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On the Dirichlet Problem with Orlicz Boundary Data.

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Sunto. – *Sia* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *una funzione di Young che soddisfa, con la sua funzione complementare* Ψ, *la condizione* Δ_2 *e siano* L^Φ *lo spazio di Orlicz generato dalla funzione* Φ *e* B *la palla unitaria di* $\mathbb{R}^n$.

Si presenta una condizione necessaria e sufficiente affinché il problema di Dirichlet per un operatore del secondo ordine ellittico in forma di divergenza:

$$\begin{cases} Lu = 0 & \text{in } B \\ u_{|_{\partial B}} = f, \end{cases}$$

sia L^Φ*-risolubile. La risolubilità per* $f \in L^\Phi$ *è intesa nel senso di* [5], [8], *dove viene trattato il caso* $\Phi(t) = t^p$.

Summary. – *Let us consider a Young's function* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying the* Δ_2 *condition together with its complementary function* Ψ, *and let us consider the Dirichlet problem for a second order elliptic operator in divergence form:*

$$\begin{cases} Lu = 0 & \text{in } B \\ u_{|_{\partial B}} = f, \end{cases}$$

B *the unit ball of* $\mathbb{R}^n$. *In this paper we give a necessary and sufficient condition for the* L^Φ*-solvability of the problem, where* L^Φ *is the Orlicz Space generated by the function* Φ. *This means solvability for* $f \in L^\Phi$ *in the sense of* [5], [8], *where the case* $\Phi(t) = t^p$ *is treated.*

1. – Introduction.

Let us consider the classical Dirichlet problem in the unit ball $B \subset \mathbb{R}^n$:

$$\begin{cases} Lu = 0 & \text{in } B \\ u_{|_{\partial B}} = f \in C^0(\partial B) \end{cases} \tag{1.1}$$

where

$$L = div(A\nabla) \tag{1.2}$$

is an elliptic operator whose coefficient matrix $A(X) \in L^\infty(\mathbb{R}^n)$, $A(X) = {}^t A(X)$ satisfies the condition

$$\frac{|\xi|^2}{K} \leq \langle A(X)\xi, \xi\rangle \leq K|\xi|^2 \tag{1.3}$$

with $K \geq 1$, for a.e. $x \in B$ and for $\xi \in \mathbb{R}^n$.

For $1 < p < \infty$, the Dirichlet problem is said to be L^p**−solvable** if for any $f \in C^0(\partial B)$ there exists a unique solution $u \in W^{1,2}_{loc}(B) \cap C^0(\bar{B})$ to (1.1) which satisfies the uniform estimate

$$\|Nu\|_{L^p(\partial B)} \leq C\|f\|_{L^p(\partial B)} \tag{1.4}$$

where Nu denotes the non-tangential maximal function ([8])

$$Nu(Q) = \sup_{X \in \Gamma_\beta(Q)} |u(X)| \tag{1.5}$$

and, for $Q \in \partial B$, $\Gamma_\beta(Q)$ is the non-tangential approach region ($\beta > 0$)

$$\Gamma_\beta(Q) = \{X \in B : |X - Q| \leq (1 + \beta)\, dist\,(X, \partial B)\} \tag{1.6}$$

To explain the known results on the L^p -solvability, we recall the definition of the harmonic measure ω_L of operator (1.2) on B.

If $f \in C^0(\partial B)$, $X \in B$, let us consider the linear functional

$$f \longrightarrow u(X) \tag{1.7}$$

on $C^0(\partial B)$ where $u \in W^{1,2}_{loc}(B)$ is the generalized solution of the classical Dirichlet problem (1.1). By the maximum principle, (1.7) is a bounded, positive continuous linear functional and $u \equiv 1$ if $f \equiv 1$. Therefore, by the Riesz representation theorem, there exists a family of regular Borel probability measures

$$\{\omega^X_L\}$$

such that u represents as

$$u(X) = \int_{\partial B} f(Q)\, d\omega^X_L(Q) \tag{1.8}$$

For fixed $X_0 \in B$ let $\omega_L = \omega^{X_0}_L$ and we refer to ω_L as the **harmonic measure** of L on B.

A key result is contained in [8]:

THEOREM 1.1. – *Let* $1 < p_0 < \infty$, $q_0 = \dfrac{p_0}{p_0 - 1}$. *The following are equivalent:*

i) *The Dirichlet problem* (1.1) *is* L^{p_0}*-solvable*;
ii) *The L-harmonic measure* ω *is absolutely continuous with respect to* σ*,*

and $k = \dfrac{d\omega}{d\sigma} \in L^{q_0}(d\sigma)$ *with*

$$\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k^{q_0}\right)^{\frac{1}{q_0}} \le C\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k\right), \tag{1.9}$$

for any surface ball $\Delta \subset \partial B$.

The first results on the L^p –solvability go back to B. E. J. Dahlberg [5], [6]. In this paper we are interested in the extension of these results to the context of the *Orlicz* Spaces (see Theorem (4.4)). Note that, thanks to the higher integrability properties of weights k satisfying 'reverse Hölder' inequalities (1.9), a better solvability automatically holds; namely, if the problem (1.1) is L^{p_0} -solvable, then it is also $L^{p_0-\delta}$ -solvable for suitable $\delta > 0$.

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Young's function that satisfies the Δ_2-condition together with its complementary function Ψ. The Dirichlet problem (1.1) is said to be L^{Φ}-**solvable** if for any $f \in C^0(\partial B)$ there exists a unique solution $u \in W^{1,2}_{loc}(B) \cap C^0(\bar{B})$ to (1.1) which satisfies the uniform estimate

$$\int_{\partial B} \Phi(Nu)d\sigma \le C\int_{\partial B} \Phi(|f|)d\sigma \tag{1.10}$$

Let us observe that for $L^{\Phi} = L^p$ the integral inequality (1.10) corresponds to the norm inequalities (1.4). We will show that condition (*ii*) of Theorem 1.1 is a necessary and sufficient condition also for the L^{Φ}-solvability of the problem (1.1), where Φ is a given Young's function such that the upper index of L^{Φ} is p_0^{-1} (see Definition 3.1).

