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Minimum Free Energy for a Rigid Heat Conductor and Application to a Discrete Spectrum Model

GIOVAMBATTISTA AMENDOLA - ADELE MANES

Sunto. – *Si considera il problema di trovare una espressione dell'energia libera minima per un conduttore di calore rigido e con memoria. Tale energia fornisce il massimo lavoro ottenibile dal materiale in un dato stato, caratterizzato in questo lavoro dalla temperatura e dalla storia passata del gradiente di questa. Una equivalente espressione viene ottenuta e applicata al particolare caso di un conduttore con spettro discreto.*

Summary. – *A general closed expression is given for the minimum free energy for a rigid heat conductor with memory effects. This formula, derived in the frequency domain, is related to the maximum recoverable work we can obtain from the material at a given state, which is characterized by the temperature and the past history of its gradient. Another explicit formula of the minimum free energy is also derived and used to obtain the results related to the particular case of a discrete spectrum model material response.*

1. – Introduction.

A generalization of the results derived by Cattaneo [4] to remove the paradox of the instantaneous heat propagation was proposed by Gurtin and Pipkin in [15]. They, using Coleman's results for materials with memory [5], proposed a non linear model for a rigid heat conductor and also considered the linearization of this theory, which yields a constitutive equation for the heat flux expressed in terms of the history of the temperature gradient.

This linearized equation has been considered by many authors to study problems connected with heat propagation. Among all the works about these investigations (see, for example, [19]) we recall, in particular, the results derived in [8], where stability and domain of dependence are established by using the maximal free energy and the maximal free entalpy functions, there explicitly constructed by means of an approximate theory of thermodynamics. Always in [8], following [16], the authors have used the integrated history of the temperature gradient, already considered in [15], to describe the states and, in particular, the temperature gradient to characterize the processes of the body.

This choice of states and processes for the rigid heat conductor has been also used in a recent article [2] to derive an explicit expression of the minimum free energy. It is well known that the minimum free energy has a great importance since it is related to the maximum recoverable work, that is the work we can obtain from a given state of the material. Therefore, such a problem has been considered by several authors, in particular, for viscoelastic solids (see, for example, [3,7] and [9-14]). Among all these articles we remember [14] and [12] because of the interesting methods there used for such investigations.

In this work we assume that the states of the rigid heat conductor are expressed by means of the past history of the temperature gradient in the place of its integrated history, to give a different formulation of the problem of finding an explicit form of the minimum free energy. The technique used to obtain such a formula starts from the study of a Wiener-Hopf integral equation, which can be solved by virtue of the thermodynamic properties of the kernel, present in the functional of the heat flux, and of some theorems on the factorization of the same kernel. Two different expressions of the minimum free energy are thus deduced and one of these is used to derive explicit formulae for the particular case of a discrete spectrum model material response.

The layout of the paper is as follows. In Sect. 2, fundamental relationships are written down. In Sect. 3, states and processes are defined as well as the continuation of histories, which allows us to introduce the notion of equivalence between histories of the temperature gradient. In Sect. 4, the thermal work is considered and some expressions corresponding to particular cases are derived; thus, another definition of equivalence between histories is given by using the boundedness of the thermal work. In Sect. 5, the maximum recoverable work is evaluated, while, in Sect. 6, a different expression is derived for it. Finally, in Sect. 7, the results for a discrete spectrum material are presented.

2. – Notations and preliminaries.

Let $\mathcal{B}$ be a rigid heat conductor with memory, which occupies a bounded and regular region $\Omega \subset \boldsymbol{R}^3$, that is simply-connected with a smooth boundary. Within the linear theory of thermodynamics developed in [15] and studied also in [8], we assume a linear relation between the internal energy e and the temperature ϑ, relative to the absolute reference temperature T_0 uniform in $\bar{\Omega}$,

$$e(\boldsymbol{x}, t) = a_0 \vartheta(\boldsymbol{x}, t) \tag{2.1}$$

and the following constitutive equation

$$\boldsymbol{q}(\boldsymbol{x}, t) = -\int_0^{+\infty} k(s)\, \boldsymbol{g}(\boldsymbol{x}, t-s) ds \tag{2.2}$$

for the heat flux $\boldsymbol{q}$, whose memory effects are expressed by means of the history up to time $t \in \boldsymbol{R}^+ = [0, +\infty)$ of the temperature gradient $\boldsymbol{g} = \nabla\vartheta$.

