Bollettino Unione Matematica Italiana

Patrizia Di Gironimo

Quasiharmonic Fields: a Higher Integrability Result

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 843–851.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_843_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 843-851

Quasiharmonic Fields: a Higher Integrability Result

PATRIZIA DI GIRONIMO

Sunto. – *In questo lavoro si studia il grado di integrabilitá dei campi quasiarmonici. Questi campi sono connessi con lo studio dell'equazione div*$(A(x)\nabla u(x)) = 0$, *dove la matrice simmetrica* $A(x)$ *soddisfa la condizione*

$$|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)\langle A(x)\xi, \xi\rangle.$$

La funzione non negativa $\mathcal{K}(x)$ *appartiene alla classe esponenziale, cioé esiste* $\beta > 0$ *tale che* $exp(\beta\mathcal{K}(x))$ *è integrabile. Si dimostra che il gradiente di una soluzione locale dell'equazione appartiene agli spazi di Zygmund* $L^2_{loc}log^{a-1}L$, $0 < a = a(\beta)$. *Inoltre si prova come il grado di migliore regolaritá dipende da* β.

Summary. – *In this paper we study the degree of integrability of quasiharmonic fields. These fields are connected with the study of the equation div*$(A(x)\nabla u(x)) = 0$, *where the symmetric matrix* $A(x)$ *satisfies the condition*

$$|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)\langle A(x)\xi, \xi\rangle.$$

The nonnegative function $\mathcal{K}(x)$ *belongs to the exponential class, i.e.* $exp(\beta\mathcal{K}(x))$ *is integrable for some* $\beta > 0$. *We prove that the gradient of a local solution of the equation belongs to the Zygmund spaces* $L^2_{loc}log^{a-1}L$, $0 < a = a(\beta)$. *Moreover we show exactly how the degree of improved regularity depends on* β.

1. – Introduction.

Let $\Omega \subset R^n$ be a connected open set.

If $B : \Omega \to R^n$, $E : \Omega \to R^n$ are integrable vector fields on Ω such that

$$\begin{aligned} div\, B &= \sum_{i=1}^{n} \frac{\partial B_i}{\partial x_i} = 0 \\ curl\, E &= \left(\frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i}\right)_{i,j=1,\dots,n} = 0, \end{aligned} \tag{1.1}$$

in the sense of distributions, the scalar product $\langle B, E\rangle$ is referred to as a div-curl product.

In this paper we shall study the degree of integrability of a class of div-curl fields (B,E) which are coupled by the distortion inequality

$$|B|^2 + |E|^2 \leq \mathcal{K}(x)\langle B,E\rangle \qquad a.e.\ in\ \Omega \tag{1.2}$$

where $1 \leq \mathcal{K}(x) < \infty$ is a measurable function in Ω.

A div-curl field (B,E) satisfying (1.2) is called a quasiharmonic field.

An example of quasiharmonic fields grew out of the study of the equation

$$div(A(x)\nabla u(x)) = 0 \tag{1.3}$$

where the symmetric matrix $A(x) \in R^{n\times n}$ satisfies the condition

$$\frac{1}{K(x)}|\xi|^2 \leq \quad \langle A(x)\xi,\xi\rangle \quad \leq K(x)|\xi|^2 \tag{1.4}$$

and $K(x)$, $K(x) \geq 1$ a.e., is a measurable function on Ω.

It is well-know (see [IS$_2$]) that it is possible to express (1.4), equivalently, by using just one inequality

$$|\xi|^2 + |A(x)\xi|^2 \leq \mathcal{K}(x)\langle A(x)\xi,\xi\rangle \tag{1.5}$$

for almost every $x \in \Omega$ and all $\xi \in R^n$, with $\mathcal{K}(x) = K(x) + \dfrac{1}{K(x)}$.

There are two vector fields associated with a solution of the equation (1.3). The first one, denoted by $E = \nabla u(x)$, is curl free, while the second $B = A(x)\nabla u(x)$ is divergence free. The condition (1.5) shows that the pair $\mathcal{F} = [B,E]$ is a quasiharmonic field.

