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DANIELA GIACHETTI, GIULIA MAROSCIA

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Porous Medium Type Equations with a Quadratic Gradient Term

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Sunto. – *In questa nota illustreremo un risultato di esistenza per il problema di Cauchy - Dirichlet in $Q_T = \Omega \times (0,T)$ per equazioni paraboliche con parte principale degenere (del tipo "mezzi porosi") aventi un termine di grado inferiore quadratico nel gradiente. Il termine noto f e il dato iniziale u_0 sono funzioni limitate non negative.*

Summary. – *We show an existence result for the Cauchy - Dirichlet problem in $Q_T = \Omega \times (0,T)$ for parabolic equations with degenerate principal part (of porous medium type) with a lower order term having a quadratic growth with respect to the gradient. The right hand side of the equation f and the initial datum u_0 are bounded nonnegative functions.*

1. – Introduction.

The model problem we refer to is:

$$(1)\qquad \begin{cases} u_t - \operatorname{div}(a(u)Du) = \beta(u)|\nabla u|^2 + f & \text{on } Q_T \\ u(x,t) = 0 & \text{on } \Sigma_T \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$, T is such that $0 < T < +\infty$, $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. The real function $a(u)$ is continuous, nonnegative and strictly increasing in $[0,+\infty)$ with $a(0) = 0$, and the real function $\beta(u)$ is nonnegative and continuous in $[0,+\infty)$. Examples of $a(s)$ and $\beta(s)$ are:

$$(2)\qquad a(s) = s^m, \quad \text{with} \quad m > 0 \qquad \beta(s) = s^{\upsilon}, \quad \text{with} \quad \upsilon \geq 0$$

In the particular case $\beta \equiv 0$, our equation reduces to the classical porous media equation, which has been widely studied: we just refer to the fundamental book of Vazquez ([9]) and references therein.

We point out that the motivation for dealing with such kind of problems comes from some applicative models in the framework of the fluid dynamics in porous media, in petroleum engineering and in hydrology (see [5]). Indeed, this kind of equations arise from the physical modelling of a simultaneous flow of two immiscible fluids in a porous medium (water and oil in the case of the so called

secondary recovery in petroleum reservoirs, or air and water in groundwater hydrology).
We point out that, while the regularity of weak solutions to problem (1) in the general form:

$$u_t - \operatorname{div} a(x,t,u,\nabla u) = b(x,t,u,\nabla u) + f \tag{3}$$

has been investigated in the literature (see [8], [4]), to our knowledge, a general existence result was still missing.
The study of equations of type (3) started in [7]. Some problems with principal part of porous medium type and lower order terms depending on u and its gradient are also studied in [1]. The techniques we employed in order to prove the existence of distributional solutions for our problem are inspired by the ones used to show the result for nonlinear uniformly parabolic problems in the early paper by Boccardo-Murat-Puel [2] and by those used in [3] for quasi-linear parabolic problems with a degenerate coercivity at infinity (i.e. with $\lim_{u\to\infty} a(u) = 0$). We use a test function method: the test functions employed to get the a priori estimates involve exponentials of a primitive of the ratio $\beta(s)/a(s)$, whose behaviour near $s = 0$ plays an essential role. An important tool in the study of the present problem is to prove the existence of a distributional solution to (1) by using the strong convergence in the space $(L^2(Q_T))^N$ of the gradient of certain truncations of the solutions of suitable approximating problems.
As a further result of our theorem we get, by suitable adaptations, an existence result for problems involving singular principal part at $u = 0$ (i.e. $\lim_{u\to 0} a(u) = +\infty$) and possibly singular first order terms (i.e. $\lim_{u\to 0} \beta(u) = +\infty$).
More specifically we deal with the case:

$$a(s) = s^m, \quad -1 < m < 0 \qquad \beta(s) = s^\upsilon, \qquad \upsilon > m - 1$$

The "mixed" problem i.e. where the principal part of the operator is of porous medium-type and the lower order term is singular in $u = 0$, is also considered.

2. – The main result.

