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A Note on $\mathcal{R}$-Maps.

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Sunto. – *Viene studiata la ricchezza della classe delle $\mathcal{R}$-mappe* [3]. *Vengono anche indicate numerose conseguenze per la teoria degli spazi quasi-$\mathcal{H}$-chiusi* (*vedi Lemma* 5).

Summary. – *Richness of the class of $\mathcal{R}$-maps* [3] *is investigated. Several consequences for the theory of quasi-$\mathcal{H}$-closed spaces are indicated.*

1. – Introduction.

In 1973, Carnahan [3] has introduced the notion of $\mathcal{R}$-maps. Mappings of stronger forms of continuity among $\mathcal{R}$-maps were studied by Noiri, see for instance [25]. In this paper, some well-known weak types of continuity with additional conditions, concerning respective forms of generalized openness or extremal disconnectednes imposed on the range space, are localized within the class of $\mathcal{R}$-maps. In the application section we show, among others, that members of the above mentioned subclasses of $\mathcal{R}$-maps preserve well-known conditions of near compactness, $\mathcal{S}$-closedness, and quasi-$\mathcal{H}$-closedness, simulaneously.

2. – Preliminaries.

No separation axioms for topological spaces (denoted with (X,τ), (Y,σ), etc.) are assumed. For a subset S of (X,τ), the closure and the interior of S have standard denotation: $\mathrm{cl}(S)$ (or $\mathrm{cl}_\tau(S)$) and $\mathrm{int}(S)$ (or $\mathrm{int}_\tau(S)$), respectively. The S is said to be *a-open* [17] (resp. *semi-open* [14]; *semi-closed* [4]; *preopen* [15]; *preclosed* [1]; *regular open*; *regular closed*) in (X,τ), if $S \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$ (resp. $S \subset \mathrm{cl}(\mathrm{int}(S))$; $\mathrm{int}(\mathrm{cl}(S)) \subset S$; $S \subset \mathrm{int}(\mathrm{cl}(S))$; $\mathrm{cl}(\mathrm{int}(S)) \subset S$; $S = \mathrm{int}(\mathrm{cl}(S))$; $S = \mathrm{cl}(\mathrm{int}(S))$). The collection of all semi-open (resp. preopen, preclosed, regular open, regular closed) subsets of (X,τ) is denoted by $\mathrm{SO}(X,\tau)$ (resp. $\mathrm{PO}(X,\tau)$, $\mathrm{PC}(X,\tau)$, $\mathrm{RO}(X,\tau)$, $\mathrm{RC}(X,\tau)$). The collection of all α-open subsets of (X,τ) forms a topology on X [17], different than the original one in general. The intersection of all semi-closed subsets of (X,τ) containing $S \subset X$ is called the *semi-closure* [4] of S and is denoted with $\mathrm{scl}\,(S)$.

A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *α-continuous* [22, 16] (resp. *semi-continuous* [14] (briefly *s.c.*; *almost continuous* in the sense of *Husain* (equiv. precontinuous) [10, 15] (briefly *a.c.H.*); *almost continuous* in the sense of *S&S* [34] (*a.c.S.*); *irresolute* [5]; an *$\mathcal{R}$-map* [3]), if $f^{-1}(V)$ is α-open in (X,τ) (resp. $f^{-1}(V) \in \mathrm{SO}\,(X,\tau)$; $f^{-1}(V) \in \mathrm{PO}\,(X,\tau)$; $f^{-1}(V) \in \tau$; $f^{-1}(V) \in \mathrm{SO}\,(X,\tau)$; $f^{-1}(V) \in \mathrm{RO}\,(X,\tau)$) for every $V \in \sigma$ (resp. $V \in \sigma$; $V \in \sigma$; $V \in \mathrm{RO}\,(Y,\sigma)$ [34, Theorem 2.2(b), see Definition 2.1]; $V \in \mathrm{SO}\,(Y,\sigma)$; $V \in \mathrm{RO}\,(Y,\sigma)$). Clearly, f is an $\mathcal{R}$-map if $f^{-1}(V) \in \mathrm{RC}\,(X,\tau)$ for any $V \in \mathrm{RC}\,(Y,\sigma)$.

