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Rational Surfaces of Kodaira Type IV (*).

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Sunto. – *Si studia la geometria di una superficie di tipo Kodaira IV descrivendo la natura delle sue curve integre di auto-intersezione minore di zero, in particolare si prova che esse sono regolari e razionali. Sotto un'opportuna ipotesi si dimostra che il monoide delle classi di divisori effettivi ad essa associato è finitamente generato e che in quasi tutti i casi il suo sistema lineare completo anticanonico è di dimensione proiettiva zero e auto-intersezione strettamente negativa. Infine si evidenzia che se tale condizione non è soddisfatta, il monoide può non essere finitamente generato.*

Summary. – *We study the geometry of a rational surface of Kodaira type IV by giving the nature of its integral curves of self-intersection less than zero, in particular we show that they are smooth and rational. Hence, under a reasonable assumption, we prove the finite generation of its monoid of effective divisor classes and in almost all cases its anticanonical complete linear system is of projective dimension zero and of self-intersection strictly negative. Furthermore, we show that if this condition is not fulfilled, the monoid may fail to be finitely generated.*

1. – Introduction.

For a given three nonnegative integers p, q and r, consider $(p+q+r)$ points $P_1, \ldots, P_p, Q_1, \ldots, Q_q, R_1, \ldots, R_r$ in the projective plane $\boldsymbol{P}_k^2$ with k an algebraically closed field of arbitrary characteristic such that the points $P_1, \ldots, P_p$ (respectively $Q_1, \ldots, Q_q$, respectively $R_1, \ldots, R_r$) are on a line L_p (respectively on a line L_q, respectively on a line L_r) and such that the plane elliptic curve $L_p + L_q + L_r$ has a triple point T not belonging to the set $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q, R_1, \ldots, R_r\}$. Here by a plane elliptic curve, we mean any nonzero effective divisor on $\boldsymbol{P}_k^2$ of arithmetic genus equals one and we allow P_i (respectively Q_i, respectively R_i) to be infinitely near P_j (respectively Q_j, respectively R_j) if i is greater than j. Now consider the

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surface obtained by blowing up the projective plane at the zero dimensional closed subscheme $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q, R_1, \ldots, R_r\}$. Such surface will be called throughout this paper a *rational surface of Kodaira type IV associated to the integers p, q and r* and will be denoted by $X_{(p,q,r)}$ or simply by X if there is no need to stress on the integers p, q and r and we refer to it as a surface of Kodaira type IV.

One may ask the following question: Is the monoid $M(X_{(p,q,r)})$ of effective divisor classes on $X_{(p,q,r)}$ finitely generated? The answer is that the finite generation of $M(X_{(p,q,r)})$ need not to hold. Indeed, it depends not only on the concrete values of p, q and r but also on the positions of the points $P_1, \ldots, P_p, Q_1, \ldots, Q_q, R_1, \ldots,$ and R_r.

For $q = r = 0$ (respectively $r = 0$), we obtain the classical examples of the blow up the projective plane at collinear points (respectively at smooth points of a degenerate conic) and for these the finite generation of the monoids of effective divisor classes is achieved. For p, q and r big enough, the anticanonical complete linear system $|-K_{X_{(p,q,r)}}|$ is reduced to a singleton, where $K_{X_{(p,q,r)}}$ denotes a canonical divisor on $X_{(p,q,r)}$. It then follows that $-K_{X_{(p,q,r)}}$ is neither ample nor numerically effective. Hence one can not use the Mori theory in the first case and the recent obtained results in the second case in order to ensure the finite generation of the monoid. Here a divisor D on a smooth projective surface S is said to be ample (respectively numerically effective) if the integers D^2 and $D.C$ (respectively $D.C$) are larger than zero (respectively larger than or equal to zero) for every nonzero effective divisor on S. The finite generation of the monoid of effective divisor classes of a smooth projective rational surface Z with an anticanonical divisor $-K_Z$ such that $K_Z^2 \geq 0$ is completely understood, see [15], [16], [13], [14], [7], [5], [9], [10], [11], and [12]. If $K_Z^2 < 0$, only some few cases are known, and in these cases there are only some sufficient conditions which are given in order to ensure the finite generation of the monoid. See also [2] and [3].