2. – Notations and preliminary results.

Now we need to recall some definitions and results on the theory of A_∞ weights.

DEFINITION 2.1. – *A function* $w : \partial B \to \mathbb{R}$ *will be called a* ***weight*** *if it is positive and if* $w \in L^1(\partial B, d\sigma)$*,* σ *being the surface measure on* ∂B.

Let μ be any non negative, Borel measure on ∂B satisfying the doubling condition

$$\mu(\Delta_{2r}(Q)) \le C\mu(\Delta_r(Q)) \tag{2.1}$$

where Q is a point on ∂B, $\Delta_r(Q)=$ B$_r(Q) \cap \partial$B, $B_r(Q)$ the ball of $\mathbb{R}^n$ with center Q and radius r (for example $\mu = \omega$, the harmonic measure associated to any elliptic operator L, or $\mu = \sigma$).

DEFINITION 2.2. – *Let υ be another non-negative measure on ∂B. Then, $\upsilon \in A_\infty(d\mu)$ if, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $E \subset \Delta_r(Q)$, $\Delta_r(Q)$ any surface ball, then*

$$\frac{\mu(E)}{\mu(\Delta_r(Q))} < \delta \Rightarrow \frac{\upsilon(E)}{\upsilon(\Delta_r(Q))} < \varepsilon.$$

The main properties of this class of measures are summarized in what follows.

THEOREM 2.1. – *The following properties hold:*

(i) *If $\upsilon \in A_\infty(d\mu)$, then υ is absolutely continuous with respect to μ and viceversa.*

(ii) *$A_\infty(d\mu) = \cup_{q>1} S_q(d\mu)$, where $\upsilon \in S_q(d\mu)$ if υ is absolutely continuous with respect to μ and $k = \frac{d\upsilon}{d\mu} \in L^q(d\mu)$, and verifies*

$$\left(\frac{1}{\mu(\Delta_r(Q))} \int\limits_{\Delta_r(Q)} k^q d\mu\right)^{\frac{1}{q}} \le C\left(\frac{1}{\mu(\Delta_r(Q))} \int\limits_{\Delta_r(Q)} k d\mu\right)$$

for all surface ball $\Delta_r(Q)$.

(iii) *$\upsilon \in S_q(d\mu)$ iff the weighted Hardy-Littlewood maximal operator M_υ defined by*

$$M_\upsilon(f)(Q) = \sup_{\Delta \ni Q} \frac{1}{\upsilon(\Delta)} \int\limits_{\Delta} |f| d\upsilon$$

verifies

$$\| M_\upsilon f\|_{L^p(d\mu)} \le C\|f\|_{L^p(d\mu)}, \qquad \frac{1}{q} + \frac{1}{p} = 1. \tag{2.2}$$

(iv) *If $\upsilon \in S_q(d\mu)$, $q > 1$, then there exists $\varepsilon > 0$ such that $\upsilon \in S_{q+\varepsilon}(d\mu)$.*

DEFINITION 2.3. – *For $1 < q < \infty$, let $\upsilon \in S_q(d\mu)$, and let k be as above. We define S_q- constant of the measure υ with respect to μ, the quantities*

$$S_{q,\mu}(\upsilon) = \sup_{\Delta} \left[\frac{\left(\frac{1}{\mu(\Delta)} \int\limits_{\Delta} k^q d\mu\right)^{\frac{1}{q}}}{\frac{1}{\mu(\Delta)} \int\limits_{\Delta} k d\mu}\right]^p, \qquad \frac{1}{q} + \frac{1}{p} = 1, \tag{2.3}$$

where the supremum is taken over all the surface ball $\Delta \subseteq \partial B$.

Combining Lemma 2.2 of [11] and Theorem 2.5 of [2] with slight modification, we have:

PROPOSITION 2.2. – *Let* $1 < q < \infty$, *and let* $\upsilon \in S_q(d\mu)$. *Then:*

$$\|M_\upsilon f\|_{L^p(d\mu)} \leq C(n,p)^{\frac{1}{p}} \left[S_{q,\mu}(\upsilon)\right]^{\frac{q}{p}} \|f\|_{L^p(d\mu)}, \qquad \frac{1}{q}+\frac{1}{p}=1, \tag{2.4}$$

and so, for all $\lambda > 0$,

$$\mu(\{M_\upsilon f > \lambda\}) \leq C(n,p) \frac{\left[S_{q,\mu}(\upsilon)\right]^q}{\lambda^p} \int_{\partial B} |f|^p d\mu. \tag{2.5}$$

Following the proof of Theorem 1 in [4], one can see that it holds:

PROPOSITION 2.3. – *Let* w *be a weight on* ∂B *such that the measure* $d\mu = w d\sigma$ *is doubling. Let* $d\upsilon = z d\mu$, $z > 0$ *on* ∂B *and* $z \in L^1(d\mu)$. *If there exist* $0 < \gamma \leq 1$ *and* $C > 0$ *such that*

$$\frac{\mu(E)}{\mu(\Delta)} \leq C \left(\frac{\upsilon(E)}{\upsilon(\Delta)}\right)^\gamma, \quad \forall \Delta, \quad \forall E \subset \Delta \tag{2.6}$$

then there exist $\delta > 0$, $K > 0$, *such that*

$$\left(\frac{1}{\mu(\Delta)} \int_\Delta z^{1+\delta} d\mu\right)^{\frac{1}{1+\delta}} \leq K \frac{1}{\mu(\Delta)} \int_\Delta z\, d\mu, \qquad \forall \Delta. \tag{2.7}$$

Moreover the constants K *and* δ *in* (2.7) *are dependent only upon the constants* C *and* γ *in* (2.6) *and upon the constant in the doubling condition of* μ.

For more details we refer the reader to the papers B. Muckenhoupt [12], R. R. Coifman and C. Fefferman [3], A. P. Calderón [4] where the theory of A_∞ weights is extensively studied.