It is known that f is α-continuous if and only if f is s.c. and a.c.H. [24, Theorem 3.2]. Also, *irresolutness $\Rightarrow$ s.c.* and *$\mathcal{R}$-mapness $\Rightarrow$ a.c.S.*, but the converses are false in general, see [31, Example 1] and [25, p. 249] respectively. In [7] the author shows that both *$\mathcal{R}$-mapness&continuity* and *$\mathcal{R}$-mapness&α-continuity* are couples of independent notions.

A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *almost open* in the sense of *S&S* [34] (briefly *a.o.S.*), if $f(U) \in \sigma$ for every $U \in \mathrm{RO}\,(X,\tau)$. A space (X,τ) is called *extremally disconnected* (*e.d.*) if $\mathrm{cl}\,(S) \in \tau$ for every $S \in \tau$.

3. – $\mathcal{R}$-mappings.

In 1972 Noiri has given the following result.

THEOREM 1. [18, Lemma 1]. – *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.S. and a.c.S., then it is an $\mathcal{R}$-map.*

An $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *weakly α-continuous* [26] (resp. *weakly continuous* [13]) (briefly *w.α.c.* and *w.c.* respectively), if $\mathrm{cl}_{\tau^\alpha}\left(f^{-1}(V)\right) \subset f^{-1}\left(\mathrm{cl}_\sigma(V)\right)$ [26, Lemma 2.2(d)] $\left(\text{resp.}\, f^{-1}(V) \subset \mathrm{int}\left(f^{-1}(\mathrm{cl}\,(V))\right)\right)$ for every $V \in \sigma$.

The following implications are known:

$$\text{continuity} \Rightarrow \text{a.c.S.} \Rightarrow \text{w.c.} \Rightarrow \text{w.}\alpha\text{.c.}$$

None of these allows the reverse (see respectively [34, Examples 2.1&2.3], [26, Example 5.4].

THEOREM 2. – *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.S., w.α.c., and a.c.H., then it is an $\mathcal{R}$-map.*

PROOF. – Let $V \in \mathrm{RO}(Y,\sigma)$. We shall show only that $\operatorname{int}\left(\operatorname{cl}\left(f^{-1}(V)\right)\right) \subset f^{-1}(V)$. By [1, Theorem 1.5(c)] we have

$$f\left(\operatorname{int}\left(\operatorname{cl}\left(f^{-1}(V)\right)\right)\right) = \operatorname{int}\left(f\left(\operatorname{int}\left(\operatorname{cl}\left(f^{-1}(V)\right)\right)\right)\right)$$
$$\subset \operatorname{int}\left(f\left(\operatorname{cl}_{\tau^a}\left(f^{-1}(V)\right)\right)\right) \subset \operatorname{int}(\operatorname{cl}(V)) = V.$$

Thus, f is an $\mathcal{R}$-map. □

We have *a-continuity* $\Rightarrow$ *w.c.* (see [16, Diagram p. 214] or [26, Lemma 5.2]) and *a-continuity* $\Rightarrow$ *a.c.H.* (see [16, Diagram p. 214] or [24, Theorem 3.2]). By [24, Example 3.10] one may easily see that there exists a w.a.c. and a.c.H. mapping which is not a.c.S.

THEOREM 3. – *Let (X,τ) be arbitrary and (Y,σ) be e.d. If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is w.a.c. and a.c.H. then it is an $\mathcal{R}$-map.*

PROOF. – Let $V \in \mathrm{RO}(Y,\sigma)$. We show that $\operatorname{int}\left(\operatorname{cl}\left(f^{-1}(V)\right)\right) \subset f^{-1}(V)$. With [26, Lemma 2.2(c)], [1, Theorem 1.5(d)], and [33, Theorem 6(3)] we obtain

$$f\left(\operatorname{int}\left(\operatorname{cl}\left(f^{-1}(V)\right)\right)\right) \subset f\left(\operatorname{cl}\left(\operatorname{int}_{\tau^a}\left(f^{-1}(\operatorname{cl}(V))\right)\right)\right) \subset f\left(\operatorname{cl}\left(\operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)\right)\right)$$
$$\subset \operatorname{cl}\left(f\left(\operatorname{int}\left(f^{-1}(\operatorname{cl}(V))\right)\right)\right) \subset \operatorname{cl}(V) = V.$$

Thus, f is an $\mathcal{R}$-map. □

The example below and [26, Example 5.6] show that w.a.c. and a.c.H. are independent of each other, even if the range space is e.d.

