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Acute Triangulations of Doubly Covered Convex Quadrilaterals

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Sunto. – *Recentemente, motivate da varie applicazioni, sono state studiate le triangolazioni di superfici utilizzando soltanto triangoli acutangoli. I triangoli e i quadrilateri possono essere triangolati, rispettivamente, con al più* 7 *e al più* 10 *triangoli acutangoli. I triangoli coperti doppiamente possono essere triangolati con al più 12 triangoli. In questo lavoro noi trattiamo le triangolazioni di quadrilateri convessi coperti doppiamente e mostriamo che tali quadrilateri possono essere triangolati con al più* 20 *triangoli acutangoli.*

Summary. – *Motivated by various applications triangulations of surfaces using only acute triangles have been recently studied. Triangles and quadrilaterals can be triangulated with at most* 7*, respectively* 10*, acute triangles. Doubly covered triangles can be triangulated with at most* 12 *acute triangles. In this paper we investigate the acute triangulations of doubly covered convex quadrilaterals, and show that they can be triangulated with at most* 20 *acute triangles.*

1. – Introduction.

The problem of producing various kinds of triangulations arises in computer graphics, physics simulation, and geographical information systems. Most applications demand not just any triangulation, but rather one with triangles satisfying certain shape and size criteria. It is generally true that large angles are undesirable, and a bound of $\frac{\pi}{2}$ on the largest angles has special importance.

A *triangulation* of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles. An *acute* (*non-obtuse*) triangulation is a triangulation whose triangles have all their angles less (not larger) than $\frac{\pi}{2}$.

What can be said about the *size*, i.e., the number of triangles, of an acute triangulation of a given polygon? In 1960, Burago and Zalgaller [3] and, independently, Goldberg [6] found out that every obtuse triangle can be triangulated into 7 acute triangles, and 7 is the minimum number. As a problem of Stover, this also appears in [5]. In 1980, Cassidy and Lord [4] showed that every square can be triangulated into 8 acute triangles, and 8 is the minimum number. This remains true for any rectangle [7]. In 2001 Maehara [10] investigated the acute triangulations of all quadrilaterals (convex or not), showing that they can be triangulated with at most 10 acute triangles. He also dealt with arbitrary polygons in [11], and his results have been refined by the first author [12]. Non-obtuse triangulations have been studied in [1], [2].

In 2000 Hangan, Itoh and Zamfirescu [7] considered the following problem: does there exist a number N such that every compact surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles? Of course, one should estimate N, if it exists. Indeed, the first compact surfaces to be investigated should be the convex ones, and among these the polyhedral surfaces play a central role. In the same year, Hangan, Itoh and Zamfirescu [7] started the investigation of acute triangulations of all Platonic surfaces (the surfaces of the five well-known Platonic solids). This was continued in [8] and [9]. At the same time, acute triangulations of some smooth convex surfaces were also considered, so for example, the sphere. However, the case of arbitrary convex surfaces is much more difficult, even for polyhedra with a small number of vertices. For instance, even the family of all tetrahedral surfaces is far from being easy to treat. A *doubly covered convex set* is a (degenerate convex) surface Γ_d homeomorphic to the sphere consisting of two planar isometric convex sets, Γ and Γ', with boundaries glued in accordance with the isometry. For any point P in Γ, let P' denote the corresponding point in Γ'. Even this case is, in full generality, still quite difficult. In 2004 the second author [13] considered acute triangulations of doubly covered triangles, and obtained the best estimate 12. In this paper we continue this task and prove that every doubly covered convex quadrilateral admits an acute triangulation of size at most 20.

2. – Preliminaries

A polygon Γ is a planar set homeomorphic to the compact disc, having as boundary $bd\Gamma$ a finite union of line-segments.

Let $\mathcal{T}$ be an acute triangulation of a polygon Γ. A vertex P of $\mathcal{T}$ is called

a *corner vertex* if P is a vertex of $bd\Gamma$;
a *side vertex* if P lies on $bd\Gamma$ but is not a corner vertex;
an *interior vertex* otherwise.

We can regard $\mathcal{T}$ as a planar graph. Clearly, a side vertex has degree at least 4 and an interior vertex has degree at least 5.

