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Bounded Solutions for Some Dirichlet Problems with $L^1(\Omega)$ Data

TOMMASO LEONORI

Sunto. – *In questo lavoro viene dimostrata l'esistenza di una soluzione per un problema il cui modello è:*

$$\begin{cases} -\Delta u + \dfrac{u}{\sigma - |u|} = \gamma|\nabla u|^2 + f(x) & \textit{in } \Omega\,, \\ u = 0 & \textit{on } \partial\Omega\,, \end{cases}$$

con $f(x)$ *in* $L^1(\Omega)$ *e* $\sigma, \gamma > 0$.

Summary. – *In this paper we prove the existence of a solution for a problem whose model is:*

$$\begin{cases} -\Delta u + \dfrac{u}{\sigma - |u|} = \gamma|\nabla u|^2 + f(x) & \textit{in } \Omega\,, \\ u = 0 & \textit{on } \partial\Omega\,, \end{cases}$$

with $f(x)$ *in* $L^1(\Omega)$ *and* $\sigma, \gamma > 0$.

1. – Introduction.

In this paper we want to prove the existence of an $L^\infty(\Omega)$ solution for a nonlinear elliptic equation whose model is:

$$\begin{cases} -\Delta u + \dfrac{u}{\sigma - |u|} = \gamma|\nabla u|^2 + f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where γ, $\sigma > 0$ and $f(x) \in L^1(\Omega)$.

For Dirichlet problem with lower order terms having quadratic growth with respect to the gradient it was proved (see [6] and references cited therein) that a bounded solution u of

$$\begin{cases} -\Delta u + a_0 u = \gamma|\nabla u|^2 + f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

exists for any γ, $a_0 > 0$, $f(x) \in L^q(\Omega)$, $q > \dfrac{N}{2}$ and it satisfies the following estimate:

$$\|u\|_{L^\infty(\Omega)} \leq C(a_0, \|f\|_{L^q(\Omega)}, \gamma, N, \Omega)\,.$$

On the other hand for semilinear equations like

$$\begin{cases} -\Delta u + g(u) = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $g(s)$ is a real function such that $g(s) \cdot s \geq 0$ and $f \in L^m(\Omega)$, $1 \leq m \leq +\infty$, it is well known (see [2],[7] for the semilinear problem and $L^1(\Omega)$ data and [1], [4] for the nonlinear one) that there exist a solution u satisfying:

$$\|g(u)\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}\,.$$

In particular if the nonlinearity is a power, i.e. $g(s) = s|s|^{r-1}$, $r > 1$, it implies that $u \in L^{mr}(\Omega)$; in other words the nonlinear term improves the summability of the solution.

In [3] it was studied the semilinear problem with a nonlinear term $g(s)$ that blows up at some point $\sigma > 0$: in this case for any $L^1(\Omega)$ data there exist a solution u such that $|u| < \sigma$ a.e. (see also [8] for general measure data).

We want to study this kind of problems in presence of both the "blow-up term" and the quadratic nonlinearity with respect to the gradient: we prove that a solution u of (1.1) exists, belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and satisfies $|u| < \sigma$ a.e., too.

2. – Assumptions and result.

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $N \geq 3$, and let us consider the following problem:

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) + H(x,u,\nabla u) = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.2}$$

where $a(x,s,\xi)$ and $H(x,s,\xi)$ are Carathéodory functions such that, for almost every $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi,\eta \in \mathbb{R}^N$, $\xi \neq \eta$:

$$\exists\, a > 0 \quad : \quad a(x,s,\xi)\cdot\xi \geq a|\xi|^2\,; \tag{2.3}$$

$$\exists\, \beta > 0 \quad : \quad |a(x,s,\xi)| \leq \beta|\xi|\,; \tag{2.4}$$

$$(a(x,s,\xi) - a(x,s,\eta))\cdot(\xi - \eta) > 0\,; \tag{2.5}$$

and

$$\exists\, h > 0 \quad \text{such that} \quad |H(x,s,\xi)| \leq h|\xi|^2\,, \tag{2.6}$$