If ω is the L-harmonic measure of ∂B, the connection between these concepts and the Problem (1.1) is also given in the following Lemma (see [8]):

LEMMA 2.4. – *Let* $f \in L^1(d\omega)$, *and, for* $X \in B$,

$$u(X) = \int_{\partial B} f d\omega^X.$$

Then, for each $P \in \partial B$

$$Nu(P) \leq C_\beta M_\omega(f)(P)$$

where β *is the aperture of the approach region* Γ_β. *Moreover, if* $f \geq 0$, $M_\omega(f)(P) \leq C'_\beta Nu(P)$.

Finally, we want just recall the following version of the Marcinkiewicz theorem (cfr. [13]). Here and below, if v is a weight on ∂B and A is a σ-measurable set, we will write $v(A) = \int_A v d\sigma$.

THEOREM 2.5. – *Let T be a sublinear operator, and let v be a weight on ∂B. Suppose that T is simultaneously of restricted weak-types (p_1, p_1) and (p_2, p_2), $1 < p_1 < p_2 < \infty$, with respect to the measure $dv = v d\sigma$, i.e.*

$$\int_{\{T\chi_E > \lambda\}} dv \leq \frac{C}{\lambda^{p_i}} v(E), \quad i = 1, 2 \tag{2.8}$$

E measurable subset of ∂B, C independent on E and on the positive constant λ. Then T is also of 'strong type' (r, r), for all $p_1 < r < p_2$, that is

$$\|Tf\|_{L^r(dv)} \leq K \|f\|_{L^r(dv)} \tag{2.9}$$

K independent on f.

If $T = M_w$, then the restricted weak type can be characterized as follows:

PROPOSITION 2.6. – *Let w, v be weights on ∂B, and let the measure dv be doubling. The weighted Hardy-Littlewood maximal operator M_w is of restricted weak-type (p, p) with respect to dv, i.e.*

$$\int_{\{M_w \chi_E > \lambda\}} dv \leq \frac{C}{\lambda^p} v(E), \quad 1 \leq p < \infty \tag{2.10}$$

with C independent on E and on the positive constant λ, iff there exists $K > 0$ such that for all Δ, and for all measurable $E \subset \Delta$,

$$\frac{w(E)}{w(\Delta)} \leq K \left(\frac{v(E)}{v(\Delta)} \right)^{\frac{1}{p}} \tag{2.11}$$

PROOF. – (2.10) $\Longrightarrow$ (2.11)

Observing that, by the definition of the operator M_w, if $E \subset \Delta$

$$M_w \chi_E(P) \geq \frac{\chi_\Delta(P)}{w(\Delta)} \int_\Delta \chi_E(Q) w d\sigma(Q) = \frac{w(E)}{w(\Delta)} \chi_\Delta(P),$$

results from (2.10)

$$v(\Delta) \leq \int_{\left\{M_w \chi_E > \frac{w(E)}{w(\Delta)}\right\}} dv \leq C v(E) \left(\frac{w(\Delta)}{w(E)} \right)^p$$

that is (2.11).

(2.11) $\Longrightarrow$ (2.10)

We have

$$M_v\chi_E(P) = \sup_{\Delta\ni P} \frac{1}{v(\Delta)} \int_{\Delta} \chi_E(Q)v(Q)d\sigma(Q) = \sup_{\Delta\ni P} \frac{v(E\cap\Delta)}{v(\Delta)}.$$

and analogously for M_w. Then, by (2.11)

$$(M_w\chi_E)^p \leq K^p M_v\chi_E,$$

so that $\{M_w\chi_E > \lambda\} \subseteq \left\{M_v\chi_E > \frac{\lambda^p}{K^p}\right\}$. Now, the measure dv doubling implies that the operator M_v is of weak-type $(1,1)$ with respect to dv; in particular,

$$\int_{\{M_v\chi_E>\lambda\}} dv(Q) \leq \frac{C_1}{\lambda} v(E)$$

and then (2.10) follows with $C = C_1K^p$. □

3. – Orlicz setting.

Let $\Phi : [0,+\infty) \to [0,+\infty)$ be a Young's function, i.e. a function of the type $\Phi(t) = \int_0^t \varphi(s)ds,\ t > 0$, where $\varphi : [0,\infty[\to\mathbb{R}$ is nondecreasing, right-continuous and such that

$$\varphi(s) > 0 \quad \forall s > 0, \quad \varphi(0) = 0, \quad \lim_{s\to\infty} \varphi(s) = +\infty.$$

The Young's function $\Psi(t)$, complementary to $\Phi(t)$, is defined by $\Psi(t) = \int_0^t \varphi^{-1}(s)ds$, where $\varphi^{-1}(s) = \sup\{u : \varphi(u) \leq s\}$. So $\varphi^{-1}(s)$ verifies the same properties of $\varphi(s)$.

If v is a weight on ∂B, the Orlicz Space $L^\Phi(dv) = L^\Phi(\partial B, v d\sigma)$ consists of all measurable functions on ∂B for which there exists $K > 0$ such that

$$\int_{\partial B} \Phi\left(\frac{|f|}{K}\right) dv \leq 1$$

and the norm $\|f\|_{L^\Phi(dv)}$ of f in $L^\Phi(dv)$ is the infimum over all such K.