In this work we give a sufficient arithmetical condition which ensures the finite generation of a rational surface of Kodaira type IV. This condition depends only on the number of points to be blown up. Our result is the following:

THEOREM 1.1. – *The monoid of effective divisor classes modulo algebraic equivalence on a rational surface of Kodaira type IV associated to the integers p, q and r is finitely generated if* $pqr - pq - pr - qr < 0$.

As a consequence, the following result holds:

COROLLARY 1.2. – *With notation as above. For the integers* $p \geq 4$, $q \geq 3$ *and* $r \geq 2$ *such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, *the monoid* $M(X_{(p,q,r)})$ *of effective divisor classes modulo algebraic equivalence on* $X_{(p,q,r)}$ *is finitely generated by the set of* (-1)-*curves,* (-2)-*curves and the three fixed components of the anticanonical divisor of* $X_{(p,q,r)}$.

As another corollary, we get the well known and classical result:

COROLLARY 1.3. – *The monoid of effective divisor classes on the surface obtained by the blow up the projective plane either at the smooth points a degenerate conic or at collinear points is finitely generated.*

Here we give an example in which the 3-tuple (p, q, r) satisfies the equality $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. This example provides also a surface of Kodaira type IV with a minimum Picard number, that is ten, for which the finite generation of the monoid fails.

EXAMPLE 1.4. – Consider the rational surface $X_{(3,3,3)}$ of Kodaira type *IV* such that the three ordinary points are sufficiently in general position. In this case one may observe that $-K_{(3,3,3)}$ is numerically effective (i.e., the intersection number of $K_{(3,3,3)}$ with any effective divisor on $X_{(3,3,3)}$ is larger than or equal to zero) and there are three smooth rational curves of self-intersection -2 on $X_{(3,3,3)}$. Henceforth by [7] there are an infinite number of smooth rational curves of self-intersection -1 on $X_{(3,3,3)}$. It then follows that the monoid of effective divisor classes on $X_{(3,3,3)}$ is not finitely generated.

The paper is organized as follows: section 2 gives some standard facts about smooth projective rational surfaces and fix our notation. Section 3 sates some general results about the integral curves of strictly negative self-intersection on anticanonical rational surfaces, i.e., those smooth projective rational surfaces carrying an effective anticanonical divisor. Section 4 contains a proof of Theorem 1.1.

2. – Preliminaries.

Let S be a smooth projective rational surface defined over an algebraically closed field of arbitrary characteristic. A canonical divisor on S, respectively the Picard group $Pic(S)$ of S will be denoted by K_S and $Pic(S)$ respectively. There is an intersection form on $Pic(S)$ induced by the intersection of divisors on S, it will be denoted by a dot, that is, for x and y in $Pic(S)$, $x.y$ is the intersection number of x and y (see [8]).

The following result known as the Riemann-Roch theorem for smooth projective rational surfaces is stated using the Serre duality.

LEMMA 2.1. – *Let D be a divisor on a smooth projective rational surface S having an algebraically closed field of arbitrary characteristic as a ground field.*

Then the following equality holds:

$$h^0(S, O_S(D)) - h^1(S, O_S(D)) + h^0(S, O_S(K_S - D)) = 1 + \frac{1}{2}(D^2 - D.K_S),$$

$O_S(D)$ being an invertible sheaf associated to the divisor D.

Here we recall some standard results, see for instance [6], [8], and [1]. A divisor class x modulo algebraic equivalence on a smooth projective rational surface Z is effective respectively numerically effective, nef in short, if an element of x is an effective, respectively numerically effective, divisor on Z. Here a divisor D on Z is nef if $D.C \geq 0$ for every integral curve C on Z. Now, we start with some properties which follow from a successive iterations of blowing up closed points of a smooth projective rational surface.

