Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 761–768.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_761_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 761-768

An Elementary Proof of the Exponential Conditioning of Real Vandermonde Matrices.

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Sunto. – *Si fornisce e si discute una dimostrazione elementare, proponibile in un corso di Matematica Numerica della Triennale, del condizionamento esponenziale di matrici di Vandermonde: si impega esclusivamente la definizione di condizionamento e l'espressione esplicita della norma infinito su* $[-1,1]$ *dei polinomi di Chebyshev di prima specie. La stessa idea dimostrativa funziona nel caso della ben nota matrice di Hilbert.*

Summary. – *We provide and discuss an elementary proof of the exponential conditioning of real Vandermonde matrices which can be easily given in undergraduate courses: we exclusively use the definition of conditioning and the sup-norm formula on* $[-1,1]$ *for Chebyshev polynomials of first kind. The same proof idea works virtually unchanged for the famous Hilbert matrix.*

1. – Introduction.

Given the grid $\mathcal{G}_n = \{x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}\}$ of pairwise distinct nodes and the set of linearly independent functions $\phi_0, \phi_1, \ldots, \phi_n$, we define the (generalized) Vandermonde matrix V_n of size $n+1$ as

$$(V_n)_{i,j} = \phi_j(x_i), \quad x_i \equiv x_i^{(n)}, \; i,j = 0, \ldots, n. \tag{1.1}$$

The classical Vandermonde matrix is obtained by putting $\phi_j(t) = t^j, j = 0, \ldots, n$, so that $(V_n)_{i,j} = x_i^j, \; i,j = 0, \ldots, n$, and is invertible whenever the nodes are pairwise distinct (see Subsection 1.1).

We wish to show that this choice of the basis functions (analogously to the case of the Hilbert matrix) is delicate if we restrict our attention to the case of real nodes i.e. $x_i \equiv x_i^{(n)} \in \boldsymbol{R}$, $i = 0, \ldots, n$. Indeed we prove that the conditioning of V_n i.e. $\mu(V_n) = \|V_n\| \|V_n^{-1}\|$ is exponential as n and this tells one that the use of such a basis makes the inherent error (related to e.g. the solution of a linear system $V_n x = b$) huge, already for moderate size of n, and therefore the Numerical Analysts must be very careful when using such a kind of matrices (see [1]). Here for a generic m-by-m matrix A, $m \in \boldsymbol{N}^+$, we consider the spectral norm $\|A\| = \sup\limits_{x \neq 0, x \in \boldsymbol{C}^m} \dfrac{\|Ax\|}{\|x\|}$, related to the Euclidean vector norm $\|x\| = \sqrt{\sum_{i=1}^m |x_i|^2}$.

In fact, in [1, Theorem 4.1] Beckermann proved that

$$\mu(V_n) \geq \sqrt{\frac{2}{n+1}}\left(\sqrt{2}+1\right)^{n-1},$$

and the latter lower-bound may be attained up to a factor $(n+1)^{3/2}$; in addition, very recently Li [3] slightly improved both these bounds. However, the proofs are quite tricky and involve somehow sophisticate tools (for further discussions see Remark 2.3 and Subsection 2.1). In Section 2 the conditioning of classical real Vandermonde matrices is discussed, by employing only tools that could be presented in detail in a classroom and whose prerequisites are reported below.

1.1 – *Prerequisites and notations.*

Let $\mathcal{G}_n = \{x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}\}$ be a grid of pairwise distinct nodes. The polynomial p of degree at most n which solves the interpolation problem $p(x_i) = g_i$, g_i given values, $x_i \equiv x_i^{(n)}$, $i = 0, \ldots, n$, exists and is unique: the existence comes from the explicit formula $p(x) = \sum_{i=0}^{n} g_i L_i(x)$, with $L_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$; the unicity follows from the fundamental theorem of algebra. Therefore the matrix V_n is invertible since the given interpolation problem is equivalent to the solution of the linear system $V_n a = g$, with $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ (i.e. p is represented in the canonical basis of polynomials, $1, x, \ldots, x^n$, in place of the Lagrange basis formed by $L_0(x), L_1(x), \ldots, L_n(x)$).