We will require that $\Phi(t)$ satisfies the Δ_2 condition, which is equivalent to the more general property:

$$\forall A > 0,\ \exists B > 0\ :\ \Phi(At) \leq B\Phi(t), \quad \forall t \geq 0. \tag{3.1}$$

Note that it implies also that

$$\forall B > 0,\ \exists A > 0\ :\ \Phi(At) \leq B\Phi(t), \quad \forall t \geq 0. \tag{3.2}$$

If also the complementary $\Psi(t)$ obey the Δ_2 condition, the Banach spaces L^Φ and L^Ψ are mutually dual, and so the *Hölder's inequality* holds. We shall require that Ψ satisfies Δ_2 too. In this case we have that $\Phi(t)$ and $\Psi(t)$ are essentially equal to $t\varphi(t)$ and $t\varphi^{-1}(t)$ respectively, for all $t \geq 0$, and then φ and φ^{-1} also satisfy the Δ_2 condition. Moreover, for the inverse functions of Φ and Ψ, we have:

$$t \leq \Phi^{-1}(t)\Psi^{-1}(t) \leq 2t, \qquad \forall t \geq 0. \tag{3.3}$$

Let us observe that, under the above hypotheses, for $\Phi(t)$ and $\Psi(t)$ we have

$$1 < \tilde{\theta} = \inf \frac{t\varphi(t)}{\Phi(t)} \leq \sup \frac{t\varphi(t)}{\Phi(t)} = \tilde{\rho} < \infty, \tag{3.4}$$

$$1 < \tilde{\tilde{\theta}} = \inf \frac{t\varphi^{-1}(t)}{\Psi(t)} \leq \sup \frac{t\varphi^{-1}(t)}{\Psi(t)} = \tilde{\tilde{\rho}} < \infty. \tag{3.5}$$

DEFINITION 3.1. – *Let*

$$h(s) = \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}. \tag{3.6}$$

The upper and lower indices ρ and θ of L^Φ are

$$\rho = \inf_{0<s<1} -\frac{log h(s)}{log s} \tag{3.7}$$

and

$$\theta = \inf_{1<s<\infty} -\frac{log h(s)}{log s} \tag{3.8}$$

respectively.

It is easy to see that $\tilde{\theta} \leq \dfrac{1}{\rho} \leq \tilde{\rho}$, and that $L^\Phi = L^p$ implies $\rho = \theta = p^{-1}$. Moreover, we want just recall some properties of these indices we will make use of below:

PROPOSITION 3.1. – *Let Φ and Ψ be as before. The following properties hold:*

i) $0 < \theta \leq \rho < 1$;

ii) *given a fixed $0 < r < \rho^{-1}$, there exists an s_0, with $0 < s_0 < 1$, such that*

$$\Phi(st) \leq \left(\frac{s}{s_0}\right)^r \Phi(t), \tag{3.9}$$

for all $t > 0$, $0 < s < 1$;

iii) *For any* s *such that* $0 < s < 1$ *we have* $h(s) \geq s^{-\rho}$; *so, for any fixed* $0 < s < 1$ *there is a* $t > 0$ *such that*

$$\frac{\Phi^{-1}(t)}{\Phi^{-1}(st)} > \frac{s^{-\rho}}{2}. \tag{3.10}$$

iv) *For any* s *such that* $0 < s \leq 1$ *and for any* $t > 0$,

$$\frac{\Phi^{-1}(st)}{\Phi^{-1}(t)} \leq s^{\tilde{\rho}^{-1}}. \tag{3.11}$$

Furthermore let us observe that, under the above hypotheses, the following holds true:

PROPOSITION 3.2. – *Let* Φ *and* Ψ *be as before and let* Φ_δ, $\delta > 0$ *such that*

$$\varphi_\delta^{-1}(t) = \left(\varphi^{-1}(t)\right)^{1+\delta};$$

then, the upper index ρ' *of* L^{Φ_δ} *is greater than the upper index* ρ *of* L^{Φ}.

Let us observe that, from the proof of Lemma 2 in [9], it is possible to compute exactly the upper index of L^{Φ_δ}, that is $\rho' = \dfrac{1 + \delta\rho - \theta_\Psi}{1 + \delta\rho}$, where θ_Ψ is the lower index of L^{Ψ}.

For a complete analysis of these properties we refer to [1], [10] and to the results obtained in [9].

4. – The main result.

In this section our aim is to extend the solvability of the Problem (1.1) to the context of Orlicz Spaces.

It holds the following:

THEOREM 4.1. – *Let* w, v *be weights on* ∂B, *such that the measures* $dv = v d\sigma$ *and* $dw = w d\sigma$ *are doubling, and let* $\Phi(t) = \int_0^t \varphi(s) ds$ *be a Young's function which, together with its complementary function* $\Psi(t)$, *satisfies the* Δ_2 *condition. Then, the following are equivalent:*

i) *There exists a constant* $C > 0$, *independent on* f, *such that:*

$$\int_{\partial B} \Phi(M_w f) v d\sigma \leq C \int_{\partial B} \Phi(|f|) v d\sigma;$$

ii) $w \in S_{\Phi}(dv)$, *that is:*

$$\left(\frac{1}{w(\Delta)}\int_{\Delta}\varepsilon v d\sigma\right)\varphi\left(\frac{1}{w(\Delta)}\int_{\Delta}\varphi^{-1}\left(\frac{w}{\varepsilon v}\right)w d\sigma\right)\leq K \tag{4.1}$$

for all surface balls Δ *and for all* $\varepsilon > 0$;

iii) $w \in S_{q_0}(dv)$, *where* $\dfrac{1}{p_0}+\dfrac{1}{q_0}=1$, p_0^{-1} *upper index of* L^{Φ}.

PROOF. – $i) \Rightarrow ii)$

Let us consider:

$$\|\chi_{\Delta}\|_{L^{\Phi}(\varepsilon dv)}=\inf\left\{k>0:\int_{\partial B}\Phi\left(\frac{\chi_{\Delta}}{k}\right)\varepsilon v d\sigma\leq 1\right\}=\frac{1}{\Phi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)} \tag{4.2}$$

and

$$T_{\varepsilon}:=\left\|\frac{w\chi_{\Delta}}{\varepsilon v}\right\|_{L^{\Psi}(\varepsilon dv)}=\inf\left\{k>0:\int_{\partial B}\Psi\left(\frac{w\chi_{\Delta}}{k\varepsilon v}\right)\varepsilon v d\sigma\leq 1\right\} \tag{4.3}$$

We can immediately observe that $T_{\varepsilon} > 0$, unless $\sigma(\Delta) = 0$, which we exclude. Indeed, $T_{\varepsilon} = 0$ implies that the function $\dfrac{w\chi_{\Delta}}{\varepsilon v}$ is zero dv-a.e., but $w, v > 0$ implies $\sigma(\Delta) = 0$. On the other hand, the converse of the Hölder's inequality implies the existence of a nonnegative function f, supported by Δ, with norm $\|f\|_{L^{\Phi}(\varepsilon dv)} = 1$ and such that $\int_{\Delta} fw d\sigma = \int_{\partial B} f\dfrac{w\chi_{\Delta}}{\varepsilon v}\varepsilon dv = T_{\varepsilon}$ and then $M_w f(P) \geq \dfrac{T_{\varepsilon}}{w(\Delta)}$, $\forall P \in \Delta$; this implies, by $i)$, $T_{\varepsilon} < \infty$.

