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Sunto. – *La famosa identità di Jacobi riguardante il prodotto triplo viene esaminata grazie alle due dimostrazioni più semplici dovute a Cauchy (1843) e Gauss (1866). Applicando il principio di induzione ed il metodo di differenze finite, lo stesso spirito ci conduce alla riconferma delle due forme finite dell'identità di prodotto quintuplo.*

Sammary. – *The simplest proof of Jacobi's triple product identity originally due to Cauchy (1843) and Gauss (1866) is reviewed. In the same spirit, we prove by means of induction principle and finite difference method, a finite form of the quintuple product identity. Similarly, the induction principle will be used to give a new proof of another algebraic identity due to Guo and Zeng (2005), which can be considered as another finite form of the quintuple product identity.*

1. – Jacobi's Triple Product Identity.

The celebrated Jacobi triple product identity states that

$$[q, x, q/x; q]_\infty = \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} x^k \quad \text{for} \quad |q| < 1 \tag{1}$$

where the q-shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n = 1, 2, \cdots$$

with the following abbreviated multiple parameter notation

$$[a, \beta, \cdots, \gamma; q]_\infty = (a; q)_\infty (\beta; q)_\infty \cdots (\gamma; q)_\infty.$$

There are several algebraic and combinatorial proofs (see [2, 10, 17, 22, 29], for examples). Here we present the simplest proof by using only the q-binomial theorem, which is originally due to Cauchy (1843) and Gauss (1866). However, it has not been well noticed up to now (cf. [4, P 497]).

Recall that the q-binomial theorem reads as

$$(2)\qquad (x;q)_m = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} x^k \quad \text{where} \quad \begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}}.$$

This can easily be established by induction principle on m.

Now replacing m and x by $m+n$ and xq^{-m} respectively, and then noting the relation

$$(3)\qquad (q^{-m}x;q)_{m+n} = (q^{-m}x;q)_m (x;q)_n = (-1)^m q^{-\binom{1+m}{2}} x^m (q/x;q)_m (x;q)_n$$

we can reformulate the q-binomial theorem as

$$(x;q)_n (q/x;q)_m = \sum_{k=0}^{m+n} (-1)^{k-m} \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\binom{k-m}{2}} x^{k-m}$$

which becomes, under summation index substitution $k \to m+k$, the following finite form of the Jacobi triple product identity

$$(4)\qquad (x;q)_n (q/x;q)_m = \sum_{k=-m}^{n} (-1)^k \begin{bmatrix} m+n \\ m+k \end{bmatrix} q^{\binom{k}{2}} x^k.$$

Letting $m, n \to \infty$ in (4), we get (1) immediately in view of limiting relation

$$\begin{bmatrix} m+n \\ m+k \end{bmatrix} = \frac{(q;q)_{m+n}}{(q;q)_{m+k}(q;q)_{n-k}} \xrightarrow{m,n\to\infty} \frac{1}{(q;q)_\infty} \quad \text{where} \quad |q<1|.$$

2. – The Quintuple Product Identity.

Just like the derivation of Jacobi's triple product identity from the q-binomial theorem, we will show that the quintuple product identity is the limiting form of the following algebraic identity.

THEOREM [Finite form of the quintuple product identity: Chen-Chu-Gu [9]]. *For a natural number m and a variable x, there holds an algebraic identity:*

$$(5)\qquad \sum_{k=0}^{m} (1+xq^k) \begin{bmatrix} m \\ k \end{bmatrix} \frac{(x;q)_{m+1}}{(q^k x^2;q)_{m+1}} x^k q^{k^2} \equiv 1.$$

Performing parameter replacements $m \to m+n$, $x \to -q^{-m}x$ and $k \to k+m$ and then simplifying the result through (3), we may restate the algebraic identity displayed in the theorem as the following finite bilateral identity

$$\sum_{k=-m}^{n}(1-xq^k)\begin{bmatrix}m+n\\m+k\end{bmatrix}\frac{(-x;q)_{1+n}(-q/x;q)_m}{(x^2;q)_{1+n+k}(q/x^2;q)_{m-k}}x^{3k}q^{k^2+\binom{k}{2}}\equiv 1. \tag{6}$$

Letting $m,n\to\infty$ in this equation and applying the relation

$$(q;q)_\infty\frac{(x^2;q)_\infty(q/x^2;q)_\infty}{(-x;q)_\infty(-q/x;q)_\infty}=[q,x,q/x;q]_\infty[qx^2,q/x^2;q^2]_\infty$$

we derive the famous quintuple product identity

$$\sum_{k=-\infty}^{+\infty}(1-xq^k)q^{3\binom{k}{2}}(qx^3)^k=[q,x,q/x;q]_\infty[qx^2,q/x^2;q^2]_\infty\quad\text{for}\quad |q|<1. \tag{7}$$

3. – Inductive Proof of the Theorem.