Throughout this paper we shall assume that

$$\langle B,E\rangle \in L^1_{loc}(\Omega). \tag{1.6}$$

The function $\mathcal{K}(x)$ in (1.5) belongs to the exponential class $Exp(\Omega)$, defined via the Orliz function $P(t) = e^t - 1$. Precisely, we assume that

$$\int_\Omega e^{\beta\mathcal{K}(x)}dx < +\infty \tag{1.7}$$

for some $\beta > 0$.

By assumptions (1.2), (1.6) and (1.7) we deduce that B and E belong to the Orlicz-Zygmund spaces $L^2_{loc}log^{a-1}L(\Omega,R^n)$ (see [RR]).

It is our goal here to investigate the degree of integrability of a quasiharmonic field, under the assumptions (1.6) and (1.7).

Our theorem not only shows a higher integrability, but it also indicates exactly how the degree of improved regularity depends on β.

THEOREM 1.1. – *Let* (B,E) *a div-curl field verifying* (1.2) *and* (1.6). *Assume that the distortion* $\mathcal{K}(x) \geq 1$ *satisfies* (1.7) *for some* $\beta > 0$. *Then there exists* $a = c(n)\beta > 0$ *such that* $B \in L^2_{loc}log^{a-1}L(\Omega,R^n)$, $E \in L^2_{loc}log^{a-1}L(\Omega,R^n)$.

As a consequence we deduce a higher integrability result for the gradient of "finite energy" solutions of the equations (1.3) verifying (1.5)(see Prop.3.3).

Recently regularity results for quasiharmonic fields have been investigated in [IS_2], [IMMP], [M]. The aim of the previous paper is to establish regularity results for B and E without fixing β in (1.7). There the result states a higher integrability of B and E, provided β is sufficiently large.

In [MM], assuming that $\mathcal{K}(x)^\gamma, \gamma > 1$, belongs to the exponential class, the authors prove that B and E belong to $L^2_{loc} log^a L$ for any $a > 0$.

When $\mathcal{K}(x)$ is bounded higher integrability results of quasiharmonic fields have been investigate in [IS_2]. (See also the references therein).

Recently a result similar to Theorem 1 has been obtained by [FKZ] for mappings of finite distortion.

2. – Preliminary results.

Define $L^s log^a L(\Omega), 1 \leq s < +\infty, a \in R$ as the Orlicz-Zygmund space generated by $\phi(t) = t^s log^a(e+t)$, at least for sufficiently large values of t, i.e. the space of all measurable functions f on Ω such that

$$\|f\|_\phi = \inf\{\lambda > 0 : \int_\Omega \phi\left(\frac{|f|}{\lambda}\right) dx \leq 1\}. \tag{2.1}$$

Let us recall that for $a \geq 0$ the non linear functional

$$[f]_{s,a} = \left[\int_\Omega |f|^s log^a \left(e + \frac{|f|}{\|f\|_s} \right) \right]^{\frac{1}{s}}$$

is comparable with the Luxemburg norm defined by (2.1).

A central ingredient in our arguments is the classical Hardy-Littlewood maximal function. Recall that, give a function $g \in L^1_{loc}(\Omega)$, we define the Hardy-Littlewood maximal function Mg of g by

$$Mg(x) = \sup_{r>0} \frac{1}{|\mathcal{B}_r|} \int_{\mathcal{B}_r} |g(y)| dy,$$

for every ball $\mathcal{B}_r$ of Ω containing the given point $x \in \Omega$.

The following proposition is classical. The proof involves Vitali's covering lemma and the Calderon-Zygmund decomposition, [S].

PROPOSITION 2.1. – *Let $h \in L^1(R^n)$. For any $t > 0$, we have*

$$\frac{1}{2^n t} \int_{|h|>t} |h(x)| dx \leq |\{x \in R^n / Mh(x) > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{|h|>\frac{t}{2}} |h(x)| dx.$$

The next Lemma is crucial to establish Theorem 1.1. For a proof see [IS_1], [GIM], [MM].