Now we claim that there exists a constant K_1 such that for all Δ and for all $\varepsilon > 0$

$$\|\chi_{\Delta}\|_{L^{\Phi}(\varepsilon dv)}\, T_{\varepsilon} \leq K_1 w(\Delta). \tag{4.4}$$

Indeed, with the same f as before, we have

$$\frac{T_{\varepsilon}}{w(\Delta)}\chi_{\Delta}(Q) \leq M_w f(Q) \qquad \forall Q \in \partial B. \tag{4.5}$$

Being $\Phi(t)$ an increasing function, yielding $i)$ and integrating we have:

$$\int_{\Delta}\Phi\left(\frac{T_{\varepsilon}}{w(\Delta)}\right)\varepsilon v d\sigma \leq \int_{\partial B}\Phi(M_w f)\varepsilon v d\sigma \leq C\int_{\partial B}\Phi(|f|)\varepsilon v d\sigma \leq C \tag{4.6}$$

that is

$$\frac{T_\varepsilon}{w(\Delta)} \le \Phi^{-1}\left(\frac{C}{\varepsilon v(\Delta)}\right)$$

Let us choose

$$K_1 = h(C^{-1}) = \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(C^{-1}t)}.$$

Taking $t = \dfrac{C}{\varepsilon v(\Delta)}$, (4.4) follows.

Now, since from (3.5) $t\varphi^{-1}(t) \le \tilde{\tilde{\rho}}\Psi(t)$ $\forall t > 0$, we have:

$$\int_{\partial B} \frac{w\chi_\Delta}{T_\varepsilon}\varphi^{-1}\left(\frac{w\chi_\Delta}{v\varepsilon T_\varepsilon}\right)d\sigma \le \tilde{\tilde{\rho}}\int_{\partial B}\Psi\left(\frac{w\chi_\Delta}{v\varepsilon T_\varepsilon}\right)\varepsilon v d\sigma \le \tilde{\tilde{\rho}}$$

and then, from (4.4) we have

$$\frac{\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}}{K_1 w(\Delta)} \int_\Delta \varphi^{-1}\left(\frac{w}{\varepsilon v}\,\frac{\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}}{K_1 w(\Delta)}\right) w d\sigma \le \tilde{\tilde{\rho}} \tag{4.7}$$

Now, let us consider the function of ε:

$$\theta(\varepsilon) = \frac{\|\chi_\Delta\|_{L^\Phi(\varepsilon dv)}}{K_1 \varepsilon w(\Delta)}$$

Let us remark that, from (3.3), it follows that $\theta(\varepsilon)$ is essentially equal to the function

$$\theta_1(\varepsilon) = \frac{v(\Delta)}{K_1 w(\Delta)}\Psi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)$$

and hence, $\lim_{\varepsilon\to o^+} \theta(\varepsilon) = +\infty$, and $\lim_{\varepsilon\to+\infty} \theta(\varepsilon) = 0$. Moreover $\theta(\varepsilon)$ is continuous, and so there exists $\varepsilon > 0$ such that $\theta(\varepsilon) = 1$, essentially equal to $\left[v(\Delta)\Psi\left(\dfrac{K_1 w(\Delta)}{v(\Delta)}\right)\right]^{-1}$.

Now, applying these results to (4.7) we obtain

$$\int_\Delta \varphi^{-1}\left(\frac{w}{v}\right) w d\sigma \le K_2 \tilde{\tilde{\rho}} v(\Delta)\Psi\left(\frac{K_1 w(\Delta)}{v(\Delta)}\right) \le K_2 \frac{\tilde{\tilde{\rho}}}{\tilde{\tilde{\theta}}} K_1 w(\Delta)\varphi^{-1}\left(\frac{K_1 w(\Delta)}{v(\Delta)}\right). \tag{4.8}$$

Then,

$$\varphi\left(\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{w}{v}\right) w d\sigma\right) \le \varphi(At) \tag{4.9}$$

by assuming $A = K_2 \dfrac{\tilde{\tilde{\rho}}}{\tilde{\tilde{\theta}}} K_1$ and $t = \varphi^{-1}\left(\dfrac{K_1 w(\varDelta)}{v(\varDelta)}\right)$. Now, from the generalized $\varDelta_2$ condition for $\varPhi$, let $B > 0$ such that $\varPhi(At) \leq B\varPhi(t)$, $t > 0$. Then we have

$$\varphi(At) \leq \frac{\tilde{\rho}}{At}\varPhi(At) \leq \frac{\tilde{\rho}B}{At}\varPhi(t) \leq \frac{\tilde{\rho}B}{\tilde{\theta}At} t\varphi(t) = \frac{\tilde{\rho}}{\tilde{\theta}}\frac{\tilde{\tilde{\theta}}}{\tilde{\tilde{\rho}}}\frac{B}{K_1K_2}\frac{w(\varDelta)}{v(\varDelta)}$$

from which the assertion *ii*) follows for $\varepsilon = 1$ with $K = \dfrac{\tilde{\rho}}{\tilde{\theta}}\dfrac{\tilde{\tilde{\theta}}}{\tilde{\tilde{\rho}}}\dfrac{B}{K_1K_2}$. The same proof applies to εv; for the constant K depends only on C, and so the assertion is proved in the general case.