The following results of Maehara will be used. (The second follows from the first one.)

PROPOSITION 1 [10]. – *Let ABC be a triangle with acute angles at B and C, and let $P \in \operatorname{relint} AC$. If the angle at A is acute (non-acute), then there is an acute triangulation of ABC with size 4 (7) such that P is the only side vertex on AC.*

PROPOSITION 2 [10]. – *Let $ABCD$ be a convex quadrilateral with acute angle at B and non-acute angle at D. Then $ABCD$ admits an acute triangulation with size at most 9 such that the only vertex of degree less than 3 is B.*

3. – Main result.

We now consider the doubly covered convex set Γ_d in the case when Γ is a convex quadrilateral $ABCD$.

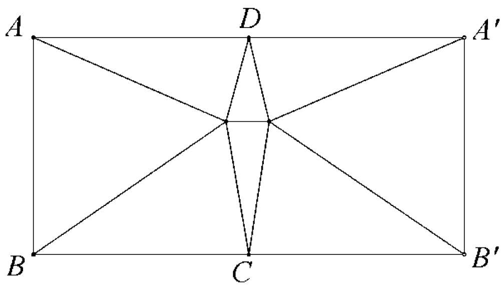
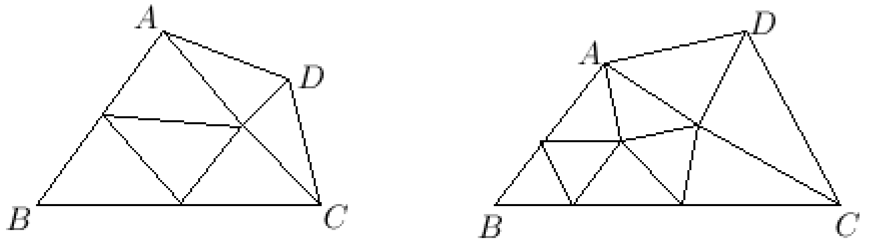


Fig. 1. – Γ is a rectangle.

THEOREM. – *Every doubly covered convex quadrilateral admits an acute triangulation with size at most 20.*

PROOF. – If $\Gamma = ABCD$ is a rectangle, then Γ_d admits an acute triangulation with size 8, as shown in Figure 1, where the two "sides" of Γ_d are unfolded on a plane.

Now suppose that Γ is not a rectangle. Then it has at least one acute corner, say B.



(a) Triangle ABC is acute (b) Triangle ABC is non-acute

Fig. 2. – Acute triangulations of a quadrilateral *ABCD*.

If $\angle D \geq \frac{\pi}{2}$, then by Proposition 2 Γ admits an acute triangulation $\mathcal{T}$ with size at most 9 [10], in which the only vertex of degree less than 3 is B (Figure 2). Thus Γ_d can be divided into at most 18 acute triangles. Notice that this division is not a triangulation. Choose the vertices F, G of $\mathcal{T}$ such that both BF and FG are edges of $\mathcal{T}$ lying in bdΓ and F is a side vertex. We slightly slide F into the interior of Γ such that all the triangles in Γ having F as a vertex remain acute, and both BFF' and GFF' are acute as well. Thus Γ_d admits an acute triangulation of size at most 20.

If $\angle D < \frac{\pi}{2}$, we first prove that Γ can be triangulated into at most 8 acute triangles such that there are at most 2 vertices with degree 2.

If both triangles ABC and ACD are acute, then clearly Γ can be triangulated into 2 acute triangles, and only B and D have degree 2.

However, in order to obtain a triangulation of Γ_d using the construction above, we have to introduce a new vertex on AC, which yields by Proposition 1 acute triangulations of size 4 for each of the triangles ABC and ACD. In this triangulation of $ABCD$, again, only B and D have degree 2.

