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Local Existence of Solutions for Perturbation Problems with Non Linear Symmetries.

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Sunto. – *Si prova l'esistenza locale di famiglie di soluzioni per un problema di perturbazione quando l'operatore imperturbato è covariante per un'azione non lineare di un gruppo di Lie.*

Summary. – *The existence of local families of solutions for perturbation equations is proved when the free operator is covariant under a non linear action of a Lie group.*

1. – Introduction.

Given two Banach spaces M and N, we are concerned with the problem of existence of local solutions for the perturbation equation [5, 6]

$$A(x) + \varepsilon B(x) = 0, \tag{1.1}$$

(A and B defined on M with values in N, $\varepsilon \in \mathbb{R}$) where the operator A commutes with two non linear actions of a Lie group $\boldsymbol{G}$ given on M and N respectively.

Supposing that the free equation $A(x) = 0$ admits a particular solution x_0, a reduction is made for the existence problem of local solutions for the perturbation equation (1.1), by showing that the differential dA_{x_0} of A at x_0 is an isomorphism of M onto the orthogonal (relatively to some inner product) of the orbit generated by the action of $\boldsymbol{G}$ on x_0 in N. Such a reduction is required, since even if the kernel of dA_{x_0} admits a topological supplementary in M, dA_{x_0} is in general not surjective (for instance A is linearisation unstable at x_0, e.g. [3, p. 244]) and the implicit function theorem fails. Such a reduction will be obtained as a consequence of the "transversality" (as described in section 2) of the mapping A at every point not too far from x_0.

The perturbation problem we consider, is expressed by equation (1.1) with the hypothesis that the kernel of the differential dA_{x_0} of A at x_0 (A and B are supposed differentiable in an open subset U of M) split and that the map B is transversal at x_0. Even then, a direct application of the implicit function theorem fails. The difficulties encountered are related to the covariance of the mapping A under the action of $\boldsymbol{G}$ and can be overcome by extending the transversality of B to a neighborhood of x_0 in U. Indeed, it will allow to show that, locally, the elements $F(x,\varepsilon) = A(x) + \varepsilon B(x)$ in N are orthogonal (relatively to some duality on $M \times N$) to the orbits under $\boldsymbol{G}$ (actually, as we shall see, under the commutator subgroup of $\boldsymbol{G}$) of the corresponding points x in M. Hence we shall obtain a theorem of local existence of solutions for the perturbation problem, by proving that $F(x,\varepsilon)$ belongs to the orbit of x_0 for sufficiently small values of ε.

2. – Transversality of the mapping A.

In this section, we introduce some notations, and in order to state Proposition 2.1 (transversality criteria for the mapping A) we recall briefly the results we need on non linear representations of Lie groups [1].

Let $\boldsymbol{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}'$ its derived algebra. We denote by $\boldsymbol{G}'$ the connected Lie subgroup of $\boldsymbol{G}$ (the commutator subgroup of $\boldsymbol{G}$) associated to $\mathfrak{g}'$. The mappings A and B, defined on a Banach space M with values in a Banach space N, are differentiable in an open subset U of M, containing the origin and connected. A duality on $M \times N$ is given by a separately continuous, nondegenerated bilinear form $\langle , \rangle$. The action of $\boldsymbol{G}$ is given by two analytic representations (S, M) and (T, N) of $\boldsymbol{G}$ on M and N respectively, as defined within the theory of non linear representations of Lie groups, presented in [1]. The representation S (resp. T) is a morphism from $\boldsymbol{G}$ to the group of invertible elements in the space of formal power series of the form $S_g = S_g^1 + \sum_{n\geq 2} S_g^n$ (resp. $T_g = T_g^1 + \sum_{n\geq 2} T_g^n$) where $g \in \boldsymbol{G}$ and S_g^n belongs to $\mathcal{L}_n(M)$ (resp. T_g^n belongs to $\mathcal{L}_n(N)$) the space of symmetric n-linear continuous mappings on M (resp. N). The maps $x \longmapsto S_g(x) = S_g^1(x) + \sum_{n\geq 2} S_g^n(x)$ and $x \longmapsto T_g(x) = T_g^1(x) + \sum_{n\geq 2} T_g^n(x)$ are supposed to be analytic in a neighborhood of the origin in M and N, respectively, if g belongs to some neighborhood of the identity in $\boldsymbol{G}$. The linear representation S^1 (resp. T^1) is supposed C^∞ in M (resp. in N) and we denote by dS (resp. dT) the analytic representation of $\mathfrak{g}$ in M (resp. in N) obtained by differentiation of S (resp. T) [1].