Moreover let us consider $g(x,s):\Omega\times(-\sigma;\sigma)\to\mathbb{R}$, $\sigma>0$, Carathéodory function and $m(s)$, $l(s):(-\sigma;\sigma)\to\mathbb{R}$ continuous, increasing functions such that:

$$g(x,s)\cdot s\geq 0\,,\quad \forall s\in(-\sigma;\sigma)\,; \tag{2.7}$$

$$|l(s)|\leq|g(x,s)|\leq\gamma(x)|m(s)|\,,\qquad \gamma(x)\in L^1(\Omega)\,,\quad \gamma(x)\geq 0\,; \tag{2.8}$$

$$m(0)=l(0)=0\,; \tag{2.9}$$

and

$$\exists\sigma>0\ :\ \lim_{s\to\pm\sigma^{\mp}} l(s)=\pm\infty\,. \tag{2.10}$$

Now we can state and prove our result:

THEOREM 2.1. – *Suppose that $a(x,s,\xi)$, $H(x,s,\xi)$ and $g(x,s)$ satisfy* (2.3)-(2.10). *Then for any $f(x)\in L^1(\Omega)$ the problem* (2.2) *admits a weak solution $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$, i.e. the following equality holds true:*

$$\int_\Omega a(x,u,\nabla u)\nabla\psi+\int_\Omega g(x,u)\psi+\int_\Omega H(x,u,\nabla u)\psi=\int_\Omega f(x)\psi\,, \tag{2.11}$$

for any $\psi\in H_0^1(\Omega)\cap L^\infty(\Omega)$.

Before proving the theorem let us recall a property of the real function $\varphi_\lambda(s)=se^{\lambda s^2}$ we will often use in what follows: $\forall a>0\,,\ b>0\,,\ \forall\lambda>\dfrac{b^2}{8a^2}$ we have:

$$a\varphi_\lambda'(s)-b\varphi_\lambda(s)\geq 1\,,\ \forall s\in\mathbb{R}\,. \tag{2.12}$$

PROOF OF THEOREM 2.1. The proof is divided into two steps:

STEP 1: $f(x)\in L^\infty(\Omega)$.

Let us denote by

$$g_n(x,s)=T_n(g(x,s))\,,\quad H_n(x,s,\xi)=\frac{H(x,s,\xi)}{1+\dfrac{1}{n}H(x,s,\xi)}\,,$$

and by $u_n\in H_0^1(\Omega)\cap L^\infty(\Omega)$ a solution (that exists by [9]) of the problem:

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n))+g_n(x,u_n)+H_n(x,u_n,\nabla u_n)=f(x) & \text{in } \Omega\,,\\ u_n=0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.13}$$

Now let us choose $\varphi_\lambda\left[(u_n-l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\right]$ as test function in the weak formulation of (2.13) and fix any $\lambda > \frac{h^2}{8a^2} = \overline{\lambda}$, so that:

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\varphi'_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] + \int_\Omega g_n(x,u_n)\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] + \int_\Omega H_n(x,u_n,\nabla u_n)\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] = \int_\Omega f(x)\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] .$$

Hence (we often omit the dependence of the function φ_λ and φ'_λ on $(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+$ for brevity) we find:

$$\int_{\{u_n \geq l^{-1}(\|f\|_{L^\infty(\Omega)})\}} |\nabla u_n|^2 (a\varphi'_\lambda - h\varphi_\lambda) + \int_\Omega [T_n(l(u_n)) - f(x)]\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] \leq 0\,,$$

and thanks to the choice of λ and (2.12), the first term is positive and we drop it. Thus, defining $l_n(s) = T_n(l(s))$,

$$\int_{\{l_n(u_n)\geq \|f\|_{L^\infty(\Omega)}\}} [l_n(u_n) - f(x)]\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] \leq 0\,.$$