If one and only one of them is acute, we may assume without loss of generality that the triangle ABC is acute and $\angle ACD \geq \frac{\pi}{2}$. Let E be the orthogonal

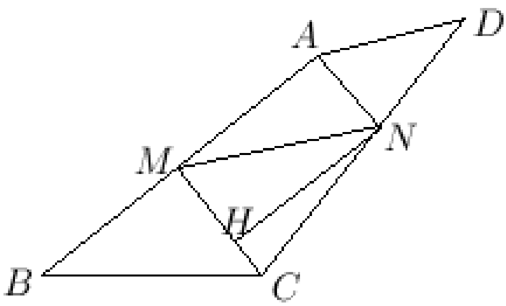


Fig. 3. – Both *ABC* and *ACD* are non-acute

projection of C on the side AD and F be the orthogonal projection of E on the side AC. Clearly $E \in \mathrm{relint} AD$ and $F \in \mathrm{relint} AC$. By Proposition 1, ABC can be triangulated into 4 acute triangles such that F is the only side vertex on AC. Now we slightly slide F in direction opposite to E and E in direction of A and obtain an acute triangulation of Γ of size 7, in which only the vertices B and D have degree 2.

If both triangles ABC and ACD are non-acute, we may assume without loss of generality that $\angle ACB \geq \frac{\pi}{2}$, $\angle CAD \geq \frac{\pi}{2}$, and the lines including BA and CD are parallel or intersect at some point closer to A than to B. Let M be the orthogonal projection of C on the side AB. Choose $N \in CD$ such that AN and MC are parallel. Then the line-segments AN, MN and CM divide $ABCD$ into 4 triangles, as shown in Figure 3. Firstly, we slightly slide M towards A and N towards C such that all triangles become acute. Let H be a point on MC such that $\angle NHC = \frac{\pi}{2}$. Clearly $H \in \mathrm{relint} CM$. By Proposition 1, there is an acute triangulation of the acute triangle BMC with size 4 such that H is the only side vertex lying on MC. After slightly sliding H opposite to N, we obtain an acute triangulation of $ABCD$ with size 8. Noticing that there are at most 2 vertices with degree 2 in each triangulation, we can conclude that Γ_d admits an acute triangulation with size at most 20. $\square$

REFERENCES

[1] B. S. Baker - E. Grosse - C. S. Rafferty, *Nonobtuse triangulations of polygons,* Discrete Comput. Geom., **3** (1988), 147-168.

[2] M. Bern - S. Mitchell - J. Ruppert, *Linear-size nonobtuse triangulations of polygons,* Discrete Comput. Geom., **14** (1995), 411-428.

[3] Y. D. Burago - V. A. Zalgaller, *Polyhedral embedding of a net (Russian), Vestnik Leningrad. Univ.*, **15** (1960), 66-80.

[4] C. Cassidy - G. Lord, *A square acutely triangulated*, J. Recreational. Math., **13** (1980/81), 263-268.

[5] M. Gardner, *New Mathematical Diversions*, Mathematical Association of America, Washington D.C., 1995.

[6] M. Goldberg, *Problem E1406: Dissecting an obtuse triangle into acute triangles*, American Mathematical Monthly, **67** (1960), 923.

[7] T. Hangan - J. Itoh - T. Zamfirescu, *Acute triangulations*, Bull. Math. Soc. Sci. Math. Roumanie, **43** (91) No. 3-4 (2000), 279-285.

[8] J. Itoh - T. Zamfirescu, *Acute triangulations of the regular dodecahedral surface*, Europ. J. Combinatorics, to appear.

[9] J. Itoh - T. Zamfirescu, *Acute triangulations of the regular icosahedral surface, Discrete Comput. Geom.*, **31** (2004), 197-206.

[10] H. Maehara, *On acute triangulations of quadrilaterals,* Proceedings of JCDCG 2000, Lecture Notes in Computer Science, **2098** (2001), 237-354.

[11] H. Maehara, *Acute triangulations of polygons*, Europ. J. Combinatorics, **23** (2002), 45-55.

[12] L. YUAN, *Acute triangulations of polygons*, Discrete Comput. Geom., **34** (2005), 697-706.
[13] C. ZAMFIRESCU, *Acute triangulations of the double triangle*, Bull. Math. Soc. Sci. Math. Roumanie, **47**, No. 3-4 (2004), 189-193.
[14] T. ZAMFIRESCU, *Acute triangulations: a short survey*, Proc. 6th Annual Conference Romanian Soc. Math. Sciences, **I** (2002), 10-18.

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