We suppose that the mapping A is covariant, in the sense that it commutes with the action of $\boldsymbol{G}$, namely

$$A \circ S_g = T_g \circ A \quad \text{for every } g \in \boldsymbol{G}\,. \tag{2.1}$$

Also, A is assumed to vanish at the origin (as occurs, for instance, if the linear part T^1 is irreducible).

We associated to the mapping A the differential form ω_A on U defined by

$\omega_A(x)(\varphi) = \langle \varphi, A(x)\rangle$, $x \in U$, $\varphi \in M$. Moreover, ω_A is supposed to be closed, i.e.,

$$\langle x_1, dA_x(x_2)\rangle = \langle x_2, dA_x(x_1)\rangle\,,\; x_1, x_2 \in M \text{ and } x \in U \tag{2.2}$$

(where dA_x denotes the differential of A at x).

As shown in [7], an inner product on N, $(|)$ and a linear mapping $\kappa\colon M \longrightarrow N$ can be associated to the duality $\langle , \rangle$ such that

$$\langle x, y\rangle = (\kappa(x)|y) \quad \text{for every } (x,y) \in M \times N\,.$$

The transversality criteria for the mapping A, presented in [2], rely on the construction of some linear representations $\widetilde{S}$ and $\widetilde{T}$ associated to S and T. For instance, $\widetilde{S}_g$ (with $g \in \boldsymbol{G}$) acts on the strict inductive limit $\widetilde{M} = \bigcup\limits_{n\geq 1} M_n$, where $M_n = \overset{n}{\underset{i=1}{\oplus}} M^i$ with $M^i = \hat{\otimes}_i M$ (the projectif tensor product of M $i-$times), as the isomorphism of $\widetilde{M}$ preserving the subspaces M_n, defined by

$$\widetilde{S}_g(x_1 \otimes \cdots \otimes x_n) = \sum_{p=1}^{n} \sum_{i_1+\cdots+i_p=n} S_g^{i_1} \otimes \cdots \otimes S_g^{i_p}(\sigma_n(x_1 \otimes \cdots \otimes x_n))$$

for every $x_1, \ldots, x_n \in M$ [1, 4]. That linear action of $\boldsymbol{G}$ on $\widetilde{M}$ and $\widetilde{N}$ is supposed to preserve the duality in the sense that $\widetilde{T}$ is the contragedient of $\widetilde{S}$, namely,

$$\langle \widetilde{S}_g(\varphi), \widetilde{T}_g(\psi)\rangle = \langle \varphi, \psi\rangle \text{ for every } (\varphi, \psi) \in \widetilde{M} \times \widetilde{N} \tag{2.3}$$

(where the duality is extended to $\widetilde{M} \times \widetilde{N}$, see [2]). In particular the linear representations S^1 and T^1 are contragedient (since $\widetilde{S}_{|M} = S^1$ and $\widetilde{T}_{|N} = T^1$).

The following proposition has been obtained [2, Proposition 3.1].

PROPOSITION 2.1. – *Suppose that A is C^∞. Then under conditions* (2.1), (2.2) *and* (2.3), *for any point $x \in B(0,r) \cap U$ (with r small enough) $A(x)$ belongs to the polar of the orbit of x under $\mathfrak{g}'$, that is to say,*

$$\langle dS_X(x), A(x)\rangle = 0 \quad \textit{for every } X \in \mathfrak{g}'\,.$$

3. – Existence of local solutions for the perturbation equation $A(x) + \varepsilon B(x) = 0$.

If $x \in M$, let us denote by $\mathfrak{g}'^S_x = \{dS_X(x)/\; X \in \mathfrak{g}'\}$ the orbit of x under $\mathfrak{g}'$ (similarly $\mathfrak{g}^S_x$ denotes the orbit of x under $\mathfrak{g}$). Also we shall write $\mathfrak{g}'^{(S,V)}_x = \{dS_X(x)/\; X \in V\}$ for the orbit along a subspace V of $\mathfrak{g}'$ (similarly $\mathfrak{g}^{(S,V)}_x$ the orbit under $\mathfrak{g}$ along a subspace V of $\mathfrak{g}$).