Note that since $\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] \geq 0$ a.e., it must be $l_n(u_n) - f \leq 0$; let us split the above integral in

$$\int_{\{n\geq l(u_n)\geq \|f\|_{L^\infty(\Omega)}\}} \big[l(u_n) - \|f\|_{L^\infty(\Omega)}\big]\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] + \int_{\{l(u_n)\geq n\geq \|f\|_{L^\infty(\Omega)}\}} \big[n - \|f\|_{L^\infty(\Omega)}\big]\varphi_\lambda\big[(u_n - l^{-1}(\|f\|_{L^\infty(\Omega)}))^+\big] \leq 0\,.$$

Observe that the arguments of the above integrals are positive, so we can deduce that $u_n < l^{-1}(\|f\|_{L^\infty(\Omega)})$, that implies by (2.10) that $\exists\, \varepsilon_1 > 0$ such that:

$$u_n(x) \leq \sigma - \varepsilon_1 \quad \text{a.e.}$$

Repeating the same argument with $\varphi_\lambda\big[(u_n + l^{-1}(\|f\|_{L^\infty(\Omega)}))^-\big]$ as test function we are able to prove that u_n is bounded from below, i.e. $\exists\, \varepsilon_2 > 0$ such that:

$$u_n(x) \geq -\sigma + \varepsilon_2 \quad \text{a.e.}$$

Putting together the above inequalities we obtain that:

$$|u_n(x)| \le \sigma - \varepsilon \text{ a.e.} \quad \varepsilon = \max\{\varepsilon_1 , \varepsilon_2\} . \tag{2.14}$$

Our next goal is to prove that $g_n(x, u_n)$ is strongly compact in $L^1(\Omega)$. Let us choose $\varphi_\lambda(u_n)$ as test function in the weak formulation of (2.13), for any $\lambda > \overline{\lambda}$,

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n \, \varphi'_\lambda(u_n) + \int_\Omega g_n(x, u_n)\varphi_\lambda(u_n) + \int_\Omega H_n(x, u_n, \nabla u_n)\varphi_\lambda(u_n) = \int_\Omega f\varphi_\lambda(u_n) ,$$

and using again (2.12) we have:

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega g_n(x, u_n)\varphi_\lambda(u_n) \le \varphi_\lambda(\sigma)\|f\|_{L^1(\Omega)} .$$

Hence (by (2.7)) $\{u_n\}$ is bounded in $H^1_0(\Omega)$ and so there exists $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that, up to a subsequence (still denoted by u_n), $u_n \to u$ weakly in $H^1_0(\Omega)$, a.e. and (by (2.14)) $*$-weakly in $L^\infty(\Omega)$. In particular $g_n(x, u_n)$ converges toward $g(x, u)$ almost everywhere and so, in order to prove that $\{g_n(x, u_n)\}_{n\in\mathbb{N}}$ is strongly compact in $L^1(\Omega)$, we want to prove that such sequence is equiintegrable. So, if E is a measurable set,

$$\int_E |g_n(x, u_n)| \le \int_E |g(x, u_n)| \le \int_E |m(u_n)|\gamma(x) \le m(\sigma - \varepsilon)\int_E \gamma(x) ,$$

and since $\gamma(x) \in L^1(\Omega)$, this implies that $\{g_n(x, u_n)\}$ is equiintegrable and thus by Vitali's Theorem

$$g_n(x, u_n) \to g(x, u) \quad \text{strongly in} \quad L^1(\Omega) . \tag{2.15}$$

We want now to prove that u_n strongly converges in $H^1_0(\Omega)$ to u.