In these relations $\boldsymbol{x} \in \Omega$ is the position vector, a_0 is a constant, positive because of physical observation, and the heat flux relaxation function $k : \boldsymbol{R}^+ \to \boldsymbol{R}$, such that $k \in L^1(\boldsymbol{R}^+) \cap H^1(\boldsymbol{R}^+)$, is defined by

$$k(t) = k_0 + \int_0^t \dot{k}(\tau)d\tau \qquad \forall t \in \boldsymbol{R}^+, \tag{2.3}$$

where $k_0 = k(0)$ is its initial value, at time $t = 0$, with $\lim_{t\to+\infty} k(t) = 0$ [8,13].

Henceforth, we shall understand that the statements are relative to any fixed $\boldsymbol{x} \in \Omega$; moreover, we shall identify the history up to time t with the couple $(\boldsymbol{g}(t), \boldsymbol{g}^t)$, where $\boldsymbol{g}(t)$ is the present value of the temperature gradient, while $\boldsymbol{g}^t(s) = \boldsymbol{g}(t - s)$ $\forall s \in \boldsymbol{R}^{++} = (0, +\infty)$ will denote its past history.

The static continuations of any history $(\boldsymbol{g}(t), \boldsymbol{g}^t)$, with duration a, defined by

$$\boldsymbol{g}^{t_{(a)}} = \begin{cases} \boldsymbol{g}^t(s - a) & \forall s > a, \\ \boldsymbol{g}(t) & \forall s \in [0, a], \end{cases} \tag{2.4}$$

yields the following heat flux

$$\boldsymbol{q}(t + a) = -\nu(a)\boldsymbol{g}(t) - \int_0^{+\infty} k(a + \rho)\boldsymbol{g}^t(\rho)d\rho, \tag{2.5}$$

where we have introduced the thermal conductivity

$$\nu(t) = \int_0^t k(\xi)d\xi. \tag{2.6}$$

We observe that in (2.5) we have $\boldsymbol{g}(t)$ because of the supposed static continuation of this value.

The thermodynamical phenomena can be specified by the introduction of the following vectorial space of the possible temperature gradients

$$\Gamma = \left\{ \boldsymbol{g}^t : (0, +\infty) \to \boldsymbol{R}^3; \quad \left| \int_0^{+\infty} k(\tau + s)\boldsymbol{g}^t(s)ds \right| < +\infty \ \forall \tau \in \boldsymbol{R}^+ \right\}, \tag{2.7}$$

where t is a parameter.

We recall the restriction imposed on the constitutive equation (2.2) by the laws of thermodynamics [8,13], that is

$$\int_0^{+\infty} k(s)\cos(\omega s)ds > 0 \qquad \forall \omega \in \boldsymbol{R}\backslash\{0\}. \tag{2.8}$$

For our considerations the Fourier transform will play an important role. Thus, we remember that for any function $f :\rightarrow \boldsymbol{R} \rightarrow \boldsymbol{R}^n$ its Fourier transform is defined by

$$(2.9)\quad f_F(\omega) = \int_{-\infty}^{+\infty} f(s)e^{-i\omega s}ds = \int_{-\infty}^{0} f(s)e^{-i\omega s}ds + \int_{0}^{+\infty} f(s)e^{-i\omega s}ds = f_-(\omega) + f_+(\omega);$$

moreover, we recall that if functions are defined on $\boldsymbol{R}^+$, these can be identified with their causal extension on $\boldsymbol{R}$, that is functions which vanish for any $s \in \boldsymbol{R}^{--} = (-\infty, 0)$, and we have

$$(2.10)\qquad f_F(\omega) = f_c(\omega) - if_s(\omega),$$

where we have introduced the half-range Fourier cosine and sine transforms

$$(2.11)\qquad f_c(\omega) = \int_0^{+\infty} f(\xi)\cos(\omega\xi)d\xi, \qquad f_s(\omega) = \int_0^{+\infty} f(\xi)\sin(\omega\xi)d\xi.$$