EXAMPLE 1. – Let $X = \{a,b\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. The identity mapping $f\colon (X,\tau) \to (X,\sigma)$ is w.c. and not a.c.H.

COROLLARY 1. – *Let a mapping $f\colon (X,\tau) \to (Y,\sigma)$ be w.c. and a.c.H. If*

- *f is a.o.S. or*
- *(Y,σ) is e.d.,*

then f is an $\mathcal{R}$-map.

Recall that *w.c.* and *a.c.H.* are independent notions too (see [26, Remark 5.8(2)] or consider Example 1 and [26, Example 5.6] for the e.d. range spaces).

In [7] the author has defined p-a.c.H. mappings: an $f\colon (X,\tau) \to (Y,\sigma)$ is p-*a.c.H.* if $f\left(\operatorname{cl}_\tau(U)\right) \subset \operatorname{cl}_\sigma(f(U))$ for every set $U \in \mathrm{PO}(X,\tau)$. By [33, Theorem 6(3)] we have p-*a.c.H.* $\Rightarrow$ *a.c.H.*, but the converse may fail [7]. In 1983, Mashhour et al.

showed that *a-continuity* $\Rightarrow$ p-*a.c.H.* [16, Corollary 1.1(i)]. The equivalence *a-continuity* $\Leftrightarrow$ *s.c. and* p-*a.c.H.* is due to the author [7]. We do not know if there is an example of p-a.c.H. mapping which is not a-continuous.

THEOREM 4. – *If a mapping* $f\colon (X,\tau)\to(Y,\sigma)$ *is* p-*a.c.H. and the range space* (Y,σ) *is e.d., then* f *is an* $\mathcal{R}$*-map.*

PROOF. – Let $V\in \mathrm{RO}\,(Y,\sigma)$. Obviously, we have $f^{-1}(V)\subset \mathrm{int}\,(\mathrm{cl}\,(f^{-1}(V)))$. On the other hand, by the assumption

$$f(\mathrm{int}\,(\mathrm{cl}\,(f^{-1}(V))))\subset f(\mathrm{cl}\,(f^{-1}(V)))\subset \mathrm{cl}\,(V)=V.$$

Hence f is an $\mathcal{R}$-map. □

COROLLARY 2. – *If a mapping* $f\colon (X,\tau)\to(Y,\sigma)$ *is a-continuous and* (Y,σ) *is e.d., then* f *is an* $\mathcal{R}$*-map.*

An $f\colon (X,\tau)\to(Y,\sigma)$ is said to be *weakly open* [32] if $f(U)\subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U)))$ for every $U\in\tau$. We have: *a.o.S.* $\Rightarrow$ *weak openness* [21, Lemma 1.4], but the converse is not true in general [21, Example 1.5]. If (X,τ) is e.d., then these notions become equivalent.

THEOREM 5. – *If a mapping* $f\colon (X,\tau)\to(Y,\sigma)$ *is weakly open* p-*a.c.H., then it is an* $\mathcal{R}$*-map.*

PROOF. – Take a $V\in \mathrm{RO}\,(Y,\sigma)$. Weak openness of f implies

$$\begin{aligned} f(\mathrm{int}\,(\mathrm{cl}\,(f^{-1}(V)))) &\subset \mathrm{int}\,(f(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(f^{-1}(V)))))) \\ &\subset \mathrm{int}\,(f(\mathrm{cl}\,(f^{-1}(V))))\subset \mathrm{int}\,(\mathrm{cl}\,(V))=V. \end{aligned}$$

Thus, f is an $\mathcal{R}$-map. □

COROLLARY 3. – *If a mapping* $f\colon (X,\tau)\to(Y,\sigma)$ *is weakly open and a-continuous then it is an* $\mathcal{R}$*-map.*

REMARK 1. – (a). For an a.o.S. and a-continuous mapping, Theorem 5 is a consequence of Theorem 2.

(b). To throw more light on Theorem 1 it is worth to recall that *a-continuity* and *a.c.S.* are independent of each other [24, Examples 3.9&3.10].