We have shown (Proposition 2.1) that $A(x) \in \mathfrak{g}'^{S\,\circ}_x$ (the polar of $\mathfrak{g}'^S_x$ relatively to the duality $\langle , \rangle$) for every $X \in \mathfrak{g}'$ and $x \in U \cap B(0,r)$ with r small enough, that is to say A is "transversal" (if not specified we intend under $\mathfrak{g}'$) in every point x of

$U \cap B(0,r)$. We assume that a solution $x_0 \in U \cap B(0,r)$ of the free equation $A(x) = 0$ exists, such that the operator B is transversal in x_0. We shall show that locally the solutions of the perturbation problem

$$A(x) + \varepsilon B(x) = 0 \quad \text{with } A \text{ and } B \text{ differentiable in } U, \tag{3.1}$$

under the hypotheses

$$Ker\, dA_{x_0} \ \text{admits a topological suplementary } E_{x_0} \text{ in } M \tag{3.2}$$

and

$$Im\, dA_{x_0} \supset \mathfrak{g}'^{S\,\circ}_{x_0}\,, \tag{3.3}$$

are generated by the representation S from a curve x_ε in U passing through x_0.

The first step will consist of extending the transversality of B to a neighborhood U_0 of x_0, in the sense that if $x \in U_0$, there exists $g \in \boldsymbol{G}$ such that $T^1_{g^{-1}} \circ B \circ S_g$ is transversal in x.

Let $(X_1, \dots, X_n)$ be a basis of $\mathfrak{g}$ with $(X_1, \dots, X_r)$, $r \le n$, such that $(dS_{X_1}, \dots, dS_{X_r})$ is a basis of $dS(\mathfrak{g})$. The transversality condition above, stands on the mapping

$$\begin{aligned} M_x : \boldsymbol{G} &\longrightarrow \mathbb{R}^r \\ g &\longmapsto M_x(g) = (\langle dS_{X_i}(x), T^1_{g^{-1}} \circ B \circ S_g(x)\rangle)_{i=1,\dots,r} \end{aligned}$$

and it will be possible to obtain it if there exists a subspace V of $\mathfrak{g}'$, $V \neq \{0\}$, for which the differential of M_x in e does not vanish. More precisely, denoting by V^S the r-dimensional subspace of $\mathfrak{g}$ generated by $(X_1, \dots, X_r)$, we give

DEFINITION 3.1. – *A point $X \in V^S$, $X \neq 0$, is "critical" for B in $x \in U$ if $dM_x(X) = 0$.*

Consider the mapping

$$\begin{aligned} \gamma : M \times N &\longrightarrow \mathbb{R}^n \\ (x,y) &\longmapsto \gamma(x,y) = (\langle dS_{X_1}(x), y\rangle, \dots, \langle dS_{X_n}(x), y\rangle)\,; \end{aligned}$$

with this notation we can write

$$M_x(g) = \pi_r \circ \gamma(x, T^1_{g^{-1}} \circ B \circ S_g(x))\,,$$

where π_r denotes the projection of $\mathbb{R}^n$ on $\mathbb{R}^r$.

LEMMA 3.1. – *Suppose that there is a subspace V (of dimension p, $p \ge 1$) of $\mathfrak{g}$ which does not contain any critical element for B at x_0 and that B is transversal at x_0 along V (i.e. $B(x_0) \in \mathfrak{g}^{(S,V)^\circ}_{x_0}$). Then there esists a neighborhood U_0 of x_0 in U*

and a unique differentiable map s in U_0 with values in V such that $T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}$ is transversal along V in x for every x in U_0. i.e.: $T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}(x) \in \mathfrak{g}_x^{(S,V)^\circ}$.