LEMMA 2.2. – *Let $\pi^\star : Pic(S) \to Pic(S')$ be the natural group homomorphism on Picard groups induced by a given birational morphism $\pi : S' \to S$ of smooth projective rational surfaces. Then $\pi^\star$ is an injective intersection-form preserving map of free abelian groups of finite rank. Furthermore, it preserves the dimensions of cohomology groups, the effective divisor classes and the numerically effective divisor classes.*

PROOF. – See [6, Lemma II.1, page 1193].

LEMMA 2.3. – *Let x be an element of the Picard group $Pic(S)$ of a smooth projective rational surface S. The effectiveness or the the nefness of x implies the noneffectiveness of $K_S - x$, where K_S denotes the element of $Pic(S)$ which contains a canonical divisor on S. Moreover, the nefness of x implies also that the self-intersection of x is greater than or equal to zero.*

PROOF. – See [6, Lemma II.2, page 1193].

3. – Integral Curves on Anticanonical Rational Surfaces.

In this section, we give some elementary properties of integral curves on a fixed anticanonical rational surface (see below for the definition). Then apply these to the case where Z is of Kodaira type IV.

DEFINITION 3.1. – *Let Z be a smooth projective rational surface having K_Z as a canonical divisor. Z is anticanonical if the sheaf $\mathcal{O}_Z(-K_Z)$ has a nonzero global section.*

Here are some standard kinds of curves on a surface.

DEFINITION 3.2. – *Let Y be a smooth projective rational surface having K_Y as a canonical divisor and let Γ be an integral curve on Y. Γ is a* (-1)-*curve* (*respectively a* (-2)-*curve*) *if Γ is a smooth, rational and of self-intersection* -1 (*respectively of self-intersection* -2).

The following describes the nature of integral curves on an anticanonical rational surface.

PROPOSITION 3.3. – *Let Z be a smooth projective anticanonical rational surface having K_Z as a canonical divisor and let Γ be an integral curve of strictly negative self-intersection on Z. Then except a finite number of cases, Γ is either a* (-1)-*curve or a* (-2)-*curve.*

PROOF. – Applying the adjunction formula to Γ and taking into account that there exists a nonzero effective divisor which is linearly equivalent to $-K_Z$ give the inequality $\Gamma.K_Z \leq 0$ except maybe for a finite number of integral curves. Hence, Γ is either a (-2)-curve (in the case where $\Gamma.K_Z = 0$) or a (-1)-curve (in the case where $\Gamma.K_Z < 0$).

As a consequence, the following result holds.

COROLLARY 3.4. – *Except at most three number of exceptions, every integral curve of strictly negative self-intersection on a surface of Kodaira type IV is either a* (-1)-*curve or a* (-2)-*curve.*

We also need the following result:

LEMMA 3.5. – *The monoid of effective divisor classes modulo algebraic equivalence on an anticanonical smooth projective rational surface X is finitely generated if and only if X has only a finite number of* (-1)-*curves and only a finite number of* (-2)-*curves.*

PROOF. – See [12, Corollary 4.2, page 109].

4. – Proof of Theorem 1.1.

In this section, we present a proof of the Theorem 1.1 stated in section one. To do so, we need to give explicitly the lattice structure of $Pic(X_{(p,q,r)})$. Firstly, the integral basis

$$(\mathcal{E}_0; -\mathcal{E}_1^{L_p}, \dots, -\mathcal{E}_p^{L_p}; -\mathcal{E}_1^{L_q}, \dots, -\mathcal{E}_q^{L_q}; -\mathcal{E}_1^{L_r}, \dots, -\mathcal{E}_r^{L_r}),$$

is defined by:

- $\mathcal{E}_0$ is the class of a line on the projective plane which does not pass through any of the assigned points $P_1, \dots, P_p; Q_1, \dots, Q_q; R_1, \dots, R_r$ in consideration,
- $\mathcal{E}_i^{L_p}$ is the class of the exceptional divisor corresponding to the i^{-th} point blown-up P_i for every $i = 1, \dots, p$,
- $\mathcal{E}_j^{L_q}$ is the class of the exceptional divisor corresponding to the j^{-th} point blown-up Q_j for every $j = 1, \dots, q$,
- $\mathcal{E}_h^{L_r}$ is the class of the exceptional divisor corresponding to the h^{-th} point blown-up R_h for every $h = 1, \dots, r$.