Concerning the Chebyshev polynomials, their definition can be given as $q_{k+1}(x) = 2xq_k(x) - q_{k-1}(x)$, $k \geq 1$, $q_0(x) = 1$, $q_1(x) = x$. By direct inspection it is clear that $q_n(x) = 2^{n-1}x^n + \tilde{q}_{n-1}(x)$ with $\tilde{q}_{n-1}$ of degree at most $n-1$ and $n \geq 1$. Moreover if $x \in [-1, 1]$ we consider the transformation $x = \cos(\theta)$, $\theta \in [0, \pi]$ and therefore a simple trigonometric identity shows that $q_n(x) = \cos(n\theta)$. As a consequence $\|q_n\|_{\infty,[-1,1]} = 1$ and $p_n^* = 2^{-(n-1)}q_n$ is such that p_n^* is monic and the famous identity

$$\|p_n^*\|_{\infty,[-1,1]} = 2^{-(n-1)}. \tag{1.2}$$

is true for $n \geq 1$.

2. – Main results.

Here we furnish an elementary proof of the exponential conditioning as n of any classical Vandermonde matrix V_n with real nodes.

THEOREM 2.1. – *Let V_n be a classical real Vandermonde matrix with pairwise distinct nodes $x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}$ and let $\mu(V_n)$ be the (spectral) condition* *number of V_n. Then there exists $c > 1$ (independent of n) such that, for every n large enough, it holds*

$$\mu(V_n) \geq c^n,$$

where we have assumed that there exists the limit l as n tends to infinity of $s_n = \max_{j=0,\ldots,n} |x_j|$, $x_j \equiv x_j^{(n)}$, $j = 0, \ldots, n$.

PROOF. – Setting $f_0 = V_n e_0$ with $(e_k)_i = 1$ if $k = i$ and zero otherwise, $k, i = 0, \ldots, n$, from the definition of the spectral condition number, we can easily see that $(f_0)_i = 1$ for every $i = 0, \ldots, n$, and

$$\|V_n\| = \sup_{x \neq 0, x \in \boldsymbol{C}^{n+1}} \frac{\|V_n x\|}{\|x\|} \geq \frac{\|V_n e_0\|}{\|e_0\|} = \|f_0\| = \sqrt{n+1}, \tag{2.1}$$

$$\|V_n^{-1}\| = \sup_{x \neq 0, x \in \boldsymbol{C}^{n+1}} \frac{\|V_n^{-1} x\|}{\|x\|} \geq \frac{\|V_n^{-1} f_0\|}{\|f_0\|} = \frac{1}{\|f_0\|} = \frac{1}{\sqrt{n+1}}.$$

Let $s_n = \max_{j=0,\ldots,n} |x_j|$, $l = \lim_{n\to\infty} s_n$ and $f_n = V_n e_n = \left(x_i^n\right)_{i=0}^n$. We have $\|V_n e_n\| = \|f_n\| \geq s_n^n$ and, if $l > 1$, for n large enough we infer

$$\|V_n\| = \sup_{x \neq 0, x \in \boldsymbol{C}^{n+1}} \frac{\|V_n x\|}{\|x\|} \geq \frac{\|V_n e_n\|}{\|e_n\|} = \|f_n\| \geq s_n^n \geq (l - \varepsilon)^n$$

with $l - 2\varepsilon > 1$. Hence, by invoking the second inequality in (2.1), we deduce

$$\mu(V_n) \geq \frac{(l-\varepsilon)^n}{\sqrt{n+1}} \geq (l - 2\varepsilon)^n$$

definitely that is the desired result with $c = l - 2\varepsilon > 1$ and independent of n. For $l < 1$ the reasoning is similar since

$$\|V_n^{-1}\| = \sup_{x \neq 0, x \in \boldsymbol{C}^{n+1}} \frac{\|V_n^{-1} x\|}{\|x\|} \geq \frac{\|V_n^{-1} f_n\|}{\|f_n\|} = \frac{1}{\|f_n\|} \geq \frac{s_n^{-n}}{\sqrt{n+1}} \geq \frac{(l+\varepsilon)^{-n}}{\sqrt{n+1}}$$

with $l + \varepsilon < 1$. Consequently, the first inequality in (2.1) implies

$$\mu(V_n) \geq (l + \varepsilon)^{-n}$$

for n large enough, and thesis follows with $c = (l + \varepsilon)^{-1}$.