So let us fix any $\lambda > \overline{\lambda}$ and choose $\psi = \varphi_\lambda(u_n - u)$ in the weak formulation of (2.13):

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla(u_n - u)\varphi'_\lambda(u_n - u) + \int_\Omega g_n(x, u_n)\varphi_\lambda(u_n - u) \\ &+ \int_\Omega H_n(x, u_n, \nabla u_n)\varphi_\lambda(u_n - u) = \int_\Omega f\varphi_\lambda(u_n - u) . \end{aligned} \tag{2.16}$$

From now on we will indicate by ε_n every quantity such that $\lim_{n\to+\infty} \varepsilon_n = 0$. If we add to both sides of (2.16)

$$B_n = \int_\Omega a(x, u_n, \nabla u) \cdot \nabla(u_n - u)\varphi'_\lambda(u_n - u)$$

and observe that $B_n = \varepsilon_n$, since $\nabla u \in L^2(\Omega)$, $\nabla(u_n - u)$ weakly converges to 0 in $L^2(\Omega)$ and $\varphi'_\lambda(u_n - u) \le \varphi'_\lambda(2\sigma)$; hence

$$\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot \nabla(u_n - u)\varphi'_\lambda(u_n - u) - h\int_\Omega |\nabla u_n|^2 \varphi_\lambda(u_n - u)$$
$$+\int_\Omega g_n(x,u_n)\varphi_\lambda(u_n - u) \le \int_\Omega f\varphi_\lambda(u_n - u) + \varepsilon_n \ .$$

From (2.3)

$$\int_\Omega |\nabla u_n|^2 \varphi'_\lambda(u_n - u) \le \frac{1}{a}\int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla u_n\, \varphi'_\lambda(u_n - u)\,,$$

so, since

$$\int_\Omega a(x,u_n,\nabla u)\cdot \nabla u_n\, \varphi_\lambda(u_n - u) = \varepsilon_n \quad \text{and} \quad \int_\Omega a(x,u_n,\nabla u)\cdot \nabla u\, \varphi_\lambda(u_n - u) = \varepsilon_n\,,$$

we obtain:

$$\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot \nabla(u_n - u)\Big(\varphi'_\lambda(u_n - u) - \frac{h}{a}\varphi_\lambda(u_n - u)\Big)$$
$$+\int_\Omega g_n(x,u_n)\varphi_\lambda(u_n - u) + \varepsilon_n \le \int_\Omega f\varphi_\lambda(u_n - u) \ .$$

Now using the property (2.12) of the function $\varphi_\lambda(s)$, (2.15) and the $*$-weak $L^\infty(\Omega)$ convergence of $\varphi_\lambda(u_n - u)$ towards 0, we have

$$\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot \nabla(u_n - u) \le \varepsilon_n\,.$$

We can now use Lemma 5 in [5] and obtain that

$$u_n \to u \quad \text{strongly in } H_0^1(\Omega)\,. \tag{2.17}$$

In particular $\nabla u_n \to \nabla u$ a.e. and $|\nabla u_n|^2$ strongly converges in $L^1(\Omega)$ towards $|\nabla u|^2$. This implies that $H_n(x,u_n,\nabla u_n)$ converges to $H(x,u,\nabla u)$ a.e. and it is dominated by a strong $L^1(\Omega)$ compact sequence (namely $\{h|\nabla u_n|^2\}$). Hence Vitali's Theorem allow us to conclude that $H_n(x,u_n,\nabla u_n)$ strongly converges in $L^1(\Omega)$ to $H(x,u,\nabla u)$.

Moreover (2.17) allows us to pass to the limit in the weak formulation of (2.13) and deduce that $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a solution of the problem. Observe that from (2.14) there exists $\varepsilon > 0$ such that $|u(x)| < \sigma - \varepsilon$.

STEP 2: $f \in L^1(\Omega)$.

Let $f_n(x)$ be a sequence of $L^\infty(\Omega)$ function that converge strongly in $L^1(\Omega)$ towards $f(x)$, $\|f_n\|_{L^1(\Omega)} \nearrow \|f\|_{L^1(\Omega)}$ (for example $f_n(x) = T_n(f(x))$). Then a weak solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem

$$\begin{cases} -\operatorname{div} a(x, u_n, \nabla u_n) + g(x, u_n) + H(x, u_n, \nabla u_n) = f_n(x) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.18}$$

exists from the previous step. In particular from the estimate (2.14) we know that $|u_n| < \sigma - \varepsilon_n$, where ε_n depends on the $L^\infty(\Omega)$ norm of $f_n(x)$. So if we try to pass to the limit we can only conclude that (let us call u as the a.e. limit of u_n) that $|u| \le \sigma$ a.e. Our first goal is to prove that $|u| < \sigma$, a.e., that is $\operatorname{meas}\{x \in \Omega : u(x) = \pm\sigma\} = 0$.