The class of a divisor on $X_{(p,q,r)}$ will be represented by the $(1+p+q+r)$-tuple

$$(a; b_1^{L_p}, \dots, b_p^{L_p}; b_1^{L_q}, \dots, b_q^{L_q}; b_1^{L_r}, \dots, b_r^{L_r})$$

Secondly, the intersection form on $Pic(X_{(p,q,r)})$ is given by:

- $\mathcal{E}_0^2 = 1 = -(\mathcal{E}_i^{L_p})^2 = -(\mathcal{E}_j^{L_q})^2 = -(\mathcal{E}_h^{L_r})^2$ for every $i = 1, \dots, p, j = 1, ..., q$ and $h = 1, \dots, r$;
- $\mathcal{E}_i^{L_p}.\mathcal{E}_{i'}^{L_p} = 0$ for every $i, i' = 1, \dots, p$, with $i \neq i'$;
- $\mathcal{E}_j^{L_q}.\mathcal{E}_{j'}^{L_q} = 0$ for every $j, j' = 1, \dots, q$, with $j \neq j'$;
- $\mathcal{E}_h^{L_r}.\mathcal{E}_{h'}^{L_r} = 0$ for every $j, j' = 1, \dots, r$, with $h \neq h'$;
- $\mathcal{E}_i^{L_p}.\mathcal{E}_j^{L_q} = \mathcal{E}_j^{L_q}.\mathcal{E}_h^{L_r} = \mathcal{E}_h^{L_r}.\mathcal{E}_i^{L_p} = 0$ for every $i = 1, ..., p$, $j = 1, \dots, q$, $h = 1, \dots, r$;
- $\mathcal{E}_0.\mathcal{E}_i^{L_p} = \mathcal{E}_0.\mathcal{E}_j^{L_q} = \mathcal{E}_0.\mathcal{E}_h^{L_r} = 0$ for every $i = 1, ..., p, j = 1, \dots, q, h = 1, \dots, r$.

REMARK 4.1. – we observe that if the class $(a; b_1^{L_p}, \dots, b_p^{L_p}; b_1^{L_q}, \dots, b_q^{L_q}; b_1^{L_r}, \dots, b_r^{L_r})$ is effective, then it represents the class of a projective plane curve of degree a and having at least multiplicity $b_1^{L_p}, ..., b_p^{L_p}$ (respectively, $b_1^{L_q}, ..., b_q^{L_q}$ and $d_1^{L_r}, ..., d_r^{L_r}$) at the points $P_1, ..., P_p$ (respectively $Q_1, ..., Q_q$ and $R_1, ..., R_r$). Also we note by assumption that the classes $\mathcal{E}_0, \mathcal{E}_i^{L_p}, \mathcal{E}_j^{L_q}, \mathcal{E}_h^{L_r}$ need not to be the classes of smooth rational curves on $X_{(p,q,r)}$.

From the above Lemma (3.5), to prove the finite generation of $M(X_{(p,q,r)})$, it is sufficient to prove that the set of (-1)-curves and the set of (-2)-curves are both finite. To do so we first show that the set of (-1)-curves on $X_{(p,q,r)}$ is finite. Indeed, let E be a general (-1)-curve on $X_{(p,q,r)}$ and let $(a; b_1^{L_p}, \dots, b_p^{L_p}; b_1^{L_q}, \dots, b_q^{L_q}; b_1^{L_r}, \dots, b_r^{L_r})$ be the $(1+p+q+r)$-tuple associated to the class of E in the Picard group $Pic(X_{(p,q,r)})$ relative to the integral basis