PROOF. – After renumbering the basis of V^S say that $(X_1, \ldots, X_p)$ is a basis of V and consider the mapping

$$\begin{aligned} \Gamma : V \times U &\longrightarrow \mathbb{R}^p \\ (X, x) &\longmapsto \pi_p \circ M_x(exp\, X) \end{aligned}$$

with π_p the projection of $\mathbb{R}^r$ on $\mathbb{R}^p$; so

$$\Gamma(X, x) = (\langle dS_{X_i}(x), T^1_{exp\,-X} \circ B \circ S_{exp\,X}(x)\rangle)_{i=1,\ldots,p}\,.$$

The mapping Γ is differentiable in $V \times U$ and the partial differential at 0 of the mapping $X \longmapsto \Gamma(X, x_0)$, which is given by $\dfrac{\partial \Gamma}{\partial X}(0, x_0) = \pi_p \circ dM_{x_0|_e}$, is an isomorphism of V on $\mathbb{R}^p$ since by hypothesis V does not contain any critical points for B at x_0 (i.e., $dM_{x_0|_e}(X) \neq 0$ for every $X \in V$ and $X \neq 0$). Now, B being supposed transversal along V at x_0, we have $\Gamma(0, x_0) = \big(\langle dS_{X_i}(x_0), B(x_0)\rangle\big)_{i=1,\ldots,p} = 0$. Therefore by the implicit function theorem there exists a neighborhood U_0 of x_0 in U and a unique differentiable map s on U_0 with values in V such that $\Gamma(s(x), x) = 0$ for every $x \in U_0$. So $\langle dS_{X_i}(x), T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}(x)\rangle = 0$ for $i = 1, \ldots, p$ and $x \in U_0$. □

REMARK 3.1. – Let us indicate by $V_{x_0}(B)$ any maximal subspace of $\mathfrak{g}'$, with basis $(X_1, \ldots, X_k)$, such that the functions defined on $U \times \boldsymbol{G}$ by $(x, g) \longmapsto \langle dS_{X_i}(x), T^1_{g^{-1}} \circ B \circ S_g(x)\rangle$ are free. Under the hypothesis of Lemma 3.1, if V contains some subspace $V_{x_0}(B)$ then $T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}$ is transversal in x since $\langle dS_{X_i}, T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}\rangle$, $i = p+1, \ldots, r$, can be written as a linear combination of $\langle dS_{X_i}, T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}\rangle_{i=1,\ldots,p}$ and so vanishes. Thus $T^1_{exp\,-s(x)} \circ B \circ S_{exp\,s(x)}(x) \in \mathfrak{g}_x^{S\,\circ}$ for every $x \in U_0$.

There exists a scalar product $(|)$ on N, and a linear mapping $\kappa\colon M \longrightarrow N$ such that $\langle x, y\rangle = (\kappa(x)|y)$ for every $(x, y) \in M \times N$ [7]. If E is a subspace of M, we shall denote by E° (resp. $\kappa(E)^\perp$) the polar (resp. the orthogonal) in N of E with respect to $\langle , \rangle$ (resp. to $(|)$). Proposition 2.1 and the previous lemma show that for $x \in U$ sufficiently close to x_0, $A(x)$ and $T^1_{g_x^{-1}} \circ B \circ S_{g_x}$ for some $g_x \in \boldsymbol{G}$, are transversal in x; so is $A(x) + \varepsilon T^1_{g_x^{-1}} \circ B \circ S_{g_x}$. The next step will be to seek if $A(x) + \varepsilon T^1_{g_x^{-1}} \circ B \circ S_{g_x}(x)$ belongs (for ε small enough) to the orbit $\kappa({\mathfrak{g}'}^S_{x_0})$ of x_0.

Hence, if one can show, that for x sufficiently close to x_0, the subspaces ${\mathfrak{g}'}^{S\,\circ}_x$ do not intersect the orbit of x_0, the perturbated equation will be solved. Indeed, since

$A(x)+\varepsilon T^1_{g_x^{-1}}\circ B\circ S_{g_x}(x)=0$ can be rewritten as $T^1_{g_x^{-1}}[A(S_{g_x}(x))+\varepsilon B(S_{g_x}(x))]=0$, $S_{g_x}(x)$ is a solution of equation (3.1). We first prove the following lemma.

We shall denote by M_∞ the space M equipped with the topology induced by $C^\infty(\boldsymbol{G},M)$ (recall that we have supposed that M coincides with the space of differential vectors for S^1).