In particular, if f and f' belong to $L^1(\boldsymbol{R}^+) \cap L^2(\boldsymbol{R}^+)$, we have

$$(2.12)\qquad f'_s(\omega) = -\omega f_c(\omega).$$

Using $(2.11)_1$, the thermodynamic restriction (2.8) assumes the form

$$(2.13)\qquad k_c(\omega) > 0 \qquad \forall \omega \in \boldsymbol{R},$$

under the hypothesis that the quantity

$$(2.14)\qquad k_c(0) = \int_0^{+\infty} k(\xi)d\xi \equiv v_\infty > 0,$$

where $v_\infty = \lim_{t\rightarrow+\infty} v(t)$ is not equal to zero.

We observe that the asymptotic value of v in (2.14) gives the heat flux due to the constant past history $\boldsymbol{g}^t(s) = \boldsymbol{g}^\dagger(s) = \boldsymbol{g}\ \forall s > 0$, since (2.2) and (2.6) yield

$$(2.15)\qquad \boldsymbol{q}(t) = -v_\infty \boldsymbol{g} = -k_c(0)\boldsymbol{g},$$

from which it follows that the constant heat flux so obtained has the opposite versus of $\boldsymbol{g}$.

We recall that if $k'' \in L^2(R^+)$ and $|\boldsymbol{k}'(0)| < +\infty$ we also have

$$(2.16)\qquad \sup_{\omega\in\boldsymbol{R}} |\omega k'_s(\omega)| < +\infty, \qquad \lim_{\omega\rightarrow\infty} \omega k'_s(\omega) = -\lim_{\omega\rightarrow\infty} \omega^2 k_c(\omega) = k'(0) < 0,$$

where we have assumed $k'(0) \neq 0$.

Finally, the inverse Fourier transform yields the following results

$$(2.17)\qquad k(t) = \frac{2}{\pi}\int_0^{+\infty} k_c(\omega)\cos(\omega t)d\omega, \qquad k(0) = \frac{2}{\pi}\int_0^{+\infty} k_c(\omega)d\omega > 0.$$

It is interesting to consider the functions $f_{\pm}$, introduced in (2.9), defined also for any $z \in \boldsymbol{C}$, the complex plane. Such functions are analytic for $z \in \boldsymbol{C}^{(\mp)}$, which exclude the real axis, but they are supposed, by assumption [14], analytic also on $\boldsymbol{R}$ and hence for any $z \in \boldsymbol{C}^{\mp}$, defined by

$$\text{(2.18)} \qquad \boldsymbol{C}^{\pm} = \left\{ z \in \boldsymbol{C}: \ \operatorname{Im} z \in \boldsymbol{R}^{\pm} \right\}, \quad \boldsymbol{C}^{(\pm)} = \left\{ z \in \boldsymbol{C}: \ \operatorname{Im} z \in \boldsymbol{R}^{\pm\pm} \right\}.$$

Thus, we shall denote by $f_{(\pm)}(z)$ a function with zeros and singularities in $\boldsymbol{C}^{\pm}$.

3. – States and processes.

The constitutive equations (2.1) and (2.2) allow us to consider the body $\mathcal{B}$ as a simple material [5,6], whose behaviour can be described in terms of states and processes. Thus, $\mathcal{B}$ can be characterized by means of the function

$$\text{(3.1)} \qquad \sigma(t) = (\vartheta(t), \boldsymbol{g}^t),$$

which is termed the thermodynamic state at time t and at any fixed point $\boldsymbol{x} \in \Omega$. We denote by Σ the set of the states of $\mathcal{B}$.

A kinetic process of duration $d \in \boldsymbol{R}^+$ is a map P piecewise continuous on the time interval $[0, d)$ and defined by

$$\text{(3.2)} \qquad P(\tau) = (\dot{\vartheta}_P(\tau), \boldsymbol{g}_P(\tau)) \qquad \forall \tau \in [0, d).$$

Let $P \in \Pi$, the set of all the processes, be a process of duration d, we can consider its restriction to any interval $[\tau_1, \tau_2) \subset [0, d)$. This restriction is denoted by $P_{[\tau_1,\tau_2)}$ and belongs to Π. Given an initial state $\sigma^i \in \Sigma$ and a process $P \in \Pi$, the function $\rho : \Sigma \times \Pi \to \Sigma$, defined by $\sigma^f = \rho(\sigma^i, P) \in \Sigma$, is called the evolution function which maps (σ^i, P) into the final state $\sigma^f \in \Sigma$.