First of all let us fix $\lambda > \overline{\lambda}$ and choose $\varphi_\lambda(u_n)$ as test function in (2.18):

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n \, \varphi_\lambda'(u_n) + \int_\Omega g(x, u_n)\varphi_\lambda(u_n) + \int_\Omega H(x, u_n, \nabla u_n)\varphi_\lambda(u_n) = \int_\Omega f\varphi_\lambda(u_n)\,.$$

Now we can use (2.12) to obtain

$$\int_\Omega \big(a\varphi_\lambda'(u_n) - h\varphi_\lambda(u_n)\big)|\nabla u_n|^2 + \int_\Omega g(x, u_n)\varphi_\lambda(u_n)$$
$$\le \varphi_\lambda(\sigma)\|f_n\|_{L^1(\Omega)} \le \varphi_\lambda(\sigma)\|f\|_{L^1(\Omega)}\,,$$

which implies that

$$\|u_n\|_{H_0^1(\Omega)} \le C \quad \text{and} \quad \int_\Omega g(x, u_n)\varphi_\lambda(u_n) \le \varphi_\lambda(\sigma)\|f\|_{L^1(\Omega)}\,. \tag{2.19}$$

So there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that, up to subsequence (not relabeled), $u_n \to u$ weakly in $H_0^1(\Omega)$, a.e. and $*$-weakly in $L^\infty(\Omega)$. In order to prove that

$$\operatorname{meas}\{x \in \Omega : |u(x)| = \sigma\} = \lim_{n\to+\infty} \lim_{s\to\sigma} \operatorname{meas}\{x \in \Omega : s < |u_n(x)| \le \sigma\} = 0\,,$$

let us consider the second of (2.19). For any $s \in]0, \sigma[$ we split the above integral into

$$\begin{aligned} \int_\Omega g(x, u_n)\varphi_\lambda(u_n) &= \int_{\{0 \le |u_n| \le s\}} g(x, u_n)\varphi_\lambda(u_n) \\ &+ \int_{\{s \le |u_n| < \sigma\}} g(x, u_n)\varphi_\lambda(u_n) \le \|f\|_{L^1(\Omega)}\varphi_\lambda(\sigma)\,. \end{aligned} \tag{2.20}$$

Using (2.28) and (2.29), we have

$$\tilde{l}(s)\varphi_\lambda(s)\,\mathrm{meas}\{s \le |u_n| < \sigma\} \le \int\limits_{\{s\le|u_n|<\sigma\}} g(x,u_n)\varphi_\lambda(u_n)$$
$$\le \int\limits_{\Omega} |f_n||\varphi_\lambda(u_n)| \le \varphi_\lambda(\sigma)\|f\|_{L^1(\Omega)}\,,$$

where $\tilde{l}(s) = \min\{l(s), -l(-\,s)\}$, that is:

$$\mathrm{meas}\{s \le |u_n(x)| < \sigma\} \le \frac{\varphi_\lambda(\sigma)}{\varphi_\lambda(s)}\frac{\|f\|}{\tilde{l}(s)}\,. \tag{2.21}$$

So we can pass to the limit in s and since $\lim\limits_{s\to\sigma}\tilde{l}(s) = +\infty$, we find

$$\mathrm{meas}\{x \in \Omega \,:\, |u(x)| = \sigma\} = 0\,; \tag{2.22}$$

we can then conclude that $|u(x)| < \sigma$ a.e.

To pass to the limit in the weak formulation of (2.18) we still need the strong $L^1(\Omega)$ convergence of $g(x,u_n)$ towards $g(x,u)$ and the strong convergence of u_n towards u in $H_0^1(\Omega)$.