LEMMA 1. [5, Theorem 1.5]. – *A mapping* $f\colon (X,\tau)\to(Y,\sigma)$ *is irresolute if and only if* $f(\mathrm{scl}\,(A))\subset \mathrm{scl}\,(f(A))$ *for any* $A\subset X$.

THEOREM 6. – *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is irresolute and a.c.H., then it is an $\mathcal{R}$-map.*

PROOF. – Let $V \in \mathrm{RO}(Y,\sigma)$. With [12, Proposition 2.7(a)] we get $f\big(\mathrm{int}\big(\mathrm{cl}\big(f^{-1}(V)\big)\big)\big) \subset f\big(\mathrm{scl}\big(f^{-1}(V)\big)\big)$, because $f^{-1}(V) \in \mathrm{PO}(X,\tau)$. From Lemma 1 we infer that

$$f\big(\mathrm{int}\big(\mathrm{cl}\big(f^{-1}(V)\big)\big)\big) \subset \mathrm{scl}(V) = \mathrm{int}(\mathrm{cl}(V)) = V.$$

Thus, f is an $\mathcal{R}$-map. □

In [7] we justify independence of irresolutness and a.c.H.

An $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *weakly quasi continuous* [28] (briefly *w.q.c.*) if $f^{-1}(V) \subset \mathrm{cl}\big(\mathrm{int}\big(f^{-1}(\mathrm{cl}(V))\big)\big)$ for every $V \in \sigma$ [26, Lemma 5.3].

One has *irresolutness* $\Rightarrow$ *w.q.c.* but by [26, Example 5.5] the reverse does not hold.

THEOREM 7. – *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is weakly open, w.q.c., and a.c.H., then it is an $\mathcal{R}$-map.*

PROOF. – Let $V \in \mathrm{RO}(Y,\sigma)$. By hypothesis we can calculate as follows

$$f\big(\mathrm{int}\big(\mathrm{cl}\big(f^{-1}(V)\big)\big)\big) \subset \mathrm{int}\big(f\big(\mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(f^{-1}(V)\big)\big)\big)\big)\big) \subset \mathrm{int}\big(f\big(\mathrm{cl}\big(\mathrm{int}\big(f^{-1}(\mathrm{cl}(V))\big)\big)\big)\big)$$

$$\subset \mathrm{int}\big(\mathrm{cl}\big(f\big(f^{-1}(\mathrm{cl}(V))\big)\big)\big) \subset \mathrm{int}(\mathrm{cl}(V)) = V.$$

It means f is an $\mathcal{R}$-map. □

[26, Examples 5.5&5.6] show the independence of w.q.c. and a.c.H.

REMARK 2. – What concerns Theorem 2, it is worth to recall that *w.a.c.* $\Rightarrow$ *w.q.c.* and that the converse is not true [26, p. 489].

A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *almost open in the sense of Wilansky* [38] (briefly *a.o.W.*), if $f^{-1}\big(\mathrm{cl}_\sigma(V)\big) \subset \mathrm{cl}_\tau\big(f^{-1}(V)\big)$ for every $V \in \sigma$. It is known that f is a.o.W. if and only if $f(U) \in \mathrm{PO}(Y,\sigma)$ for every $U \in \tau$ [33, Theorem 11] (and hence, a.o.W. coincides with preopenness [15]). Recall that both *a.o.W.&weak openness* and *a.o.W.&a.o.S.* are pairs of independent notions [21, p. 315].

THEOREM 8. – *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.W. and* p*-a.c.H., then it is an $\mathcal{R}$-map.*

PROOF. – Let $V \in \mathrm{RO}(Y,\sigma)$. With [33, Theorem 11] we get

$$f\big(\mathrm{int}\,\big(\mathrm{cl}\,\big(f^{-1}(\mathrm{cl}\,(V))\big)\big)\big) \subset \mathrm{int}\,\big(\mathrm{cl}\,\big(f\big(\mathrm{cl}\,\big(f^{-1}(\mathrm{cl}\,(V))\big)\big)\big)\big) \subset \mathrm{int}\,\big(\mathrm{cl}\,\big(f\big(\mathrm{cl}\,\big(f^{-1}(V)\big)\big)\big)\big).$$

But f is p-a.c.H., whence $f\big(\mathrm{int}\,\big(\mathrm{cl}\,\big(f^{-1}(V)\big)\big)\big) \subset \mathrm{int}\,(\mathrm{cl}\,(V)) = V$. Thus, f is an $\mathcal{R}$-map. □

REMARK 3. – The reader is advised to compare Theorem 8 with Theorem 5.