Now to prove that *ii*) implies *iii*) we need some preliminary results:

LEMMA 4.2. – *Let $\varPhi$, p_0, w, and v be as in Theorem* 4.1. *Then,* $w \in S_{\varPhi}(dv)$ *implies that the weighted Hardy-Littlewood maximal operator M_w is bounded from $L^r(dv)$ to itself, for all $r > p_0$.*

PROOF. – By the interpolation criterion (Theorem 2.5) it is enough to prove that M_w is of restricted weak-type (p_0, p_0) that is (2.11) with $p = p_0$. We have, by duality between $L^{\varPhi}$ and $L^{\varPsi}$:

$$\frac{w(E)}{w(\varDelta)} = \frac{1}{w(\varDelta)}\int_{\partial B} \chi_E\left(\frac{\chi_E w}{\varepsilon v}\right)\varepsilon dv \leq \frac{1}{w(\varDelta)}\|\chi_E\|_{L^{\varPhi}(\varepsilon dv)}\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^{\varPsi}(\varepsilon dv)}.$$

We claim that

$$\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^{\varPsi}(\varepsilon dv)} \leq C_1 w(\varDelta)\varPhi^{-1}\left(\frac{1}{\varepsilon v(\varDelta)}\right) \tag{4.10}$$

Indeed, let us observe that (4.1) is equivalent to

$$\int_{\varDelta} \varphi^{-1}\left(\frac{w}{\varepsilon v}\right) w d\sigma \leq w(\varDelta)\varphi^{-1}\left(\frac{Kw(\varDelta)}{\varepsilon v(\varDelta)}\right) \tag{4.11}$$

Observing that $t\varphi^{-1}(t) \geq \varPsi(t)$, we have:

$$\begin{aligned}\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^{\varPsi}(\varepsilon dv)} &\leq \inf\left\{k > 0 | \int_{\varDelta} \varPsi\left(\frac{w}{k\varepsilon v}\right)\varepsilon v d\sigma \leq 1\right\} \\ &\leq \inf\left\{k > 0 | \int_{\varDelta} \frac{w}{k}\varphi^{-1}\left(\frac{w}{k\varepsilon v}\right) d\sigma \leq 1\right\},\end{aligned}$$

and from (4.11)

$$\left\|\frac{w\chi_E}{\varepsilon v}\right\|_{L^{\Psi}(\varepsilon dv)} \le \inf\left\{k>0 \mid \frac{1}{k}w(\Delta)\varphi^{-1}\left(\frac{Kw(\Delta)}{k\varepsilon v(\Delta)}\right)\le 1\right\}$$

$$\le \inf\left\{k>0 \mid \frac{K}{\varepsilon v(\Delta)} \le \Phi\left(\frac{k}{w(\Delta)}\right)\right\}$$

$$= w(\Delta)\Phi^{-1}\left(\frac{K}{\varepsilon v(\Delta)}\right) \le C_1 w(\Delta)\Phi^{-1}\left(\frac{1}{\varepsilon v(\Delta)}\right)$$

(where $C_1 = h(K^{-1})$), i.e. the (4.10). So we obtain

$$\frac{w(E)}{w(\Delta)} \le C_1 \frac{\Phi^{-1}\left(\dfrac{1}{\varepsilon v(\Delta)}\right)}{\Phi^{-1}\left(\dfrac{1}{\varepsilon v(E)}\right)}. \tag{4.12}$$

Then the statement follows from Proposition 3.1, (iii), taking $s = \dfrac{v(E)}{v(\Delta)} < 1$ and $\varepsilon = \dfrac{1}{tv(E)}$. $\square$

LEMMA 4.3. – *Let Φ, p_0, w and v be as in Theorem* 4.1, *and let Φ_δ, $\delta > 0$ be such that*

$$\varphi_\delta^{-1}(t) = \left(\varphi^{-1}(t)\right)^{1+\delta}$$

Then, $w \in S_\Phi(dv)$ implies $w \in S_{\Phi_\delta}(dv)$, for small δ.

PROOF. – Suppose $w \in S_\Phi(dv)$. We want to prove that $w \in S_{\Phi_\delta}(dv)$ for some $\delta > 0$, i.e.

$$\left(\frac{1}{w(\Delta)}\int_\Delta \varepsilon v d\sigma\right)\varphi_\delta\left(\frac{1}{w(\Delta)}\int_\Delta \varphi_\delta^{-1}\left(\frac{w}{\varepsilon v}\right) w d\sigma\right) \le K \tag{4.13}$$

for all surface balls Δ and for all $\varepsilon > 0$. For this purpose, it is sufficient to prove, for $z_\varepsilon = \varphi^{-1}\left(\dfrac{w}{\varepsilon v}\right)$,

$$\left(\frac{1}{w(\Delta)}\int_\Delta z_\varepsilon^{1+\delta} dw\right)^{\frac{1}{1+\delta}} \le C\frac{1}{w(\Delta)}\int_\Delta z_\varepsilon dw \quad \forall \varepsilon > 0,\ \forall \Delta. \tag{4.14}$$

Indeed, if (4.14) holds true, by $\varphi_\delta(t) = \varphi(t^{\frac{1}{1+\delta}})$, we have

$$\left(\frac{1}{w(\Delta)}\int_\Delta \varphi_\delta^{-1}\left(\frac{w}{\varepsilon v}\right) dw\right)^{\frac{1}{1+\delta}} \le C\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\left(\frac{w}{\varepsilon v}\right) dw$$

and than,

$$\varphi_\delta\left(\frac{1}{w(\Delta)}\int_\Delta \varphi_\delta^{-1}\Big(\frac{w}{\varepsilon v}\Big)dw\right) \le \varphi\left(C\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\Big(\frac{w}{\varepsilon v}\Big)dw\right)$$

$$\le C'\varphi\left(\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\Big(\frac{w}{\varepsilon v}\Big)dw\right) \le K\frac{w(\Delta)}{\varepsilon v(\Delta)}$$

i.e. (4.13). So let us prove (4.14). To begin, we prove (4.14) for $\varepsilon = 1$. Denote $z = z_1 = \varphi^{-1}\Big(\dfrac{w}{v}\Big)$ and set

$$\tilde\varphi(t) = \frac{1}{\varphi^{-1}(t^{-1})} \qquad \left(\tilde\varphi^{-1}(t) = \frac{1}{\varphi(t^{-1})}\right).$$