LEMMA 2.2. – *Let $\langle B, E\rangle$ be a nonnegative div-curl product such that $B \in L^p log^{-1}L(\Omega, R^n)$, $E \in L^q log^{-1}L(\Omega, R^n)$ with $1 < p, q < +\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then for any $0<\sigma<1$*

$$\fint_{\sigma\mathcal{B}_\rho} \langle B, E\rangle dx \leq c\Big(\fint_{\mathcal{B}_\rho} |B|^r dx\Big)^{\frac{1}{r}}\Big(\fint_{\mathcal{B}_\rho} |E|^s dx\Big)^{\frac{1}{s}}$$

where $\mathcal{B}_\rho = \mathcal{B}(x,\rho) \subset\subset \Omega$, $\frac{1}{r}+\frac{1}{s}=1+\frac{1}{n}$, $1 \leq r \leq p$, $1 \leq s \leq q$ and $c = c(n,p,q)$.

3. – Proof of Theorem 1.1.

Now we fix a ball $\mathcal{B}_0 = \mathcal{B}(x_0, r_0) \subset\subset \Omega$ and assume that

$$\int_{\mathcal{B}_0} \langle B, E\rangle dx = 1 \tag{3.1}$$

for homogeneity property, under our assumptions this condition is not restrictive.

Define the following auxiliary functions

$$h_1(x) = d^n(x)\langle B, E\rangle \tag{3.2}$$

$$h_2(x) = d(x)(|B|^{\frac{2}{n}} + |E|^{\frac{2}{n}}) \tag{3.3}$$

$$h_3(x) = \chi_{\mathcal{B}_0}(x) \tag{3.4}$$

where $d(x) = dist(x, R^n \backslash \mathcal{B}_0)$ and χ_E is the characteristic function of the set E.

The following Lemma will be useful to establish Theorem 1.1.

LEMMA 3.1. – *If* (3.1) *holds, then we have*

$$\Big(\fint_{\mathcal{B}} h_1(x)dx\Big)^{\frac{1}{n}} \leq c(n)\Big(\fint_{2\mathcal{B}} h_2^q(x)dx\Big)^{\frac{1}{q}} + c(n)\Big(\fint_{2\mathcal{B}} h_3(x)dx\Big)^{\frac{1}{n}} \tag{3.5}$$

for all balls $\mathcal{B} \subset R^n$, where $h_1(x), h_2(x), h_3(x)$ are defined by (3.2)-(3.4) *and $q = \frac{n^2}{n+1}$.*

PROOF. – We need to prove it only if $\mathcal{B}$ intersects $\mathcal{B}_0$, otherwise one can easily see that (3.5) is trivial. We split the proof of (3.5) in two cases, precisely when $3\mathcal{B}$ is or is not contained in $\mathcal{B}_0$

CASE 1: $3\mathcal{B} \subset \mathcal{B}_0$. By a geometric consideration we have that

$$\max_{x\in\mathcal{B}} d(x) \le 4 \min_{x\in 2\mathcal{B}} d(x).$$

Applying Lemma 2.2

$$\begin{aligned}
\Big(\fint_{\mathcal{B}} h_1(x)dx\Big)^{\frac{1}{n}} &\le \max_{\mathcal{B}} d(x)\Big(\fint_{\mathcal{B}} \langle B, E\rangle dx\Big)^{\frac{1}{n}} \\
&\le c(n) \min_{2\mathcal{B}} d(x)\Big(\fint_{2\mathcal{B}} |B|^{\frac{2n}{n+1}} dx\Big)^{\frac{n+1}{2n^2}} \Big(\fint_{2\mathcal{B}} |E|^{\frac{2n}{n+1}} dx\Big)^{\frac{n+1}{2n^2}} \\
&\le c(n) \min_{2\mathcal{B}} d(x)\Big[\Big(\fint_{2\mathcal{B}} |B|^{\frac{2n^2}{(n+1)n}} dx\Big)^{\frac{n+1}{n^2}} + \Big(\fint_{2\mathcal{B}} |E|^{\frac{2n^2}{(n+1)n}} dx\Big)^{\frac{n+1}{n^2}}\Big] \\
&\le c(n)\Big\{\fint_{2\mathcal{B}} \Big[d(x)\Big(|B|^{\frac{2}{n}} + |E|^{\frac{2}{n}}\Big)\Big]^{\frac{n^2}{n+1}} dx\Big\}^{\frac{n+1}{n^2}} \\
&= c(n)\Big(\fint_{2\mathcal{B}} h_2^q(x)dx\Big)^{\frac{1}{q}}
\end{aligned}$$