Let us consider Ω a bounded open set in $\mathbb{R}^N$, T such that $0 < T < +\infty$ and the sets $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. We state the problem related to the equation (1) in the general form:

$$\textbf{(P)}\begin{cases} u_t - \operatorname{div}(a(x,t,u,\nabla u)) = b(x,t,u,\nabla u) + f(x,t) & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \Sigma_T \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

where f and u_0 are nonnegative functions such that $f \in \mathrm{L}^\infty(\mathrm{Q_T})$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Moreover $a(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $b(x,t,s,\xi) : \Omega \times (0,T) \times$

$\mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions (i.e. they are measurable with respect to (x,t) for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) for almost every $(x,t) \in Q_T$) and they satisfy the growth conditions:

A1) There exists a continuous real function $a : \mathbb{R} \longrightarrow [0,+\infty)$ strictly increasing on $[0,+\infty)$, such that $a(0) = 0$ and moreover:

$$a(x,t,s,\xi) \cdot \xi \geq a(s)|\xi|^2 \quad \text{a.e. in } Q_T, \text{ and } \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

A2) there exists $\Lambda \in \mathbb{R}$, with $\Lambda > 1$, such that:

$$|a(x,t,s,\xi)| \leq \Lambda a(s)|\xi| \quad \text{a.e. in } Q_T, \text{ and } \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

A3) the following property holds, $\forall s > 0$ and a.e. in Q_T:

$$[a(x,t,s,\xi) - a(x,t,s,\eta)][\xi - \eta] > 0 \ \forall (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N \text{ s.t. } \xi \neq \eta.$$

B) There exists a continuous function $\beta : \mathbb{R} \longrightarrow [0,+\infty)$ such that $\beta(s) > 0$ if $s > 0$, and there exists $0 < \lambda \leq 1$ such that , for $k > 0$, one of the following hypotheses holds a.e. in Q_T and $\forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N$:

B1) if $\dfrac{\beta}{a} \in L^1(0,k)$: $|b(x,t,s,\xi)| \leq \beta(s)|\xi|^2$;

B2) if $\dfrac{\beta}{a} \notin L^1(0,k)$: $\lambda\beta(s)|\xi|^2 \leq b(x,t,s,\xi) \leq \beta(s)|\xi|^2$.

REMARK 2.1. – We point out that in the following we will show that a distributional solution u of problem (**P**) verifies $u \in [0,k]$, for $k = k(\|u_0\|_{L^\infty}, \|f\|_{L^\infty}, T)$.

Furthermore we have to add a technical hypothesis we will need in the following:

H) if $\dfrac{\beta}{a}$ is unbounded near $s = 0$: $\exists\, \varepsilon > 0$ such that $\dfrac{\beta}{a}$ is decreasing in $(0,\varepsilon)$.

REMARK 2.2. – We observe that condition **H)** is always attained if the functions a and β are power functions which is the model case we refer to.

Then we define some functions we will use in the following:

$$A(s) := \int_0^s a(\sigma)d\sigma, \quad \psi(s) := \int_0^s a(\sigma)e^{\gamma(\sigma)}d\sigma, \tag{4}$$

$$\gamma(s) := \begin{cases} \displaystyle\int_0^s \frac{\beta(\sigma)}{a(\sigma)}d\sigma & \text{if } \dfrac{\beta}{a} \in L^1(0,k) \\ \displaystyle -\int_s^k \frac{\beta(\sigma)}{a(\sigma)}d\sigma & \text{if } \dfrac{\beta}{a} \notin L^1(0,k) \end{cases} \tag{5}$$

and for $M > 0$ the truncation near the origin $s = 0$:

$$\mathcal{T}_M(s) := \max\{s, M\}.$$

REMARK 2.3. – We point out that γ is an increasing function. If $\beta/a \in L^1(0,k)$, $\gamma \in L^\infty(0,k)$ and, for $s \in [0,k]$, $\gamma \geq 0$, $\gamma(0) = 0$; if $\beta/a \notin L^1(0,k)$, $\gamma \underset{s \to 0}{\longrightarrow} -\infty$ and, for $s \in (0,k]$, $\gamma \leq 0$. Hence $e^{\gamma(s)}$ is bounded, so that function $\psi(s)$ is well defined. This is a fundamental remark since in the following we will deal with test functions of this form.

Let us give the definition of a *distributional solution* of problem (**P**):

DEFINITION. – *We say that $u \in L^\infty(Q_T)$ is a* distributional solution *for problem* (**P**) *if, $\forall \eta \in C_0^\infty(\Omega)$: $\eta^2 u \in L^1(Q_T) \cap C(0,T; L^2_{loc}(\Omega))$, $\eta\, a(x,t,u,\nabla u) \in L^2(Q_T)$ and $\eta^2 b(x,t,u,\nabla u) \in L^1(Q_T)$. Moreover $\psi(u) \in L^2(0,T; H_0^1(\Omega))$, $u(x,0) = u_0(x)$ a.e. in Ω and $\forall M > 0, \mathcal{T}_M(u) \in L^2(0,T; H^1(\Omega))$. Finally $\forall \phi \in C_0^\infty(Q_T)$ the following identity holds:*

$$-\iint_{Q_T} u\phi_t + \iint_{Q_T} a(x,t,u,\nabla u)\nabla\phi = \iint_{Q_T} b(x,t,u,\nabla u)\phi + \iint_{Q_T} f\phi.$$

In the case that $\eta \equiv 1$ we say that u is a distributional solution regular up to the boundary *of Ω.*

Under the above hypotheses we are able to state the following:

THEOREM.– *Under the assumptions* **A1)-A3)**, **B)**, **H)** *and* **D1)-D2)**, *there exists at least one bounded distributional solution to problem* (**P**). *If $\beta/a \in L^1(0,k)$ the solution is regular up to the boundary in the sense of the above definition.*

3. – Sketch of the proof.