If $l = 1$ then we use a different idea. First we consider the auxiliary Vandermonde matrix W_n with nodes $x_j' = x_j / s_n$. In such a way we have $x_j' \in [-1, 1]$ and

$$W_n = V_n D_n, \quad D_n = \operatorname{diag}_{j=0,\ldots,n}(s_n^{-j}), \quad s_n = \max_{j=0,\ldots,n} |x_j|.$$

From the sub-multiplicativity of the (matrix) norms (i.e. $\|AB\| \leq \|A\| \cdot \|B\|$ for any choice of m-by-m matrices A, B) we obtain $\mu(W_n) \leq \mu(V_n)\mu(D_n)$ with

$\mu(D_n) = s_n^n$ if $s_n \geq 1$ and $\mu(D_n) = s_n^{-n}$ otherwise. Therefore

$$\mu(V_n) \geq \mu(W_n)\mu^{-1}(D_n) \tag{2.2}$$

where, by virtue of the relation $l = 1$, for every $\varepsilon > 0$ there exists $\bar{n}$ for which

$$s_n^{\pm n} \geq (1-\varepsilon)^n, \quad \forall n \geq \bar{n}. \tag{2.3}$$

Since $\varepsilon > 0$ is arbitrary, it suffices to prove the exponential conditioning of W_n (which means that we are assuming all the nodes in the interval $[-1,1]$).

The inequalities in (2.1) still hold with W_n in place of V_n since they do not depend on l. In this new context the key bound is from below and concerns $\|W_n\| \geq \sqrt{n+1}$. We use the monic Chebyshev polynomials of first kind p_n^* over $[-1,1]$ having degree $n \geq 1$ (see [4] and Subsection 1.1). Hence, by calling x^* the vector of the coefficients of p_n^* in the canonical ascending polynomial basis, by monocity of p_n^*, we deduce $\|x^*\| \geq 1$ and then

$$\begin{aligned}
\|W_n^{-1}\| = \sup_{x\neq 0,\ x\in \boldsymbol{C}^{n+1}} \frac{\|W_n^{-1}x\|}{\|x\|} \\
\geq \frac{\|W_n^{-1}(W_n x^*)\|}{\|W_n x^*\|} \\
\geq \frac{\|x^*\|}{\|W_n x^*\|} \\
\geq_{p_n^* \text{ monicity}} \frac{1}{\|W_n x^*\|}.
\end{aligned}$$

Now, by direct inspection, the expression of the vector $W_n x^*$ is such that its i-th entry is given by $(W_n x^*)_i = \sum_{j=0}^n (x_i')^j (x^*)_j = p_n^*(x_i')$ and consequently, by (2.1) and by the latter inequality on $\|W_n^{-1}\|$, we deduce

$$\begin{aligned}
\mu(W_n) =& \|W_n\|\|W_n^{-1}\| \\
\geq& \sqrt{n+1}\frac{1}{\|W_n x^*\|} \\
=& \sqrt{n+1}\frac{1}{\sqrt{\sum_{i=0}^n |p_n^*(x_i')|^2}} \\
\geq& \sqrt{n+1}\frac{1}{\sqrt{(n+1)\|p_n^*\|^2_{\infty,[-1,1]}}} \\
\geq_{(1.2)}& 2^{n-1}
\end{aligned}$$

and finally the claimed result follows from (2.2) and (2.3). □

REMARK 2.2. – Theorem 2.1 is still valid under the assumption that $\max_{j=0,\ldots,n} |x_j^{(n)}|$ has no limit as n tends to infinity. Indeed, if the above sequence has k accumulation points, $k < \infty$, then $\max_{j=0,\ldots,n} |x_j^{(n)}|$ can be partitioned in exactly k subsequences having limit: in other words $\boldsymbol{N} = \bigcup_{j=1}^{k} \{n_q^{(j)} : q \in \boldsymbol{N}\}$ and on every Vandermonde subsequence $V_{n_q^{(j)}}$ we can directly apply Theorem 2.1 (since for every $j = 1, \ldots, k$, the subsequence $\max_{j=0,\ldots,n_q^{(j)}} |x_j^{(n_q^{(j)})}|$ has limit as q tends to infinity). Therefore there exist positive numbers $c_j > 1$, $j = 1, \ldots, k$, for which $\mu(V_{n_q^{(j)}}) \geq c_j^{n_q^{(j)}}$ with q large enough. Putting together the partial information it is true that