CASE 2: $3\mathcal{B} \not\subset \mathcal{B}_0, \mathcal{B} \cap \mathcal{B}_0 \neq \emptyset$. We have that

$$\max_{x\in\mathcal{B}} d(x) \le \max_{x\in 2\mathcal{B}} d(x) \le c(n)|2\mathcal{B} \cap \mathcal{B}_0|^{\frac{1}{n}}.$$

By using (3.1), we conclude that

$$\begin{aligned}
\Big(\fint_{\mathcal{B}} h_1(x)dx\Big)^{\frac{1}{n}} &\le \max_{\mathcal{B}} d(x)\Big(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}\cap\mathcal{B}_0} \langle B, E\rangle dx\Big)^{\frac{1}{n}} \\
&\le c(n)\Big(\frac{|2\mathcal{B}\cap\mathcal{B}_0|}{|\mathcal{B}|}\int_{\mathcal{B}_0} \langle B, E\rangle dx\Big)^{\frac{1}{n}} \\
&\le c(n)\Big(\frac{1}{|2\mathcal{B}|}\int_{2\mathcal{B}} h_3(x)dx\Big)^{\frac{1}{n}}.
\end{aligned}$$

Combining these two cases we get the inequality (3.5). □

PROOF OF THEOREM 1.1. – According to Lemma (3.1), we observe that (3.5) is true for all balls $\mathcal{B} \subset R^n$. So the following point-wise inequality for the maximal functions yields

$$[M(h_1)(y)]^{\frac{1}{n}} \le c(n)[M(h_2^q)(y)]^{\frac{1}{q}} + c(n)[M(h_3)(y)]^{\frac{1}{n}}, \quad \forall y \in R^n$$

from which, for $\lambda > 0$, we also deduce that

$$|\{x \in R^n / M(h_1)(x) > \lambda^n\}| \leq |\{x \in R^n / c(n)M(h_2^q)(x) > \lambda^q\}| \\ + |\{x \in R^n / c(n)M(h_3)(x) > \lambda^n\}|.$$

The definition of h_3 implies that $M(h_3)(x) \leq 1$ in R^n, then the set $\{x \in R^n / M(h_3)(x) > \lambda^n\}$ is empty for $\lambda > \lambda_1 = \lambda_1(n)$. Hence

$$|\{x \in R^n / M(h_1)(x) > \lambda^n\}| \leq |\{x \in R^n / c(n)M(h_2^q)(x) > \lambda^q\}|$$

for all $\lambda > \lambda_1$. We use Proposition 2.1 to deduce

$$\int_{h_1 > \lambda^n} h_1(x)dx \leq c(n)\lambda^{n-q} \int_{c(n)h_2 > \lambda} h_2^q(x)dx \tag{3.6}$$

for all $\lambda > \lambda_1$. We may assume that the constant $c(n)$ in (3.6) is bigger than one.

Let us define the function

$$\psi(\lambda) = \frac{n-q}{a} log^a \lambda + log^{a-1} \lambda,$$

where $q = \dfrac{n^2}{n+1}$ as above, and a is a positive constant that will be fixed in equation (3.9) below.