$(\mathcal{E}_0; -\mathcal{E}_1^{L_p}, \ldots, -\mathcal{E}_p^{L_p}; -\mathcal{E}_1^{L_q}, \ldots, -\mathcal{E}_q^{L_q}; -\mathcal{E}_1^{L_r}, \ldots, -\mathcal{E}_r^{L_r})$. Since E is general, it follows that the degree $a \geq 1$. From the two equalities $E^2 = -1$ and $E.K_{X_{(p,q,r)}} = -1$, one may obtain the following two equalities:

$$(4.1)\quad (b_1^{L_p})^2 + \ldots + (b_p^{L_p})^2 + (b_1^{L_q})^2 + \ldots + (b_q^{L_q})^2 + (b_1^{L_r})^2 + \ldots + (b_r^{L_r})^2 = a^2 + 1,$$

$$(4.2)\quad (a - b_1^{L_p} - \ldots - b_p^{L_p}) + (a - b_1^{L_q} - \ldots - b_q^{L_q}) + (a - b_1^{L_r} - \ldots - b_r^{L_r}) = 1.$$

Hence either the following case which we refer to as the case 1,

$$(4.3)\quad (b_1^{L_p})^2 + \ldots + (b_p^{L_p})^2 + (b_1^{L_q})^2 + \ldots + (b_q^{L_q})^2 + (b_1^{L_r})^2 + \ldots + (b_r^{L_r})^2 = a^2 + 1,$$

and

$$(4.4)\quad b_1^{L_p} + \ldots + b_p^{L_p} = a - 1,$$

and

$$(4.5)\quad b_1^{L_q} + \ldots + b_q^{L_q} = b_1^{L_r} + \ldots + b_r^{L_r} = a,$$

or the following case which we refer to as the case 2

$$(4.6)\quad (b_1^{L_p})^2 + \ldots + (b_p^{L_p})^2 + (b_1^{L_q})^2 + \ldots + (b_q^{L_q})^2 + (b_1^{L_r})^2 + \ldots + (b_r^{L_r})^2 = a^2 + 1,$$

and

$$(4.7)\quad b_1^{L_q} + \ldots + b_q^{L_q} = a - 1,$$

and

$$(4.8)\quad b_1^{L_p} + \ldots + b_p^{L_p} = b_1^{L_r} + \ldots + b_r^{L_r} = a,$$

or the following case which we refer to as the case 3

$$(4.9)\quad (b_1^{L_p})^2 + \ldots + (b_p^{L_p})^2 + (b_1^{L_q})^2 + \ldots + (b_q^{L_q})^2 + (b_1^{L_r})^2 + \ldots + (b_r^{L_r})^2 = a^2 + 1,$$

and

$$(4.10)\quad b_1^{L_r} + \ldots + b_r^{L_r} = a - 1,$$

and

$$(4.11)\quad b_1^{L_p} + \ldots + b_p^{L_p} = b_1^{L_q} + \ldots + b_q^{L_q} = a.$$

holds.

It follows that if either p, q or r vanishes, then a also vanishes. So we may consider the integers p, q and r to be all not equal to zero. Assume that we are in the case 1, and consider the new scalars $(a_i^{L_p})_{i\in\{1,\ldots,p\}}$, $(\beta_j^{L_q})_{j\in\{1,\ldots,q\}}$ and

$(\gamma_h^{L_r})_{h\in\{1,\dots,r\}}$ defined by $\alpha_i^{L_p} = b_i^{L_p} - \dfrac{a-1}{p}$ for every $i = 1,\dots,p$, $\beta_j^{L_q} = b_j^{L_q} - \dfrac{a}{q}$ for every $j = 1,\dots,q$, and $\gamma_h^{L_r} = b_h^{L_r} - \dfrac{a}{r}$ for every $h = 1,\dots,r$.

Then the equations (4.3), (4.4) and (4.5) give

$$(4.12)\quad (\alpha_1^{L_p})^2 + \ldots + (\alpha_p^{L_p})^2 + (\beta_1^{L_q})^2 + \ldots + (\beta_q^{L_q})^2 + (\gamma_1^{L_r})^2 + \ldots + (\gamma_r^{L_r})^2 = 1 + a^2 - \frac{(a-1)^2}{p} - \frac{a^2}{q} - \frac{a^2}{r},$$

and

$$(4.13)\quad \alpha_1^{L_p} + \ldots + \alpha_p^{L_p} = 0,$$

and

$$(4.14)\quad \beta_1^{L_q} + \ldots + \beta_q^{L_q} = 0,$$

and

$$(4.15)\quad \gamma_1^{L_r} + \ldots + \gamma_r^{L_r} = 0.$$

It follows then from the equation (4.12) that a is bounded. With the same method, we prove that a is bounded in the case 2 and in the case 3.