Now, for $\varepsilon = 1$ the inequality

$$\left(\frac{1}{w(\Delta)}\int_\Delta v\,d\sigma\right)\varphi\left(\frac{1}{w(\Delta)}\int_\Delta \varphi^{-1}\Big(\frac{w}{v}\Big)w\,d\sigma\right) \le K \tag{4.15}$$

becomes, also by the Δ_2-condition for φ^{-1}

$$\frac{1}{w(\Delta)}\int_\Delta zw\,d\sigma \le K'\varphi^{-1}\left(\left(\frac{1}{w(\Delta)}\int_\Delta \frac{1}{\varphi(z)}w\,d\sigma\right)^{-1}\right) \tag{4.16}$$

that is

$$\left(\frac{1}{w(\Delta)}\int_\Delta zw\,d\sigma\right)\tilde\varphi\left(\frac{1}{w(\Delta)}\int_\Delta \tilde\varphi^{-1}\Big(\frac{1}{z}\Big)w\,d\sigma\right) \le K', \quad \forall\Delta. \tag{4.17}$$

Now, we claim that (4.17) implies that for all Δ there exists $\lambda = \lambda_\Delta$ such that

$$\left\|\frac{\chi_\Delta}{\lambda z}\right\|_{L^{\tilde\Psi}(\lambda z dw)} \le Cw(\Delta)\tilde\Phi^{-1}\left(\frac{1}{\lambda (zw)(\Delta)}\right) \tag{4.18}$$

(where $\lambda z dw$ stays for the measure $\lambda z w d\sigma$). To prove (4.18) let us observe that $\tilde\varphi$ and $\tilde\varphi^{-1}$ satisfy both the Δ_2-condition, and then $\tilde\Phi$ and $\tilde\Psi$ obey Δ_2 too. Then, from (4.17)

$$\left(\frac{1}{w(\Delta)}\int_\Delta \tilde\varphi^{-1}\Big(\frac{1}{z}\Big)w\,d\sigma\right) \le \tilde\varphi^{-1}\left(K'\frac{w(\Delta)}{(zw)(\Delta)}\right) \le C'\tilde\varphi^{-1}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right), \tag{4.19}$$

for all Δ. Now, let $\lambda = \lambda_\Delta$ such that

$$\frac{1}{\lambda} = (zw)(\Delta)\tilde{\Psi}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right). \tag{4.20}$$

We have

$$\frac{1}{\lambda} \sim w(\Delta)\tilde{\varphi}^{-1}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right)$$

that is

$$\tilde{\varphi}^{-1}\left(\frac{1}{\lambda w(\Delta)}\right) \sim \frac{w(\Delta)}{(zw)(\Delta)}$$

and then

$$\tilde{\Phi}\left(\frac{1}{\lambda w(\Delta)}\right)\lambda \sim \frac{1}{(zw)(\Delta)}.$$

So,

$$\frac{1}{\lambda w(\Delta)} \sim \tilde{\Phi}^{-1}\left(\frac{1}{\lambda(zw)(\Delta)}\right). \tag{4.21}$$

We have, from (4.19),

$$\int_\Delta \frac{\tilde{\Psi}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right)}{\frac{w(\Delta)}{(zw)(\Delta)}\tilde{\varphi}^{-1}\left(\frac{w(\Delta)}{(zw)(\Delta)}\right)}\tilde{\varphi}^{-1}\left(\frac{1}{z}\right)\lambda w d\sigma \le C'$$

then, if $\tilde{\tilde{\rho}} = \tilde{\tilde{\rho}}(\tilde{\Psi})$,

$$\int_\Delta \frac{1}{\tilde{\tilde{\rho}}}\left(\frac{1}{z}\tilde{\varphi}^{-1}\left(\frac{1}{z}\right)\right)\lambda z w d\sigma \le C'$$

hence,

$$\int_\Delta \frac{1}{C'}\,\frac{\tilde{\tilde{\theta}}}{\tilde{\tilde{\rho}}}\,\tilde{\Psi}\left(\frac{1}{z}\right)\lambda z w d\sigma \le 1.$$

Then, by (3.2), there exists $C > 0$ such that

$$\int_\Delta \tilde{\Psi}\left(\frac{C}{z}\right)\lambda z w d\sigma \le 1$$

and so, by (4.21), (4.18) follows.

Now, $\forall E \subset \Delta$, by the equality

$$\|\chi_E\|_{L^{\tilde{\Phi}}(\lambda z dw)} = \frac{1}{\tilde{\Phi}^{-1}\left(\dfrac{1}{(\lambda z w)(E)}\right)}, \tag{4.22}$$

and by (4.18), we have

$$\frac{w(E)}{w(\Delta)} \le C \frac{\tilde{\Phi}^{-1}\left(\dfrac{1}{\lambda (zw)(\Delta)}\right)}{\tilde{\Phi}^{-1}\left(\dfrac{1}{\lambda (zw)(E)}\right)}$$

and then, by (3.11) with $s = \dfrac{(zw)(E)}{(zw)(\Delta)}$, we obtain

$$\frac{w(E)}{w(\Delta)} \le C \left(\frac{(zw)(E)}{(zw)(\Delta)}\right)^{\tilde{\rho}^{-1}}, \qquad \forall \Delta, \quad \forall E \subset \Delta. \tag{4.23}$$

Now, observing that $w \in S_{\Phi}(dv)$ implies that $z_\varepsilon \in L^1(dw)$ for all $\varepsilon > 0$, from Proposition 4.23 with $d\mu = w d\sigma$ and $dv = zw d\sigma$, we have that there exist $\delta > 0$ and $K > 0$ such that

$$\left(\frac{1}{w(\Delta)} \int_{\Delta} z^{1+\delta} dw\right)^{\frac{1}{1+\delta}} \le K \left(\frac{1}{w(\Delta)} \int_{\Delta} z dw\right) \tag{4.24}$$

for all Δ, that is (4.14) with $\varepsilon = 1$. Now, let us observe that the constant K' in (4.17) is independent on ε; then, also the constants in (4.23) are independent on ε too. So, by the last assertion in Theorem 2.3, the constants δ and K in (4.24) are independent on ε, and then the proof holds true also in the general case. □

