1) = f(a\mu 1) = a\mu 1 = \delta(f(a1)) = \delta(f(c + a1)) = \delta f(x)$. So $\mathrm{Aut}(A^-)_0$ is really a direct product of its subgroups $\mathrm{Aut}(A)$ and $j(\mathbb{C}^*)$. From this follows easily that the map given by (3) is a continuous isomorphism of topological groups hence a Lie groups isomorphism. □

Completing the previous results, we can exhibit an epimorphism of Lie groups $q : \operatorname{Aut}(A^-)_0 \to \operatorname{Aut}(A)$ such that $qi = 1$. This is given by $q(g) = f$ (the unique $f \in \operatorname{Aut}(A)$ given by Proposition 3.2).

We can now extract some consequences for Lie derivations in topologically simple associative H^*-algebras. In the infinite dimensional case we proved before that $\operatorname{Der}(A^-) = \operatorname{Der}(A)$ and as the elements of $\operatorname{Der}(A)$ have also been described in section 2, there is nothing more to say. In the finite dimensional case, $\operatorname{Der}(A^-)$ is the Lie algebra of the group $\operatorname{Aut}(A^-)_0$, while $\operatorname{Der}(A)$ is the Lie algebra of $\operatorname{Aut}(A)$. In the finite dimensional case, any Lie groups monomorphism is an immersion, thus the inclusion map $i : \operatorname{Aut}(A) \to \operatorname{Aut}(A^-)$ induces by differentiation, a Lie algebras monomorphism $di_1 : \operatorname{Der}(A) \to \operatorname{Der}(A^-)$. On the other hand $dp_1 : \operatorname{Der}(A^-) \to \mathbb{C}$ is a Lie algebras epimorphism (since $dp_1 dj_1 = 1$), and $dp_1 di_1 = 0$. So $di_1(\operatorname{Der}(A)) \subset \ker(dp_1)$ but a dimensional argument proves the equality $di_1(\operatorname{Der}(A)) = \ker(dp_1)$ (the isomorphism (3) also says that $\dim(\operatorname{Der}(A^-)) = 1 + \dim(\operatorname{Der}(A)))$. Summarizing we have a short exact sequence

$$0 \to \operatorname{Der}(A) \overset{di_1}{\to} \operatorname{Der}(A^-) \overset{dp_1}{\to} \mathbb{C} \to 0 \tag{4}$$

which is also split. As a corollary, we have a Lie algebras isomorphism $\operatorname{Der}(A^-) \cong \operatorname{Der}(A) \oplus \mathbb{C}$ which is the infinitesimal version of (3). Taking differentials it is easy to check that $dp_1 : \operatorname{Der}(A^-) \to \mathbb{C}$ acts in the following way: $dp_1(D) = D(1)$ for any $D \in \operatorname{Der}(A^-)$. In fact, any Lie derivation $D \in \operatorname{Der}(A^-)$ maps $\mathrm{Z}(A)$ to $\mathrm{Z}(A)$. Moreover the fact that $\operatorname{im}(di_1) = \ker(dp_1)$ means that for $D \in \operatorname{Der}(A^-)$, we have $D(1) = 0$ if and only if $D \in \operatorname{Der}(A)$. The map $dj_1 : \mathbb{C} \to \operatorname{Der}(A^-)$ acts mapping any $\lambda \in \mathbb{C}$ to the Lie derivation δ_λ which annihilates commutators an $\delta_\lambda(1) = \lambda 1$. Thus $\operatorname{Der}(A^-) = \operatorname{Der}(A) \oplus \delta_{\mathbb{C}}$ where $\delta_{\mathbb{C}}$ is the ideal of $\operatorname{Der}(A^-)$ of all maps δ_λ. Also $\operatorname{Der}(A)$ is an ideal of $\operatorname{Der}(A^-)$. Summarizing the results in the last paragraph we have:

THEOREM 3.2. – *Let $A = \mathcal{HS}(H)$ be the complex H^*-algebra of Hilbert-Schmidt operators in the Hilbert space H. If H is infinite dimensional then* $\operatorname{Der}(A^-) = \operatorname{Der}(A)$. *If H is finite dimensional, there is a split short exact sequence* (4) *which proves that* $\operatorname{Der}(A)$ *and* $\delta_{\mathbb{C}}$ *are ideals in* $\operatorname{Der}(A^-)$ *and* $\operatorname{Der}(A^-) = \operatorname{Der}(A) \oplus \delta_{\mathbb{C}}$. *Thus any Lie derivation D of $\mathcal{HS}(H)$ for a finite dimensional H is of the form $D = D' + \delta$ with $D \in$* $\operatorname{Der}(A)$ *and $\delta : A \to$* $\mathrm{Z}(A)$ *a linear map annihilating commutators.*