LEMMA 3.2. – *Let $x_1\in U$, there exists a subset U_1 of U, open in M_∞ and containing x_1, such that*

$$\forall x\in U_1\,,\quad \kappa({\mathfrak g'}^S_{x_1})\cap\kappa({\mathfrak g'}^S_x)^\perp=\{0\}$$

(actually $\kappa({\mathfrak g'}^S_{x_1})\cap\kappa(\mathfrak g'^{(S,V)}_x)^\perp=\{0\}$ for every subspace $V\neq\{0\}$ of $\mathfrak g'$).

REMARK 3.2. – Similarly, we have also that

$$\kappa(\mathfrak g^S_{x_1})\cap\kappa(\mathfrak g^{(S,V)}_x)^\perp=\{0\}\quad\text{for every } x\in U_1.$$

[PROOF OF LEMMA 3.2] – Let us consider the mapping $a\colon M_\infty\longrightarrow\mathcal L(\kappa({\mathfrak g'}^S_{x_1}),\mathbb R^n)$ where $a(x)$ is defined by $a(x)(y)=\gamma(x,y)$ for any $y=\kappa(dS_X(x_1))$ with $X\in\mathfrak g'$. That mapping is continuous, since in M_∞ the operator dS^1_X, $X\in\mathfrak g'$, are continuous. We have $a(x_1)(y)=(\langle dS_{X_i}(x_1),y\rangle)_{i=1,\dots,n}\neq 0$ if $y\in\kappa({\mathfrak g'}^S_{x_1})$, so $a(x_1)$ belongs to the set of invertible elements of $\mathcal L(\kappa({\mathfrak g'}^S_{x_1}),\mathbb R^n)$ which is open in it. Thus, a being continuous, there exists an open U_1 in M_∞ (and $U_1\subset U$)) containing x_1 such that $a(x)$ is invertible for every $x\in U_1$. Hence, if $y\in\kappa({\mathfrak g'}^S_{x_1})\cap\kappa({\mathfrak g'}^S_x)^\perp$ for $x\in U_1$ then $a(x)(y)=\gamma(x,y)=0$, thus $y=0$. So $\kappa({\mathfrak g'}^S_{x_1})\cap\kappa({\mathfrak g'}^S_x)^\perp=\{0\}$ for every $x\in U_1$. If V is a subspace of $\mathfrak g'$ of dimension p, by considering the mapping $\pi_p\circ a$ (π_p being the projection of $\mathbb R^p$ on $\mathbb R^n$) we obtain in the same way that $\kappa({\mathfrak g'}^S_{x_1})\cap\kappa(\mathfrak g'^{(S,V)}_x)^\perp=\{0\}$ for every $x\in U_1$. □

We place ourself under the assumptions of Proposition 2.1. Accordingly (as we shall see shortly) $Im\,dA_{x_0}\subset{{\mathfrak g'}^S_{x_0}}^\circ$, and so by condition (3.3) we can write $N=\kappa({\mathfrak g'}^S_{x_0})\oplus Im\,dA_{x_0}$.

We shall show that to each ε sufficiently close to 0, it can be associated a point x_ε close to x_0 in U such that there is one solution of (3.1) located on the orbit of x_ε (by the representation S restricted to $\boldsymbol{G}'$).

THEOREM 3.1. – *Let $V\neq\{0\}$, be a p-dimensional subspace of $\mathfrak g'$ with no critical points for B at x_0 and suppose that the operators dS^1_X, with $X\in V$, are continuous. Assume the hypotheses of Proposition* 2.1 *to be satisfied and suppose that*

$$S\circ A=T^1\circ A\,. \tag{3.4}$$

Then, under conditions (3.2) *and* (3.3), *in a neighborhood* U_0 *of* x_0 *in* U, *there exists a differentiable family* z_ε, *for* ε *small enough, of solutions of the perturbation equation* (3.1), *generated by the representation* S *restricted to the commutator* $\boldsymbol{G}'$ *of* $\boldsymbol{G}$.