$$\mu(V_n) \geq c^n$$

with n sufficiently large and $c = \min_{j=1,\ldots,k} c_j > 1$. If the set of accumulation points is not finite then the preceding reduction to Theorem 2.1 cannot be done in the same manner since $\inf_{j \in \boldsymbol{J}} c_j$, $\#\boldsymbol{J} = \infty$, may equal 1 although every c_j is strictly larger than 1.

In such a situation we can reduce the reasoning again to Theorem 2.1, by exploiting a contradiction argument. Indeed, if the desired result does not hold then we can define a subsequence n_q for which

$$\mu(V_{n_q}) \leq c^{n_q} \tag{2.4}$$

for every $c > 1$ and for every $q \geq \bar{q}_c$. By Theorem 2.1 it has to be true that $\max_{j=0,\ldots,n_q} |x_j^{(n_q)}|$ has no limit as q tends to infinity. However we can extract a subsequence n_{q_t} from n_q for which $\max_{j=0,\ldots,n_{q_t}} |x_j^{(n_{q_t})}|$ has limit as t diverges infinity. Therefore, again by Theorem 2.1, we have $\mu(V_{n_{q_t}}) \geq d^{n_{q_t}}$, $d > 1$, definitely: it is clear that the last inequality contradicts the relation in (2.4).

2.1 – *A further comparison with the literature.*

We recall that on this topic there exist many contributions where the exponential conditioning is proven, and also with best constants (see Beckermann [1]). Other results can be found in [2, 5, 7, 8]. In all these works the findings are more general, but the approach is more sophisticate and is not elementary: here for not elementary we mean that advanced knowledge is required and therefore the reasoning is not easily understandable e.g. by undergraduate students. For instance Krylov vectors, Gram matrices, Hankel matrices, expansion coefficients of orthogonal polynomials, and nontrivial inequalities involving these objects are basic tools in [1, 2, 3, 5, 6, 7, 8]. On the other hand, also some of these more delicate proofs can be simplified, by only considering further polynomials associated to the Chebyshev ones. As an example, let s_n be the maximum of the moduli of the nodes like in Theorem 2.1. Since the spectral norm of V_n cannot be less than the Euclidean norm of any of its rows, it follows that $\|V_n\| \geq s_n^n$. In analogy with our proof, take $q_n = 2^{n-1}p_n^*$ as the normalized Chebyshev polynomial of degree n with unitary infinity norm on $[-1, 1]$ and leading coefficient equal to 2^{n-1}, see Subsection 1.1. Furthermore, arrange the coefficients of $\tilde{q}_n(x) = q_n(x/b)$ in the basis of the ascending monomials as a column vector y (see [1, Proof of Theorem 4.1]): as a consequence, $\|y\| \geq (2/s_n)^n/2$ (the leading coefficient of the new polynomial $\tilde{q}_n$) and $V_n y$ is the column vector containing the samplings of $\tilde{q}_n$ at the given nodes. Hence $\|V_n\| \geq \sqrt{n+1}$ so that

$$\mu(V_n) \geq s_n^n \frac{\|y\|}{\|V_n y\|} \geq \frac{2^{n-1}}{\sqrt{n+1}}. \tag{2.5}$$

We remark that similar bounds were obtained by Gautschi and Inglese [2] and by Tyrtyshnikov [7].