Observe that

$$\phi(\lambda) = \frac{d}{d\lambda}\psi(\lambda) = \frac{n-q}{\lambda} log^{a-1}\lambda + \frac{a-1}{\lambda} log^{a-2}\lambda > 0$$

for all $\lambda > \lambda_2 = exp\left(\dfrac{n+1}{n}\right)$ and that

$$\lambda^{n-q}\phi(\lambda) = \frac{d}{d\lambda}(\lambda^{n-q} log^{a-1}\lambda).$$

So we can multiply both sides of (3.6) by $\phi(\lambda)$, integrated with λ over (λ_0, j), for j large and $\lambda_0 = max(\lambda_1, \lambda_2)$. Changing the order of the integration we get

$$\int_{j^n > h_1 > \lambda_0^n} h_1(x)dx \int_{\lambda_0}^{h_1^{\frac{1}{n}}} \phi(\lambda)d\lambda \leq c(n) \int_{j > c(n)h_2 > \lambda_0} h_2^q(x)dx \int_{\lambda_0}^{c(n)h_2} \lambda^{n-q}\phi(\lambda)d\lambda,$$

that is,

$$\int_{j^n > h_1 > \lambda_0^n} (\psi(h_1^{\frac{1}{n}}(x)) - \psi(\lambda_0))h_1(x)dx \leq c(n) \int_{j > c(n)h_2(x) > \lambda_0} h_2^n(x) log^{a-1}(c(n)h_2(x))dx.$$

Then it follows easily from (3.1) that

$$\frac{1}{a}\int_{j^n>h_1>\lambda_0^n} h_1(x)log^a h_1^{\frac{1}{n}}(x)dx \le c(n)\int_{j>c(n)h_2>\lambda_0} h_2^n(x)log^{a-1}(c(n)h_2(x))dx + c(n,a)|\mathcal{B}_0| \tag{3.7}$$

where $c(n) \ge 1$.

In the remaining part of the proof, by using the distortion inequality, we want to absorb the right hand side of (3.7) on the left.

Using the following elementary inequality (see [FKZ])

$$ab\ log^{a-1}(C(n)(ab)^{\frac{1}{n}}) \le \frac{C(n)}{\beta} alog^a(a^{\frac{1}{n}}) + C(a,\beta,n)exp(\beta b)$$

with $a = h_1(x), b = K(x)$, we deduce by (1.2) that

$$\begin{aligned} &\frac{1}{a}\int_{j^n>h_1>\lambda_0^n} h_1(x)log^a h_1^{\frac{1}{n}}(x)dx \\ &\le \frac{c(n)}{\beta}\int_{j^n>h_1>\lambda_0^n} h_1(x)log^a h_1^{\frac{1}{n}}(x)dx + c(n,a,\beta)\int_{\mathcal{B}_0} exp(\beta K(x))dx \\ &+ c(n,a)|\mathcal{B}_0| \\ &\le \frac{c(n)}{\beta}\int_{j^n>h_1>\lambda_0^n} h_1(x)log^a h_1^{\frac{1}{n}}(x)dx + c(n,a,\beta)\int_{\mathcal{B}_0} exp(\beta K(x))dx. \end{aligned} \tag{3.8}$$

By (3.8), setting

$$a = \frac{\beta}{2c(n)} \tag{3.9}$$

we obtain

$$\int_{j^n>h_1>\lambda_0^n} h_1(x)log^a h_1^{\frac{1}{n}}(x)dx \le c(n,\beta)\int_{\mathcal{B}_0} exp(\beta K(x))dx. \tag{3.10}$$

By letting $j \to +\infty$, in both sides of (3.10), and using the monotone convergence theorem we deduce that

$$\int_{\mathcal{B}_0} d^n(x)\langle B,E\rangle log^a(e + d^n(x)\langle B,E\rangle)dx \le c(n,\beta)\int_{\mathcal{B}_0} exp(\beta K(x))dx.$$

Finally, noticing that in $\sigma\mathcal{B}_0 = \mathcal{B}(x_0,\sigma r_0)$ with $0<\sigma<1$ we have $d^n(x) \ge (1-\sigma)^n r_0^n \ge c(n,\sigma)|\mathcal{B}_0|$, and taking into account the normalization (3.1), we have the inequality

$$\fint_{\sigma\mathcal{B}_0}\langle B,E\rangle log^a\Big(e+\frac{\langle B,E\rangle}{\fint_{\mathcal{B}_0}\langle B,E\rangle dx}\Big)dx$$
$$\leq c(n,\beta,\sigma)\Big(\fint_{\mathcal{B}_0} exp(\beta K(x))dx\Big)\Big(\fint_{\mathcal{B}_0}\langle B,E\rangle dx\Big)$$

which concludes the proof, by means of (1.2). □

At this point we apply this result to the study of equation (1.3).