PROOF. – From Lemma 3.1, there exists an open neighborhood U_0 of x_0 in U and a mapping $s: U_0 \longrightarrow V$ such that $T^1_{g_x^{-1}} \circ B \circ S_{g_x} \in \kappa(\mathfrak{g}'^{(S,V)}_x)^\perp$, with $g_x = exp\, s(x)$, for every $x \in U_0$. By Proposition 2.1 one has $A(x) \in \kappa(\mathfrak{g}'^S_x)^\perp \subset \kappa(\mathfrak{g}'^{(S,V)}_x)^\perp$ for every $x \in U \cap B(0,r)$ with r small enough. So up to take a subneighborhood of x_0 in U_0 we have that $A(x) + \varepsilon T^1_{g_x^{-1}} \circ B \circ S_{g_x}(x) \in \kappa(\mathfrak{g}'^{(S,V)}_x)^\perp$ for every $x \in U_0$. Let us show that $A(x) + \varepsilon\, T^1_{g_x^{-1}} \circ B \circ S_{g_x}(x)$ remains on the orbit of x_0 if ε is small enough.

Denote by $P_{x_0}: N \longrightarrow \kappa(\mathfrak{g}'^S_{x_0})$ the projection on the orbit of x_0 in N and consider the mapping $q_{x_0}: N \longrightarrow \kappa(\mathfrak{g}'^S_{x_0})^\perp$ defined by $q_{x_0}(y) = y - P_{x_0}(y)$. Recall that $Ker\, dA_{x_0}$ admits a topological supplementary space E_{x_0} in M, and consider the mapping Λ from the open $E_{x_0} \cap (U \setminus \{x_0\}) \times \mathbb{R}$, of the Banach space $E_{x_0} \times \mathbb{R}$, into $\kappa(\mathfrak{g}'^S_x)^\perp$ defined by $\Lambda(x,\varepsilon) = q_{x_0}\{A(x_0 + x) + \varepsilon\, T^1_{g^{-1}_{x_0+x}} \circ B \circ S_{g_{x_0+x}}(x_0 + x)\}$. The mapping Λ is differentiable and if we indicate by Λ_0 the mapping $x \longmapsto \Lambda(x,0) = q_{x_0}(A(x_0 + x))$, one has $d\Lambda_0(0) = dA_{x_0}$. Proposition 2.1 once again gives that $Im\, dA_{x_0} \subset \mathfrak{g}'^{S\,\circ}_{x_0}$. Indeed, as $A(x_0) = 0$ and $\langle dS_X(x), A(x)\rangle = 0$ for all x in a neighborhood of x_0, by differentiation at x_0 of the function $\psi(x) = \langle dS_X(x), A(x)\rangle$, one gets $d\psi_{x_0}(h) = \langle dS_X(x_0, dA_{x_0}(h)\rangle = 0$, for every $h \in M$. Thus $Im\, dA_{x_0} = \mathfrak{g}'^{S\,\circ}_{x_0}$ by (3.3) and $dA_{x_0|E_{x_0}}$ is a linear homeomorphism of E_{x_0} onto the Banach space $\kappa(\mathfrak{g}'^S_{x_0})^\perp$. Since $\Lambda(0,0) = 0$, according to the implicit function theorem there exist $\eta\rangle 0$, $\eta \in \mathbb{R}$, and a unique differentiable mapping u from $]-\eta,\eta[$ into $E_{x_0} \cap U_0 \backslash \{x_0\}$ such that $\Lambda(u(\varepsilon),\varepsilon) = 0$ for every $\varepsilon \in]-\eta,\eta[$. So $A(x_0 + u(\varepsilon)) + \varepsilon\, T^1_{g^{-1}_{x_0+u(\varepsilon)}} \circ B \circ S_{g_{x_0+u(\varepsilon)}}(x_0 + u(\varepsilon)) \in \kappa(\mathfrak{g}'^S_{x_0})$ if $\varepsilon \in]-\eta,\eta[$. Therefore, dS^1_X being continuous if $X \in V$ (by hypothesis), we can choose η such that $x_0 + u(\varepsilon) \in U_0 \cap U_1$, where U_1 is given by Lemma 3.2 and is open in M; so

$$(\star) \qquad A(x_0 + u(\varepsilon)) + \varepsilon\, T^1_{g^{-1}_{x_0+u(\varepsilon)}} \circ B \circ S_{g_{x_0+u(\varepsilon)}} = 0$$

for every $\varepsilon \in]-\eta,\eta[$. Now, by (3.4), the relation $(\star)$ is equivalent to

$$A(S_{g_{x_\varepsilon}}(x_\varepsilon)) + \varepsilon B(S_{g_{x_\varepsilon}}(x_\varepsilon)) = 0\,,$$

with $x_\varepsilon = x_0 + u(\varepsilon)$. □

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