Step 0: *The approximating problems.*

Let us define the approximating problems:

$$(\mathbf{P}_n) \quad \begin{cases} (u_n)_t - \operatorname{div}(a_n(x,t,u_n,\nabla u_n)) = b_n(x,t,u_n,\nabla u_n) + f(x,t) & \text{in } \mathrm{Q_T} \\ u_n(x,t) = 0 & \text{on } \Sigma_\mathrm{T} \\ u_n(x,0) = u_{0,n}(x) & \text{in } \Omega \end{cases}$$

where $u_{0,n} \in L^\infty(\Omega) \cap H_0^1(\Omega)$ is a suitable regularization of the datum obtained by a standard technique of convolution such that it satisfies the property

$$\lim_{n\to\infty} \frac{1}{n} \|u_{0,n}\|_{H^1_0(\Omega)} = 0$$

We point out that we can always assume that $u_{0,n}$ is bounded uniformly in $L^\infty(\Omega)$. The functions $a_n,\ b_n,\ \alpha_n,\ \beta_n$ are defined as follows:

$$a_n(x,t,s,\xi) := a(x,t,T_n(\mathcal{T}_{1/n}(s)),\xi), \qquad b_n(x,t,s,\xi) := T_n(b(x,t,T_n(\mathcal{T}_{1/n}(s)),\xi)),$$

$$\alpha_n(s) = \alpha(T_n(\mathcal{T}_{1/n}(s))), \qquad \beta_n(s) = \beta(T_n(\mathcal{T}_{1/n}(s))),$$

where

$$T_n(s) := \max\{-n, \min\{n,s\}\}$$

For every fixed $n \in \mathbb{N}$ there exists at least one weak solution $u_n \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^2(\Omega))$ to problem $(\mathbf{P}_n)$ (see [2], [6]) The approximating functions of the ones defined in (4) and (5) are:

$$A_n(s) := \int_0^s \alpha_n(\sigma)d\sigma, \qquad \psi_n(s) := \int_0^s \alpha_n(\sigma)e^{\gamma_n(\sigma)}d\sigma,$$

$$\gamma_n(s) := \begin{cases} \displaystyle\int_0^s \frac{\beta_n(\sigma)}{\alpha_n(\sigma)}d\sigma & \text{if} \quad \dfrac{\beta}{\alpha} \in L^1(0,k) \\ \displaystyle -\int_s^k \frac{\beta_n(\sigma)}{\alpha_n(\sigma)}d\sigma & \text{if} \quad \dfrac{\beta}{\alpha} \notin L^1(0,k) \end{cases}$$

In order to prove that the nonnegativity of the data implies $u_n \geq 0$ a.e. in Q_T it suffices to take $\phi_n = -(u_n)_-$ as test function in $(\mathbf{P_n})$ in the case that **B2)** holds and $\phi_n = -(u_n)_- e^{-\gamma_n(u_n)}$ in the case that **B1)** holds.

Step 1: *A priori estimates.*

1) *Uniform L^∞-estimates for $\{u_n\}_n$*: by rewriting $(\mathbf{P_n})$ in the new variable $v_n = e^{-t}u_n$ and by multiplying the new equation by the test function $\phi_n = e^{\Gamma_n(v_n)}(v_n - \tilde{k})_+$ where $\tilde{k} = \max\big(\sup_{n\in\mathbb{N}} \|u_{0,n}\|_{L^\infty}, \|f\|_{L^\infty}\big)$ and

$$\tilde{\alpha}_n(s) := \min_{0\le t\le T} \alpha_n(e^t s), \quad \tilde{\beta}_n(s) := \max_{0\le t\le T} \beta_n(e^t s), \quad \Gamma_n(v_n) := \int_0^{v_n} \frac{\tilde{\beta}_n(s)}{e^{-T}\tilde{\alpha}_n(s)}ds$$

we get that there exists $k > 0$ such that $\|u_n\|_{L^\infty(Q_T)} \leq k$.