In terms of basic hypergeometric series, the identity (5) can be derived as the limiting case $M\to\infty$ of the terminating very-well poised ${}_6\phi_5$-series identity (cf. [19, II-21]):

$${}_6\phi_5\left[\begin{matrix}x^2,qx,-qx,&q^{-m},&x,&M\\x,&-x,q^{1+m}x^2,&qx,&qx^2/M\end{matrix}\,\middle|\,q;\frac{q^{1+m}x}{M}\right]=\frac{(qx^2;q)_m(qx/M;q)_m}{(qx;q)_m(qx^2/M;q)_m}. \tag{8}$$

For those who are not familiar with basic hypergeometric series, we offer a very elementary proof of (5) based on induction principle on m.

When $m=0$, it is trivial to see that (5) is true. Suppose that (5) holds for a natural number m. Then we need to check it also for $m+1$.

Replacing x by qx and then k by $k-1$, we can restate (5) as

$$x\equiv\sum_{k=1}^{1+m}(1+xq^k)\begin{bmatrix}m\\k-1\end{bmatrix}\frac{(qx;q)_{m+1}}{(q^{1+k}x^2;q)_{m+1}}x^kq^{k^2-k}. \tag{9}$$

Then the linear combination

$$\frac{1-xq^{m+1}}{1-x^2q^{m+1}}\mathrm{Eq}(5)+\frac{(1-x)q^{m+1}}{1-x^2q^{m+1}}\mathrm{Eq}(9)$$

leads us to the following

$$\begin{aligned}1\equiv\frac{1-xq^{m+1}}{1-x^2q^{m+1}}&+\frac{(1-x)xq^{m+1}}{1-x^2q^{m+1}}\\=\sum_{k=0}^{1+m}(1+xq^k)&\begin{bmatrix}1+m\\k\end{bmatrix}\frac{(x;q)_{m+2}}{(q^kx^2;q)_{m+2}}x^kq^{k^2}\\\times\frac{(1-q^{1+m-k})(1-x^2q^{1+m+k})+q^{1+m-k}(1-q^k)(1-x^2q^k)}{(1-q^{m+1})(1-x^2q^{m+1})}&\end{aligned}$$

which corresponds to the case $m+1$ of (5) on account of the fact that the last rational-factor reduces to one. According to induction principle, this confirms (5).

4. – Constructive Proof of the Theorem.

For a natural number m and a variable x, we will investigate the finite sum $\Omega_m(x)$ given by

$$\Omega_m(x) := \sum_{k=0}^{m}(1+xq^k)\begin{bmatrix}m\\k\end{bmatrix}\frac{(x;q)_{m+1}}{(q^kx^2;q)_{m+1}}x^kq^{k^2}. \tag{10}$$

Let $\mathcal{A}_k$ and $\mathcal{B}_k$ be two sequences defined respectively by

$$\mathcal{A}_k := \frac{(-1)^kx^kq^{\binom{k+1}{2}}}{1-xq^k} \quad \text{and} \quad \mathcal{B}_k := (-1)^k\begin{bmatrix}m-1\\k\end{bmatrix}\frac{(x;q)_m}{(q^{k+1}x^2;q)_m}q^{\binom{k+1}{2}}.$$