Finally we can give a version of the previous result not only for topologically simple complex H^*-algebras, but for H^*-algebras with zero annihilator:

THEOREM 3.3. – *Let A be an associative H^*–algebra with zero annihilator and let D be a Lie derivation on A. Then there exists a derivation d on A such that* *if we denote by $\{I_a\}$ the family of the minimal closed ideals of A we have:*

1. *If I_a is infinite dimensional then $D|_{I_a} = d|_{I_a}$.*
2. *If I_a is finite dimensional then $\delta_a := D|_{I_a} - d|_{I_a}$ is a linear mapping from I_a into the center of A sending commutators to zero.*

PROOF. – Denote by $\{I_a\}_{a\in\Lambda}$ the family of minimal closed ideals of A. Let us consider $I_{a_0} \in \{I_a\}_{a\in\Lambda}$.

If I_{a_0} is infinite dimensional and if we denote by d_{a_0} the restriction of D to I_{a_0}, since I_{a_0} is an infinite dimensional topologically simple associative H^*-algebra, from the classifications of topologically simple associative ([1]) and Lie H^*-algebras ([6]), we have that I_{a_0} is also a topologically simple Lie H^*-algebra and therefore $I_{a_0} = \overline{[I_{a_0}, I_{a_0}]}$. Hence, as D is continuous ([15]), we conclude easily that I_{a_0} is invariant under D. Theorem 3.2 now shows that $d_{a_0} : I_{a_0} \to I_{a_0}$ is a derivation, being also clear that $\|d_{a_0}\| \leq \|D\|$.

If I_{a_0} is finite dimensional with $\dim I_{a_0} > 1$, as I_{a_0} is isomorphic to an associative algebra of the type $\mathcal{M}_n(\mathbb{C})$, $n > 1$, then $[I_{a_0}, I_{a_0}]$, the vector span of $\{[x, y] : x, y \in I_{a_0}\}$, is a simple Lie algebra of type A_l and $Z(I_{a_0}) \simeq \mathbb{C}Id_n$. If we denote by D_{a_0} the restriction of D to $[I_{a_0}, I_{a_0}]$, by [8, Theorem 9, p. 80] D_{a_0} extends to a derivation $d_{a_0} : I_{a_0} \to I_{a_0}$. If we call

$$\delta_{a_0} := D|_{I_{a_0}} - d_{a_0} : I_{a_0} \to A,$$

we assert that $\delta_{a_0}(I_{a_0}) \subset Z(A)$ and that $\delta_{a_0}([I_{a_0}, I_{a_0}]) = 0$. Indeed, let us write any element $x \in I_{a_0}$ as $x = c + a$ with $c \in Z(I_{a_0}) \subset Z(A)$ and $a \in [I_{a_0}, I_{a_0}]$, (note that this decomposition is unique). We have that the character of derivation of d_{a_0} implies $d_{a_0}(c) = 0$. As D is a Lie derivation then $D|_{I_{a_0}}(c) \in Z(A)$. Finally, as $d_{a_0}(a) = D|_{I_{a_0}}(a)$ for any $a \in [I_{a_0}, I_{a_0}]$ we conclude $\delta_{a_0}(I_{a_0}) \subset Z(A)$ and $\delta_{a_0}([I_{a_0}, I_{a_0}]) = 0$. Let us observe that we also have in this case $\|d_{a_0}\| \leq \|D\|$.

Finally, if $\dim I_{a_0} = 1$ we define $d_{a_0} = 0$. As $A = \overline{\perp_{a\in\Lambda} I_a}$, the fact $\|d_a\| \leq \|D\|$ for all $a \in \Lambda$, and the continuous character of any d_a, allow us to extend $\{d_a\}_{a\in\Lambda}$ to a continuous derivation d on A. It is clear that d satisfies the conditions of Theorem 3.3. □

COROLLARY 3.1. – *Let A be an associative H^*–algebra with zero annihilator and let D be a continuous Lie derivation on A. Then there exists a derivation d on A and a linear mapping τ from A into the center of A such that $D = d + \tau$.*

PROOF. – As d is also continuous ([15]), $A = \overline{\perp_{a\in\Lambda} I_a}$ and $Z(A)$ is closed (the product in any H^*-algebra is continuous, see [7]), Theorem 3.3 gives us easily that $D - d$ is a linear mapping from A into $Z(A)$, and the proof is complete. □

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