COROLLARY 4. – *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.W. and a-continuous then it must be an $\mathcal{R}$-map (see Corollary 3).*

The next lemma is obvious, hence the proof is omitted.

LEMMA 2. – *$f\colon (X,\tau) \to (Y,\sigma)$ is an $\mathcal{R}$-map if and only if* $\mathrm{int}\,\big(\mathrm{cl}\,\big(f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(A)))\big)\big) = f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(A)))$ *for every $A \subset Y$.*

LEMMA 3. [19, Theorem 1(2)]. – *A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is s.c. if and only if* $\mathrm{int}\,\big(\mathrm{cl}\,\big(f^{-1}(B)\big)\big) \subset f^{-1}(\mathrm{cl}\,(B))$ *for every subset $B \subset Y$.*

A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *contra-precontinuous* [11] if $f^{-1}(V) \in \mathrm{PC}\,(X,\tau)$ for every $V \in \sigma$.

THEOREM 9. – *Let (Y,σ) be e.d. If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is s.c. and contra-precontinuous then it is an $\mathcal{R}$-map.*

PROOF. – Let $V \in \mathrm{RO}\,(Y,\sigma)$. By hypothesis we have $f^{-1}(\mathrm{cl}\,(V)) = \mathrm{int}\,\big(\mathrm{cl}\,\big(f^{-1}(\mathrm{cl}\,(V))\big)\big)$. Put $V = \mathrm{int}\,(\mathrm{cl}\,(A))$ for an $A \subset Y$ and apply Lemma 2. □

REMARK 4. – Since in an e.d. space (Y,σ),

$$\mathrm{RO}\,(Y,\sigma) = \mathrm{RC}\,(Y,\sigma),$$

Theorem 9 can be obtained by the following observation:

a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is s.c. and contra-precontinuous if and only if $f^{-1}(V) \in \mathrm{RC}\,(X,\tau)$ for every $V \in \sigma$ (i.e., iff f is RC-continuous [6]) [11, Theorem 3.9].

We complete our investigation with considering the notion of θ-continuity. An $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *θ-continuous* [8] if for every $x \in X$ and $V \in \sigma$ with $f(x) \in V$ there exists a $U \in \tau$ such that $x \in U$ and $f\big(\mathrm{cl}_\tau(U)\big) \subset \mathrm{cl}_\sigma(V)$. In [34, 9] it was shown that *a.c.S. $\Rightarrow$ θ-continuity $\Rightarrow$ w.c.*, but none of these implications is reversible. Notice that also: *a-continuity $\Rightarrow$ θ-continuity* [16].

Straight from the respective definitions one obtains the following.

REMARK 5. – Let a mapping $f\colon (X,\tau) \to (Y,\sigma)$ be weakly open or a.o.W. If it is θ-continuous, then it is a.c.S.

Thus for the a.o.S. case, Theorem 1 can be reformulated as follows: *a.o.S. together with θ-continuity imply $\mathcal{R}$-mapness*. It is worth to notice that in the a.o.S. case, Remark 5 can be proved using [23, Theorem 2.1].

4. – Applications.

In this section some corollaries following from what obtained above are presented.

THEOREM 10. – *Let an $f\colon (X,\tau) \to (Y,\sigma)$ be surjective and $g\colon (Y,\sigma) \to (Z,\gamma)$ be arbitrary. If the composition $g \circ f$ is a.o.S., the function f and the space (Y,σ) fulfil one of the assumptions of Theorems* 2-9 *or Corollaries* 1-4, *then g is a.o.S.*

PROOF. – The proofs are similar to that of [18, Theorem 2(a)] and thus omitted. □