Now, we can conclude that $ii) \Longrightarrow iii)$. Indeed, we have that $w \in S_{\Phi}(dv)$ implies $w \in S_{\Phi_\delta}(dv)$ (Lemma 4.3); then the maximal function M_w is bounded from $L^r(dv)$ to itself, for all $r > \dfrac{1}{\rho'}$, ρ' upper index of L^{Φ_δ} (Lemma 4.2). In particular, (Proposition 3.2), M_w is bounded from $L^{p_0}(dv)$ to itself, that is $w \in S_{q_0}(dv)$.

Finally, $iii) \Longrightarrow i)$:

Let $w \in S_{q_0}(dv)$, $\dfrac{1}{p_0} + \dfrac{1}{q_0} = 1$. Hence, by Theorem 2.1, (iv), there exists an $\varepsilon > 0$ such that $w \in S_{q_0+\varepsilon}(dv)$. Assume $m = q_0 + \varepsilon$; from Proposition 2.2, we have, f measurable,

$$\begin{aligned}\int_{\partial B} \Phi(M_w f) v d\sigma &= \int_{\partial B} \left(\int_0^{M_w f} \varphi(s) ds \right) dv \\ &\leq \tilde{\rho} \int_{\partial B} \left(\int_0^{M_w f} \frac{\Phi(s)}{s} ds \right) dv \\ &= \tilde{\rho} \int_0^{+\infty} \left(\int_{\{P \in \partial B : M_w f(P) > s\}} dv \right) \frac{\Phi(s)}{s} ds \\ &\leq 2^{m'} c(n,m) \tilde{\rho} S \int_0^{+\infty} \left(\frac{1}{s} \right)^{m'} \left(\int_{\{|f| > \frac{s}{2}\}} |f|^{m'} dv \right) \frac{\Phi(s) ds}{s},\end{aligned}$$

where $S = S_{m,v}(w)^m$. Then, by Fubini's theorem

$$\int_{\partial B} \Phi(M_w f) v d\sigma \leq c(n,m) \tilde{\rho} S C' \int_{\partial B} \left(\int_0^1 \frac{\Phi(|f| s)}{s^{m'}} \frac{ds}{s} \right) v d\sigma$$

where C' is the doubling constant of $\Phi(t)$. Let now $m_0' = \dfrac{m' + p_0}{2}$. So $m_0' < p_0$, and then, as already mentioned, there is an s_0, $0 < s_0 < 1$ such that $\Phi(st) \leq \left(\dfrac{s}{s_0} \right)^{m_0'} \Phi(t)$, $t > 0$, $0 < s < 1$. Then,

$$\begin{aligned}\int_{\partial B} \Phi(M_w f) v d\sigma &\leq \frac{c(n,m) \tilde{\rho} S C'}{s_0^{m_0'}} \int_{\partial B} \Phi(|f|) \left(\int_0^1 s^{m_0' - (m'+1)} ds \right) v d\sigma \\ &= \frac{c(n,m) \tilde{\rho} S C'}{(m_0' - m') s_0^{m_0'}} \int_{\partial B} \Phi(|f|) v d\sigma\end{aligned}$$

This completes the proof of Theorem 4.1. □

As a corollary of the above theorem we have the following extension of Theorem 1.1. Namely:

THEOREM 4.4. – *Let B be the unit ball of $\mathbb{R}^n$ and let $\Phi(t) = \int_0^t \varphi(\tau) d\tau$ be a Young's function that satisfies the Δ_2-condition together with its complementary function $\Psi(s) = \int_0^s \varphi^{-1}(\tau) d\tau$, and let p_0^{-1} be the upper index of the Orlicz Space $L^{\Phi}(\partial B, d\sigma)$. Then the following are equivalent:*

i) *The Dirichlet problem* (1.1) *is* L^{Φ}*-solvable.*

ii) *The L-harmonic measure* ω *is absolutely continuous with respect to* σ, *and* $k = \dfrac{d\omega}{d\sigma} \in S_{\Phi}(d\sigma)$, *that is:*

$$\left(\frac{1}{k(\Delta)}\int_{\Delta} \varepsilon d\sigma\right)\varphi\left(\frac{1}{k(\Delta)}\int_{\Delta}\varphi^{-1}\left(\frac{k}{\varepsilon}\right)k d\sigma\right) \leq K \tag{4.25}$$

for all surface balls Δ *and for all* $\varepsilon > 0$.

iii) *The L-harmonic measure* ω *belongs to* $S_{q_0}(d\sigma)$, $\dfrac{1}{p_0} + \dfrac{1}{q_0} = 1$, *i.e.* ω *is absolutely continuous with respect to* σ, *and* $k = \dfrac{d\omega}{d\sigma} \in L^{q_0}(d\sigma)$, *with*

$$\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k^{q_0}\right)^{\frac{1}{q_0}} \leq C\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k\right), \qquad \forall \Delta.$$

PROOF. – The implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ follow directly from Theorem 4.1. For $(i) \Rightarrow (ii)$ we only need to prove that L^{Φ}-solvability of the Problem (1.1) implies that the harmonic measure ω is absolutely continuous with respect to the surface measure σ, and it follows from standard arguments. □

Note that from the 'openness' of the condition $\omega \in S_q(d\sigma)$ (Theorem 2.1, iv)), it follows that the L^{Φ}-solvability implies also the L^{Φ_δ}-solvability of the Problem (1.1), with suitable $\delta > 0$, and the upper index of L^{Φ_δ} is bigger than the one of L^{Φ}.

Moreover, we observe that, in the case $\Phi(t) = tlg^{a}(e+t)$, $0 \leq a < 1$, Theorem 1.1 does not hold (see [14]).

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