In view of the boundary condition $\mathcal{B}_{-1} = \mathcal{B}_m = 0$ and the finite differences

$$\mathcal{A}_k - \mathcal{A}_{k+1} = (-1)^k\frac{1-x^2q^{2k+1}}{(1-xq^k)(1-xq^{k+1})}x^kq^{\binom{k+1}{2}}$$

$$\mathcal{B}_k - \mathcal{B}_{k-1} = (-1)^k\{1-x^2q^{2k}\}\begin{bmatrix}m\\k\end{bmatrix}\frac{(x;q)_{m+1}}{(q^kx^2;q)_{m+1}}q^{\binom{k}{2}}$$

we can reformulate the Ω-sum defined in (10) as follows:

$$\begin{aligned}\Omega_m(x) &= \sum_{k=0}^{m}\mathcal{A}_k\{\mathcal{B}_k-\mathcal{B}_{k-1}\} = \sum_{k=0}^{m}\mathcal{A}_k\mathcal{B}_k - \sum_{k=1}^{m}\mathcal{A}_k\mathcal{B}_{k-1}\\ &= \sum_{k=0}^{m-1}\mathcal{A}_k\mathcal{B}_k - \sum_{k=0}^{m-1}\mathcal{A}_{k+1}\mathcal{B}_k = \sum_{k=0}^{m-1}\mathcal{B}_k\{\mathcal{A}_k-\mathcal{A}_{k+1}\}\end{aligned}$$

which leads us to the following relation:

$$\Omega_m(x) = \sum_{k=0}^{m-1}\frac{1-x^2q^{2k+1}}{(1-xq^k)(1-xq^{k+1})}\begin{bmatrix}m-1\\k\end{bmatrix}\frac{(x;q)_{m+1}}{(q^{k+1}x^2;q)_m}x^kq^{k^2+k}. \tag{11}$$

From this expression, we can derive the following interesting result.

LEMMA [Recurrence relation]. *For the Ω-function defined in* (10), *there holds the recursion*

$$\Omega_m(x) = \Omega_{m-1}(qx) \quad \textit{where} \quad m = 1,2,\cdots. \tag{12}$$

PROOF. – We further define two sequences $\mathcal{C}_k$ and $\mathcal{D}_k$ respectively by

$$\mathcal{C}_k := \frac{(q^{m-k};q)_k (x;q)_{m+1}}{(x^2;q)_{k+m+1}} x^{2k} q^{k^2+k}$$

$$\mathcal{D}_k := \frac{(x^2;q)_{k+2}}{(q;q)_k} \frac{x^{-k}}{(1-x)(1-xq^{k+1})}.$$

In view of the boundary condition $\mathcal{C}_m = \mathcal{D}_{-1} = 0$ and the finite differences

$$\mathcal{C}_k - \mathcal{C}_{k+1} = \{1 - x^2 q^{2k+2}\} \frac{(q^{m-k};q)_k (x;q)_{m+1}}{(x^2;q)_{k+m+2}} x^{2k} q^{k^2+k}$$

$$\mathcal{D}_k - \mathcal{D}_{k-1} = \frac{1 - x^2 q^{2k+1}}{(1-xq^k)(1-xq^{k+1})} \frac{(x^2;q)_{k+1}}{(q;q)_k} x^{-k}$$

we can manipulate Ω-sum displayed in (11) as follows:

$$\begin{aligned}\Omega_m(x) &= \sum_{k=0}^{m-1} \mathcal{C}_k \{\mathcal{D}_k - \mathcal{D}_{k-1}\} = \sum_{k=0}^{m-1} \mathcal{C}_k \mathcal{D}_k - \sum_{k=1}^{m-1} \mathcal{C}_k \mathcal{D}_{k-1} \\ &= \sum_{k=0}^{m-1} \mathcal{C}_k \mathcal{D}_k - \sum_{k=0}^{m-1} \mathcal{C}_{k+1} \mathcal{D}_k = \sum_{k=0}^{m-1} \mathcal{D}_k \{\mathcal{C}_k - \mathcal{C}_{k+1}\} \\ &= \sum_{k=0}^{m-1} \{1 + xq^{k+1}\} \begin{bmatrix} m-1 \\ k \end{bmatrix} \frac{(qx;q)_m}{(q^{2+k}x^2;q)_m} x^k q^{k^2+k}.\end{aligned}$$

This leads us to the recursion stated in the lemma. □

Iterating for m-times the recurrence relation:

$$\Omega_m(x) = \Omega_{m-1}(qx)$$

we find the following algebraic identity:

$$\Omega_m(x) = \Omega_{m-1}(qx) = \Omega_{m-2}(q^2x) \cdots = \cdots \Omega_0(q^m x) = 1 \tag{13}$$

which leads us immediately to the algebraic identity stated in the Theorem.