A space (X,τ) is called *quasi-$\mathcal{H}$-closed* [29] (resp. *nearly compact* [35]; *$\mathcal{S}$-closed* [36]), briefly: $\mathcal{QHC}$ (resp. $\mathcal{NC}$; $\mathcal{SC}$), if for every cover $\mathcal{F} = \{V_a : a \in \nabla\} \subset \tau$ (resp. $\mathcal{F} \subset \tau$; $\mathcal{F} \subset \mathrm{SO}(X,\tau)$) there exists a finite subset $\nabla_0 \subset \nabla$ such that $X = \bigcup_{a\in\nabla_0} \mathrm{cl}(V_a)$ (resp. $X = \bigcup_{a\in\nabla_0} \mathrm{int}(\mathrm{cl}(V_a))$; $X = \bigcup_{a\in\nabla_0} \mathrm{cl}(V_a)$). The first name for $\mathcal{QHC}$ spaces was *almost-compact* spaces [30]. It is known that $\mathcal{NC} \Rightarrow \mathcal{QHC}$ and $\mathcal{SC} \Rightarrow \mathcal{QHC}$, but the converses fail, see [35, Section 1, p. 702] and [20, Remark 1.5] respectively. For e.d. spaces we have $\mathcal{NC} \Leftrightarrow \mathcal{SC} \Leftrightarrow \mathcal{QHC}$ (compare [2, Corollary 1]).

LEMMA 4. – *A space (X,τ) is $\mathcal{NC}$* [35, Theorem 2.1(c)] *(resp. $\mathcal{SC}$* [2, Theorem 2]*) if and only if every regular open (resp. regular closed) cover of X allows a finite subcover.*

By Lemma 4 and [27, Lemma 3.9] we obtain what follows.

LEMMA 5. – *Let a surjective $f\colon (X,\tau) \to (Y,\sigma)$ be an $\mathcal{R}$-map and the space (X,τ) be $\mathcal{NC}$ (resp. $\mathcal{SC}$; $\mathcal{QHC}$). Then (Y,σ) is $\mathcal{NC}$ (resp. $\mathcal{SC}$; $\mathcal{QHC}$).*

THEOREM 11. – *Assume $f\colon (X,\tau) \to (Y,\sigma)$ is a surjection. Let it be*

- ([35, Theorem 3.1] *for the $\mathcal{NC}$ case) a.o.S. and a.c.S. (see Remark* 5*), or*
- *a.o.S., w.a.c., and a.c.H. or*

- *weakly open and* p*-a.c.H., or*
- *irresolute and a.c.H., or*
- *weakly open, w.q.c., and a.c.H., or*
- *a.o.W. and* p*-a.c.H.*

If (X,τ) *is* $\mathcal{NC}$ *(resp.* $\mathcal{SC}$*;* $\mathcal{QHC}$*) then* (Y,σ) *is* $\mathcal{NC}$ *(resp.* $\mathcal{SC}$*;* $\mathcal{QHC}$*).*

PROOF. – These follow at once from Lemma 5 and Theorems 1, 2, 5, 6, 7, 8 respectively. □

REMARK 6. – Since continuity implies a-continuity and these notions do not coincide [22], the 6th case of Theorem 11 is an extension to [20, Theorem 2.2].

Recall that an irresolute (resp. θ-continuous) surjection preserves the $\mathcal{SC}$ [37, Theorem 3.5] (resp. $\mathcal{QHC}$ [27, Lemma 3.9]) property. Obviously, by [27, Lemma 3.9] an a.c.S. surjection preserves the $\mathcal{QHC}$ property [37, Theorem 3.3].

THEOREM 12. – *Let a space* (Y,σ) *be e.d. and let a surjection* $f\colon (X,\tau)\to(Y,\sigma)$ *fulfil the following:*

- f *is w.a.c., and a.c.H. or*
- f *is* p*-a.c.H., or*
- f *is s.c. and contra-precontinuous.*

If (X,τ) *is* $\mathcal{NC}$ *or* $\mathcal{SC}$ *or* $\mathcal{QHC}$*, then* (Y,σ) *is* $\mathcal{QHC}$ *(*$=\mathcal{NC}=\mathcal{SC}$*).*

PROOF. – Lemma 5 and Theorems 3, 4, 9 respectively.

It must be reminded that Thompson [37, Theorem 3.2] proved, in fact, that for every s.c. surjection, if the domain space is $\mathcal{SC}$ then the range space is $\mathcal{QHC}$.

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