Note that (1.3) is the Euler-Lagrange equation of the variational integral

$$\mathcal{E}[u]=\int_{\Omega} < A(x)\nabla u, \nabla u > dx.$$

We deal with solution of (1.3) having"finite energy", namely $\mathcal{E}[u]$ is finite.

If $u\in W^{1,1}_{loc}(\Omega)$ is a local solution of (1.3), we set $B=A\nabla u$ and $E=\nabla u$ so that $divB=0$ and $curlE=0$. Let us remark that $\langle B,E\rangle$ is locally integrable on Ω, since u is a local solution of (1.3) with"finite energy". By assumptions (1.5), (1.7) the gradient of a finite energy solution belongs to the Orlicz-Zygmund space $L^2_{loc}log^{a-1}L(\Omega,R^n)$. From Theorem 1.1 we deduce the following

PROPOSITION 3.3. – *Let u be a local solution of* (1.3) *with "finite energy". Assume that the distortion $\mathcal{K}(x)\geq 1$ satisfies* (1.7) *for some $\beta>0$. Then there exists $a=c(n)\beta>0$ such that $|\nabla u|\in L^2_{loc}log^{a-1}L(\Omega,R^n)$, $|A\nabla u|\in L^2_{loc}log^{a-1}L(\Omega,R^n)$.*

REFERENCES

[FKZ] D. FARACO - P. KOSKELA - X. ZHONG, *Mappings of finite distortion: the degree of regularity*, Advances in Mathematics, **190** (2005), 300-318.

[GV] V. GOL'DSTEIN - S. VODOP'YANOV, *Quasiconformal mappings and spaces of funtions with generalized first derivatives*, Sibirsk. Mat. Z., **17** (1976), 515-531.

[GIM] L. GRECO - T. IWANIEC - G. MOSCARIELLO, *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. Journ., n. 2 (1995), 305-339.

[HM] S. HENCL - J. MALY, *Mappings of finite distortion: Hausdorff measure of zero sets*, Math. Ann., **324** (2002), 451-464.

[I] T. IWANIEC, *p-Harmonic tensors and quasiregular mappings*, Annals of Math., **136** (1992), 651-685.

[IMMP] T. IWANIEC - L. MIGLIACCIO - G. MOSCARIELLO - A. PASSARELLI DI NAPOLI, *A priori estimates for non linear elliptic complexes*, Advances in Diff. Eq., 8 (2003), 513-546.

[IS_1] T. IWANIEC - C. SBORDONE, *On the integrability of the Jacobians under minimal hypothesis*, Arch. Rat. Mech. Anal., **119** (1992), 129-143.

[IS_2] T. IWANIEC - C. SBORDONE, *Quasiharmonic fields*, Ann. Inst. H. Poincaré, AN18, **5** (2001), 519-527.

[MM] L. MIGLIACCIO - G. MOSCARIELLO, *Higher integrability of div-curl products*, Ricerche di Matematica, (1) XLIX (2000), 151-161.

[M] G. MOSCARIELLO, *On the integrability of finite energy solutions for p-harmonic equations*, Nodea, 11 (2004) 393-406.

[RR] M. M. RAO - Z. D. REN, *Theory of Orlicz spaces*, Marcel Dekker, City, 1991.

[Sb] C. SBORDONE, *New estimates for div-curl products and very weak solutions of P.D.E.'s*, Ann. Scuola Norm. Sup. Pisa, **25** (1997), 739-756.

[S] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

Dipartimento di Matematica e Informatica,
Università di Salerno, Via Ponte Don Melillo-84084 Fisciano (SA)
E-mail: pdigironimo@unisa.it

Pervenuta in Redazione
il 28 gennaio 2006 e in forma rivista il 10 ottobre 2007