The informed reader will notice that the procedure just employed is the so-called "Abel's lemma on summation by parts". This method has been shown powerful to evaluate classical and basic hypergeometric series. The interested reader may refer to Chu and Jia [11, 12, 13, 14] for more details and further developments.

5. – Another Finite Form of the Quintuple Product Identity.

In a recent paper [30], Guo and Zeng found another finite form of the quintuple product identity. Its reduced case $m = 2m$ has appeared in Paule [24, Eq 27].

PROPOSITION (Guo and Zeng [30, Theorem 9.1] *For a natural number m and a variable x, there holds an algebraic identity:*

$$\sum_{k=0}^{m}(1-x^2q^{1+2k})\begin{bmatrix}m\\k\end{bmatrix}\frac{(qx;q)_m}{(q^{1+k}x^2;q)_{m+1}}x^kq^{k^2}\equiv 1. \tag{14}$$

In terms of basic hypergeometric series, the identity (14) can be derived as the limiting case $M \to \infty$ of the terminating very-well poised ${}_6\phi_5$-series identity (cf. [19, II-21]):

$${}_6\phi_5\left[\begin{matrix}qx^2,\ q^{3/2}x,\ -q^{3/2}x,\ \ q^{-m},\ \ qx,\ \ \ M\\ q^{1/2}x,\ -q^{1/2}x, q^{2+m}x^2, qx, q^2x^2/M\end{matrix}\middle| q;\frac{q^{1+m}x}{M}\right]=\frac{(q^2x^2;q)_m(qx/M;q)_m}{(qx;q)_m(q^2x^2/M;q)_m}.$$

Analogously, an inductive proof of (14) can be reproduced as follows.

When $m = 0$, it is trivial to see that (14) is true. Suppose that (14) holds for a natural number m. Then we need to check it also for $m + 1$.

Replacing x by qx and then k by $k - 1$, we can restate (14) as

$$x \equiv \sum_{k=1}^{1+m}(1-x^2q^{1+2k})\begin{bmatrix}m\\k-1\end{bmatrix}\frac{(q^2x;q)_m}{(q^{2+k}x^2;q)_{m+1}}x^kq^{k^2-k}. \tag{15}$$

Then the linear combination

$$\frac{1-xq^{m+1}}{1-x^2q^{m+2}}\mathrm{Eq}(14)+\frac{(1-qx)q^{m+1}}{1-x^2q^{m+2}}\mathrm{Eq}(15)$$

leads us to the following

$$\begin{aligned}1 &\equiv \frac{1-xq^{m+1}}{1-x^2q^{m+2}} \quad + \quad \frac{(1-qx)xq^{m+1}}{1-x^2q^{m+2}}\\ &= \sum_{k=0}^{1+m}(1-x^2q^{1+2k})\begin{bmatrix}1+m\\k\end{bmatrix}\frac{(qx;q)_{m+1}}{(q^{1+k}x^2;q)_{m+2}}x^kq^{k^2}\\ &\times\frac{(1-q^{1+m-k})(1-x^2q^{2+m+k})+q^{1+m-k}(1-q^k)(1-x^2q^{1+k})}{(1-q^{m+1})(1-x^2q^{m+2})}\end{aligned}$$

which corresponds to the case $m+1$ of (14) on account of the fact that the last rational-factor reduces to one. According to induction principle, this confirms (14).

Similar to the last section, a constructive proof of (14) can be provided either. We leave it to the reader as an exercise.

REMARK For the historical note about the quintuple product identity, the reader can refer to [8]. More comprehensive coverage has been provided recently by Cooper [15]. Compared with the known proofs of this identity due to Watson [27, 28] based on functional equations and elliptic functions, Atkin and Swinnerton-Dyer [5] via function theoretic methods, Gordan [20] through functional equations, Carlitz and Subbarao [8] by multiplying two triple products as well as Paule [24] by the WZ-method, the proof presented in this paper is much simpler and more elementary, which requires only some high school algebra.

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REFERENCES

[1] K. ALLADI, *The quintuple product identity and shifted partition functions*, J. Comput. Appl. Math., **68** (1996), 3-13.

[2] G. E. ANDREWS, *A simple proof of Jacobi's triple product identity*, Proc. Amer. Math., **16** (1965), 333-334.

[3] G. E. ANDREWS, *Applications of basic hypergeometric functions*, SIAM Review, **16** (1974), 441-484.