2) *Uniform L^1-estimates (in the case* **B1**)) *or L^1_{loc}-estimates (in the case* **B2**)) *for* $\{b_n(x,t,u_n,\nabla u_n)\}_n$: in the case **B1**) we use the test function $\phi_n := (e^{\gamma_n(u_n)}-1) \in L^2(0,T;H^1_0(\Omega))$; in the case **B2**) we use $\phi_n := (e^{\gamma_n(u_n)}-1)\eta^2(x) \in L^2(0,T;H^1_0(\Omega))$, where $\eta \in C^\infty_0(\Omega)$.

3) *Uniform $L^2(0,T;H^1_0(\Omega))$ estimates for* $\{\psi_n(u_n)\}_n$: the estimate can be obtained by multiplying equation ($\mathbf{P_n}$) by the test function $\phi_n = e^{\gamma_n(u_n)}\psi_n(u_n)$. Hence:

$$\|\psi_n(u_n)\|_{L^2(0,T;H^1_0(\Omega))} \le C \tag{6}$$

4) *Uniform L^2-estimates (in the case* **B1**)) *or L^2_{loc}-estimates (in the case* **B2**)) *for* $\{a_n(x,t,u_n,\nabla u_n)\}_n$: in the case that **B1**) holds, (6) implies the estimate:

$$\|\nabla A_n(u_n)\|_{L^2(Q_T)} \le C, \quad \forall \text{ n} \in \mathbb{N} \tag{7}$$

since $|\nabla A_n(u_n)| \le |\nabla \psi_n(u_n)|$. Then the boundedness of $a_n(x,t,u_n,\nabla u_n)$ follows by **A2**). If **B2**) holds the estimate can be proved by considering the inequality $|a_n(x,t,u_n,\nabla u_n)|^2 \le \Lambda^2 a_n(u_n)\frac{a_n(u_n)}{\beta_n(u_n)}\beta_n(u_n)|\nabla u_n|^2$ and by hypothesis **H**) and **Step1** and **Step 2**.

Step 2: *Convergence results.*

• *Strong convergence of $\{u_n\}_n$ in $L^1(Q_T)$ (case* **B1**)) *or $L^1_{loc}(Q_T)$ (case* **B2**)): in order to get such convergence we use an Aubin-type theorem (see [3]).

• *Strong convergence of* $\{\nabla T_M(u_n)\}_n$ in $(L^2(Q_T))^N$ (in the case **B1**)) or in $L^2_{loc}(0,T;(L^2_{loc}(\Omega))^N)$ (in the case **B2**)), for any $M>0$: we regularize the derivative in time and use again suitable test functions of exponential type.

• *Passing to the limit.* To pass to the limit on the terms $a_n(x,t,u_n,\nabla u_n)$ and $b_n(x,t,u_n,\nabla u_n)$ we use, as main tool, the following properties, that are verified uniformly in n and for any $C = K \times [0,T]$ with $K \subset\subset E' \subset\subset \Omega$:

$$\lim_{M\to 0} \iint_{C\cap\{u_n\le M\}} |a_n(x,t,u_n,\nabla u_n)|^2 = 0, \ \lim_{M\to 0} \iint_{C\cap\{u_n\le M\}} b_n(x,t,u_n,\nabla u_n) = 0. \tag{8}$$

In order to prove the results in (8) we use for the first one on the left the test function $-[u_n - M]_-\rho^2(x)$ and for the second one on the right the tests $-[e^{\gamma_n(M)-\gamma_n(u_n)}-1]_+\rho^2(x)$ (case **B1**)) and $-[e^{\gamma_n(u_n)-\gamma_n(M)}-1]_-\rho^2(x)$ (case **B2**)), where the function ρ satisfies $\rho \in C^\infty_0(\Omega)$, $\rho \le 1$, $\rho \equiv 1$ on K and $supp(\rho) = E'$.
The requirements in the Definition are satisfied by the previous estimates. As far as the assumption on the initial datum is concerned, we prove that $u_n \to u$ strongly in $C([0,T];L^2_{loc}(\Omega))$.

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Daniela Giachetti: Dip. Metodi e Modelli Matematici per le Scienze Applicate,
Univ. degli Studi di Roma "La Sapienza", Via Scarpa 16, 00161 Roma, Italia
e-mail: daniela.giachetti@uniroma1.it

Giulia Maroscia: Dip. Metodi e Modelli Matematici per le Scienze Applicate,
Univ. degli Studi di Roma "La Sapienza", Via Scarpa 16, 00161 Roma, Italia
e-mail: gmaroscia@dmmm.uniroma1.it

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