Let us prove the first one: $\forall s \in \,]0\,,\,\sigma[$ we can use $\psi = \varphi_\lambda(G_{\sigma-s}(u_n))$ as test function in the weak formulation of (2.18), where $G_h(t) = t - T_h(t)$ and $\lambda > \overline{\lambda}$. So we have:

$$a\int\limits_{\{s\le|u_n|<\sigma\}} |\nabla u_n|^2\varphi_\lambda'(G_{\sigma-s}(u_n)) + \int\limits_{\Omega} g(x,u_n)\varphi_\lambda(G_{\sigma-s}(u_n))$$
$$\le \varphi_\lambda(\sigma)\int\limits_{\{s\le|u_n|<\sigma\}} |f_n(x)| + h\int\limits_{\{s\le|u_n|<\sigma\}} |\nabla u_n|^2\varphi_\lambda(G_{\sigma-s}(u_n))\,.$$

Hence using the sign condition on $g(x,s)$ and (2.12) we get:

$$\int\limits_{\{s\le|u_n|<\sigma\}} |\nabla u_n|^2 \le \varphi_\lambda(\sigma)\int\limits_{\{s\le|u_n|<\sigma\}} |f_n(x)|\,. \tag{2.23}$$

Let us now take as test function $\psi = \dfrac{1}{\varepsilon}T_\varepsilon(G_s(u_n))$ in the weak formulation of (2.18), $s \in \,]0\,,\,\sigma[$; so

$$\frac{a}{\varepsilon}\int\limits_{\{s\le|u_n|<s+\varepsilon\}} |\nabla u_n|^2 + \int\limits_{\{s+\varepsilon\le|u_n|<\sigma\}} g(x,u_n) \le \int\limits_{\{s\le|u_n|<\sigma\}} |f_n(x)| + h\int\limits_{\{s\le|u_n|<\sigma\}} |\nabla u_n|^2\,.$$

We can drop the first term (since it is positive), take the limit as $\varepsilon \to 0$ and substitute (2.23) in last inequality, so that:

$$\int_{\{s\le|u_n|<\sigma\}} g(x,u_n) \le (h\varphi_\lambda(\sigma)+1)\int_{\{s\le|u_n|<\sigma\}} |f_n(x)|\,. \tag{2.24}$$

Now we can prove that $g(x,u_n)$ is equiintegrable. Take any $\varepsilon>0$ and $E\subset\Omega$ and consider

$$\int_E |g(x,u_n)| = \int_{E\cap\{0\le|u_n|\le s\}} |g(x,u_n)| + \int_{E\cap\{s\le|u_n|<\sigma\}} |g(x,u_n)|\,,$$

so we can use (2.24) on the last term and choose s close enough to σ such that, since $f_n(x)$ is strongly compact in $L^1(\Omega)$ and by (2.21),

$$\int_{E\cap\{s\le|u_n|<\sigma\}} |g(x,u_n)| \le \int_{\{s\le|u_n|<\sigma\}} g(x,u_n)\varphi_\lambda \le (\varphi_\lambda(\sigma)+1)\int_{\{s\le|u_n|<\sigma\}} |f_n(x)| \le \frac{\varepsilon}{2}\,.$$

Moreover

$$\int_{E\cap\{0\le|u_n|\le s\}} |g(x,u_n)| \le \tilde m(s)\int_E |\gamma(x)|\,,\qquad \tilde m(s)=\max\{m(s),-m(-s)\}\,;$$

so, thanks to the fact that $\gamma(x)\in L^1(\Omega)$, there exists $\delta_\varepsilon>0$ such that

$$\forall E: |E|\le\delta_\varepsilon \quad \tilde m(s)\int_E |\gamma(x)| \le \frac{\varepsilon}{2}\,.$$

Since $\{g(x,u_n)\}$ is equiintegrable and it converges a.e. towards $g(x,u)$, we can apply Vitali's Theorem to conclude that $g(x,u_n)\to g(x,u)$ strongly in $L^1(\Omega)$.