[4] G. E. ANDREWS - R. ASKEY - R. ROY, *Special Functions*, Cambridge University Press, Cambridge, 2000.

[5] A. O. L. ATKIN - P. SWINNERTON-DYER, *Some properties of partitions*, Proc. London Math. Soc., **4** (1954), 84-106.

[6] W. N. BAILEY, *Series of hypergeometric type which are infinite in both directions*, Quart. J. Math. (Oxford) **7** (1936), 105-115.

[7] W. N. BAILEY, *On the simplification of some identities of the Rogers-Ramanujan type*, Proc. London Math. Soc., **1** (1951), 217-221.

[8] L. CARLITZ - M. V. SUBBARAO, *A simple proof of the quintuple product identity*, Proc. Amer. Math. Society, **32**, 1 (1972), 42-44.

[9] W. Y. C. CHEN - W. CHU - N. S. S. GU, *Finite form of the quintuple product identity*, Journal of Combinatorial Theory (Series A), **113**, 1 (2006), 185-187.

[10] W. CHU, *Durfee rectangles and the Jacobi triple product identity*, Acta Math. Sinica [New Series], **9**, 1 (1993), 24-26.

[11] W. CHU, *Abel's Lemma on summation by parts and Ramanujan's* ${}_1\psi_1$*-series Identity*, Aequationes Mathematicae, **72**, 1/2 (2006), 172-176.

[12] W. CHU, *Abel's Method on summation by parts and Hypergeometric Series*, Journal of Difference Equations and Applications, **12**, 8 (2006), 783-798.

[13] W. CHU, *Abel's Lemma on summation by parts and Basic Hypergeometric Series*, Advances in Applied Mathematics, **39**, 4 (2007), 490-514.

[14] W. CHU - C. Z. JIA, *Abel's Method on summation by parts and Terminating Well-Poised q-Series Identities*, Journal of Computational and Applied Mathematics, **207**, 2 (2007), 360-370.

[15] S. COPPER, *The quintuple product identity*, International J. of Number Theory, **2**, 1 (2006), 115-161.

[16] R. J. EVANS, *Theta function identities*, J. Math. Anal. Appl., **147**, 1 (1990), 97-121.

[17] J. A. EWELL, *An easy proof of the triple product identity*, Amer. Math. Month., 88 (1981), 270-272.

[18] H. M. FARKAS - IRWIN KRA, *On the quintuple product identity*, Proc. Amer. Math. Soc., **127**, 3 (1999), 771-778.

[19] G. GASPER - M. RAHMAN, *Basic Hypergeometric Series* (2nd edition), Cambridge University Press, 2004.

[20] B. GORDON, *Some identities in combinatorial analysis*, Quart. J. Math. Oxford, **12** (1961), 285-290.

[21] M. D. HIRSCHHORN *A generalization of the quintuple product identity*, J. Austral. Math. Soc., **A44** (1988), 42-45.

[22] R. P. LEWIS, *A combinatorial proof of the triple product identity*, Amer. Math. Month., **91** (1984), 420-423.

[23] L. J. MORDELL, *An identity in combinatorial analysis*, Proc. Glasgow Math. Ass., **5** (1961), 197-200.

[24] P. PAULE, *Short and easy computer proofs of the Rogers-Ramanujan identities and of identities of similar type*, Electronic J. of Combinatorics, **1** (1994), R#10.

[25] D. B. SEARS, *Two identities of Bailey*, J. London Math. Soc., **27** (1952), 510-511.

[26] M. V. SUBBARAO - M. VIDYASAGAR, *On Watson's quintuple product identity*, Proc. Amer. Math. Society, **26**, 1 (1970), 23-27.

[27] G. N. WATSON, *Theorems stated by Ramanujan VII: Theorems on continued fractions*, J. London Math. Soc., **4** (1929), 39-48.

[28] G. N. WATSON, *Ramanujan's Vertumung über Zerfallungsanzahlen*, J. Reine Angrew. Math., **179** (1938), 97-128.

[29] E. M. WRIGHT, *An enumerative proof of an identity of Jacobi*, J. of London Math. Soc., **40** (1965), 55-57.

[30] V. J. W. GUO - J. ZENG, *Short proofs of summation and transformation formulas for basic hypergeometric series*, Journal of Mathematical Analysis and Applications, **327**, 1 (2007), 310-325.

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