Now we proceed to prove that the set of (-2)-curves on $X_{(p,q,r)}$ is finite. Indeed, let N be a general (-2)-curve on $X_{(p,q,r)}$. Let $(a; b_1^{L_p}, \ldots, b_p^{L_p}; b_1^{L_q}, \ldots, b_q^{L_q}; b_1^{L_r}, \ldots, b_r^{L_r})$ be the $(1+p+q+r)$-tuple associated to the class of N in the Picard group $Pic(X_{(p,q,r)})$ relative to the integral basis $(\mathcal{E}_0; -\mathcal{E}_1^{L_p}, \ldots, -\mathcal{E}_p^{L_p}; -\mathcal{E}_1^{L_q}, \ldots, -\mathcal{E}_q^{L_q}; -\mathcal{E}_1^{L_r}, \ldots, -\mathcal{E}_r^{L_r})$. Since $N^2 = -2$, it follows that the degree $a \geq 1$. From the two equalities $N^2 = -2$ and $N.K_{X_{(p,q,r)}} = 0$, one may obtain the following equalities:

$$(4.16)\quad (b_1^{L_p})^2 + \ldots + (b_p^{L_p})^2 + (b_1^{L_q})^2 + \ldots + (b_q^{L_q})^2 + (b_1^{L_r})^2 + \ldots + (b_r^{L_r})^2 = a^2 + 2,$$

and

$$(4.17)\quad b_1^{L_p} + \ldots + b_p^{L_p} + b_1^{L_q} + \ldots + b_q^{L_q} + b_1^{L_r} + \ldots + b_r^{L_r} = 3a.$$

It follows that if either p, q or r vanishes, then a also vanishes. Consequently, we assume that p, q and r do not vanish. From the two equalities (4.16) and (4.17), one may obtain the following three equalities:

$$(4.18)\quad (b_1^{L_p})^2 + \ldots + (b_p^{L_p})^2 + (b_1^{L_q})^2 + \ldots + (b_q^{L_q})^2 + (b_1^{L_r})^2 + \ldots + (b_r^{L_r})^2 = a^2 + 2,$$

and

$$(4.19)\quad b_1^{L_p} + \ldots + b_p^{L_p} = a,$$

and

$$b_1^{L_q} + \ldots + b_q^{L_q} = a, \tag{4.20}$$

and

$$b_1^{L_r} + \ldots + b_r^{L_r} = a. \tag{4.21}$$

We claim that the integer a is bounded. To see this, we argue as follows. Define $x_i^{L_p}$, $y_j^{L_q}$ and $z_h^{L_r}$ for every $i = 1, \ldots, p$, for every $j = 1, \ldots, q$ and for every $h = 1, \ldots, r$ as follows.

$$x_i^{L_p} = \left(b_i^{L_p} - \frac{a}{p}\right), \tag{4.22}$$

and

$$y_j^{L_q} = \left(b_j^{L_q} - \frac{a}{q}\right), \tag{4.23}$$

and

$$z_h^{L_r} = \left(b_h^{L_r} - \frac{a}{r}\right). \tag{4.24}$$

Then the equations (4.19) and (4.21) become respectively:

$$x_1^{L_p} + \ldots + x_p^{L_p} = 0, \tag{4.25}$$

and

$$y_1^{L_q} + \ldots + y_q^{L_q} = 0, \tag{4.26}$$

and

$$z_1^{L_r} + \ldots + z_r^{L_r} = 0. \tag{4.27}$$

Whereas the equation (4.18) gives the following equation:

$$(x_1^{L_p})^2 + \ldots + (x_p^{L_p})^2 + (y_1^{L_q})^2 + \ldots + (y_q^{L_q})^2 + (z_1^{L_r})^2 + \ldots + (z_r^{L_r})^2 = 2 + a^2 - \frac{a^2}{p} - \frac{a^2}{q} - \frac{a^2}{r}, \tag{4.28}$$

which implies by our assumption that the nonnegative integer a is bounded.

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