Let us prove that $\{u_n\}$ is strongly compact in $H_0^1(\Omega)$; take $\psi=\varphi_\lambda(u_n-u)$ as test function in the weak formulation of (2.18) for any $\lambda>\overline{\lambda}$, so:

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n-u)\varphi_\lambda'(u_n-u) + \int_\Omega g(x,u_n)\varphi_\lambda(u_n-u)$$
$$-h\int_\Omega |\nabla u_n|^2\varphi_\lambda(u_n-u) \le \int_\Omega f_n\varphi_\lambda(u_n-u)\,.$$

As before let us add on both sides of the above inequality

$$\int_\Omega a(x,u_n,\nabla u)\cdot\nabla(u_n-u)\varphi_\lambda'(u_n-u) = \varepsilon_n\,,$$

thus, since $\varphi'_\lambda(u_n - u)$ and $\varphi_\lambda(u_n - u)$ are uniformly bounded, $\nabla u \in L^2(\Omega)$ and $\nabla(u_n - u)$ weakly converges to 0 in $L^2(\Omega)$, we have:

$$\int_\Omega (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot \nabla(u_n - u)\varphi'_\lambda(u_n - u) - h\int_\Omega |\nabla u_n|^2 \varphi_\lambda(u_n - u)$$
$$+\int_\Omega g(x, u_n)\varphi_\lambda(u_n - u) \leq \int_\Omega f\varphi_\lambda(u_n - u) + \varepsilon_n\,.$$

Moreover observe that

$$\frac{h}{a}\int_\Omega a(x, u_n, \nabla u) \cdot \nabla u_n\, \varphi_\lambda(u_n - u) = \varepsilon_n \;\text{ and }\; \frac{h}{a}\int_\Omega a(x, u_n, \nabla u) \cdot \nabla u\, \varphi_\lambda(u_n - u) = \varepsilon_n\,,$$

so:

$$\int_\Omega a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u) \cdot \nabla(u_n - u)(\varphi'_\lambda(u_n - u) - \varphi_\lambda(u_n - u))$$
$$+\frac{h}{a}\int_\Omega g(x, u_n)\varphi_\lambda(u_n - u) \leq \int_\Omega f\varphi_\lambda(u_n - u) + \varepsilon_n\;.$$

Since $g(x, u_n)$ and $f_n(x)$ strongly converge in $L^1(\Omega)$ to $g(x, u)$ and $f(x)$ respectively, and $\varphi_\lambda(u_n - u)$ to 0 $*$-weakly in $L^\infty(\Omega)$ (recall that both u_n and u are bounded by σ), we can apply again Lemma 5 in [5] to obtain the strong $H_0^1(\Omega)$ convergence of u_n towards u.

Let us follow the same idea of the previous step in order to pass to the limit in the weak formulation of (2.18): in fact the strong convergence of u_n in $H_0^1(\Omega)$ towards u implies the a.e. convergence of the gradients and the convergence of $\|u_n\|^2_{H_0^1(\Omega)} \to \|u\|^2_{H_0^1(\Omega)}$. So we can apply, once again, Vitali's Theorem to prove that the lower order term that has natural growth with respect to the gradient strongly converges in $L^1(\Omega)$ to $H(x, u, \nabla u)$. Hence we can pass to the limit in the weak formulation of (2.2) and so $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (2.2) such that $|u(x)| < \sigma$ a.e. □

REMARK 1. – Actually hypothesis (2.6) can be weakened and in particular we can assume that $\exists \gamma > 0$ and $\psi \in L^1(\Omega)$ such that

$$H(x, s, \xi) \leq \gamma|\xi|^2 + \psi(x)\,.$$

Such condition is more general than (2.6) and allow us to choose nonlinearity of the type

$$H(x, s, \xi) = \rho(x)|\xi|^q$$

where $1 < q < 2$ and $\rho(x) \in L^{\frac{2}{2-q}}(\Omega)$.

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