Now if we require that our students know also the explicit expression of the Chebyshev polynomials [4] and the associated recurrence coefficients (see Subsection 1.1), then we are allowed to make a slightly more careful analysis. Following [1, Proof of Theorem 4.1] we infer

$$\|V_n\| \geq \|(1, s_n, s_n^2, \ldots, s_n^n)\|$$

and thus, setting $\mathbf{i}^2 = -1$, the use of Cauchy-Schwartz leads to the following estimates

$$\begin{aligned}
\mu(V_n) \geq & \|(1, s_n, s_n^2, \ldots, s_n^n)\| \frac{\|y\|}{\|V_n y\|} \\
\geq & \frac{|(1, \boldsymbol{i}s_n, (\boldsymbol{i}s_n)^2, \ldots, (\boldsymbol{i}s_n)^n))y|}{\sqrt{n+1}} \\
= & \frac{|q_n(\boldsymbol{i})|}{\sqrt{n+1}} \geq \frac{(1+\sqrt{2})^n - (1+\sqrt{2})^{-n}}{2\sqrt{n+1}} \\
\geq & \frac{(1+\sqrt{2})^n - (1+\sqrt{2})^{n-1}}{2\sqrt{n+1}} = \frac{(1+\sqrt{2})^{n-1}}{2\sqrt{n+1}}.
\end{aligned}$$

REMARK 2.3. – A philosophical (or may be qualitative) motivation for this bad behavior of the polynomial canonical basis (irrespectively of the choice of $\mathcal{G}_n \subset \boldsymbol{R}$) can be traced in the "quasi" linear dependence of the functions $1, x, \ldots, x^n$ over the domain $\boldsymbol{R}$ with large n. In fact, while the notion of linear dependence is theoretically boolean (on a given set of vectors, this property holds or does not hold and only one of these facts is true), in Numerical Analysis we can talk of "quasi" linear dependence when a vector of the set can be approximated sufficiently well by a linear combination of the others. In this respect we should observe that the conditioning of a matrix gives a measure of the degree of linear dependence of the row or column vectors, which compose the given matrix. In our case, e.g. on $[0,1]$, when k is large the functions x^k and x^{k+1} are "quasi" linearly dependent because $\|x^k - x^{k+1}\|_{\infty,[0,1]} = (1-1/(k+1))^k/(k+1)$ is infinitesimal as k diverges but $\|x^j\|_{\infty,[0,1]} = 1$ for every $j \in \boldsymbol{N}$. In fact, with regard to a generic Haar system $\{\phi_0, \phi_1, \dots, \phi_n\}$ on $I \subset \boldsymbol{R}$ and in order to deduce an exponential conditioning of the associated Vandermonde sequence $\left\{\left(\phi_j(x_i)\right)_{i,j=0}^{n}\right\}_{n\in\boldsymbol{N}}$, the precise formal condition is that

$$\forall b > 0, \qquad \frac{\displaystyle\min_{a_0,a_1,\dots,a_{n-1}} \|\phi_n - \sum_{j=0}^{n-1} a_j\phi_j\|_{\infty,[-b,b]\cap I}}{\max_{j=0,\dots,n} \|\phi_j\|_{\infty,[-b,b]\cap I}}$$

is exponentially decreasing as a function of n. We notice that this condition is satisfied for $\phi_j(x) = x^j$ and for a nontrivial real interval I.

As already observed, the latter bad functional behavior can be read directly in the conditioning of the matrices V_n and also in the conditioning of the famous Hilbert matrix H_n [8], which comes from the same set of polynomial monomials $\phi_j(x) = x^j$ via integrals, and for which a version of Theorem 2.1 can be stated and proved in the same way (but working directly on H_n which is symmetric positive definite). For generalizations of the Hilbert matrix on unbounded intervals, the analysis can be much more involved and in that case it is probably much easier to follow Todd [6] and to prove first an explicit formula for the inverse of a Cauchy matrix; an other alternative is the use of the techniques by Taylor [5] or finally the application of the tools in [1].

A this point we should observe that the pathological behavior of the polynomial canonical basis $\phi_j(x) = x^j$, $j = 0, \dots, n$, is in actuality related to the domain only. Indeed if we shift to the complex field and we consider $\mathcal{G}_n$ formed by the $(n+1)$-th roots of the unity, then the corresponding Vandermonde matrix is the Fourier matrix F_n, which is perfectly conditioned since F_n is unitary and therefore $\mu(F_n) = 1$: indeed on the unit complex circle (where each node of $\mathcal{G}_n$ lies) the functions $1, z, \dots, z^n$ are substantially linearly independent and more precisely they are orthonormal with respect to the L^2 scalar product expressed in the variable θ with $z^j = \exp(\mathbf{i}j\theta)$, $\theta \in [0, 2\pi)$.

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Pervenuta in Redazione
il 17 luglio 2007