We observe that the heat flux (2.2) depends on the past history of the temperature gradient; the present value of this quantity characterizes the process P, defined by (3.2), for $\tau = 0$.

Given an initial state, if we apply a process P, we obtain, in particular, a prolongation of the past history of $\boldsymbol{g}$.

Firstly, we consider the case when a process is applied at time $t = 0$. Thus, we have the process $P(t) = (\dot{\vartheta}_P(t), \boldsymbol{g}_P(t)) \ \forall \ t \in [0, d)$, since $\tau \equiv t \in [0, d)$, and the initial state $\sigma(0) = (\vartheta_*(0), \boldsymbol{g}_*^0)$, which yields the temperature and the past history of its gradient at time $t = 0$. This process induces a set of states denoted by $\sigma(t) = (\vartheta(t), \boldsymbol{g}^t)$ and defined by

$$\text{(3.3)} \qquad \vartheta(t) = \vartheta_*(0) + \int_0^t \dot{\vartheta}_P(s) ds, \qquad \boldsymbol{g}^t(s) = \begin{cases} \boldsymbol{g}_P(t-s) & 0 < s \le t, \\ \boldsymbol{g}_*^0(s-t) & s > t. \end{cases}$$

Now, we consider the process P applied at time $t > 0$. Denoting by $\sigma(t) = (\vartheta(t), \boldsymbol{g}^t)$ the initial state, the process $P(\tau) = (\dot{\vartheta}_P(\tau), \boldsymbol{g}_P(\tau))$, defined for any $\tau \in [0, d)$, is related to

$$\vartheta_P : (0, d] \to \boldsymbol{R}, \qquad \vartheta_P(\tau) = \vartheta(t) + \int_0^{\tau} \dot{\vartheta}_P(\xi) d\xi, \tag{3.4}$$

while the continuation of the past history of the temperature gradient is defined by means of the final value $\boldsymbol{g}_f(\tau') = (\boldsymbol{g}_P * \boldsymbol{g})(\tau')$, $\forall \tau' \equiv t + \tau < t + d$ and given by

$$\boldsymbol{g}_f\,(t + d - s) = (\boldsymbol{g}_P * \boldsymbol{g})(t + d - s) = \begin{cases} \boldsymbol{g}_P(d - s) & 0 < s \leq d, \\ \boldsymbol{g}(t + d - s) & s > d, \end{cases} \tag{3.5}$$

because of the function

$$\boldsymbol{g}_P : [0, d) \to \boldsymbol{R}^3, \qquad \boldsymbol{g}_P(\tau) = \boldsymbol{g}(t + \tau) \tag{3.6}$$

assigned by P.

The constitutive equation (2.2) yields the linear functional $\tilde{\boldsymbol{q}} : \Gamma \to \boldsymbol{R}^3$, which gives the possible heat fluxes corresponding to the past histories of the temperature gradient

$$\tilde{\boldsymbol{q}}\,(\boldsymbol{g}^t) = - \int_0^{+\infty} k(s) \boldsymbol{g}^t(s) ds, \tag{3.7}$$

where $\boldsymbol{g}^t \in \Gamma$.

DEFINITION 3.1. – *Let $\boldsymbol{g}_j^t$, $j = 1, 2$, be two past histories of the temperature gradient, corresponding to the same instantaneous value $\vartheta(t)$ of the temperature. They are said to be equivalent if for every $\boldsymbol{g}_P : [0, \tau) \to \boldsymbol{R}^3$ and for every $\tau > 0$ we have*

$$\tilde{\boldsymbol{q}}\,((\boldsymbol{g}_P * \boldsymbol{g}_1)^{t+\tau}) = \tilde{\boldsymbol{q}}\,((\boldsymbol{g}_P * \boldsymbol{g}_2)^{t+\tau}) \tag{3.8}$$

whatever may be $\dot{\vartheta}_P : [0, \tau) \to \boldsymbol{R}$.

It is interesting to consider the zero past history $\boldsymbol{g}^t(s) = \boldsymbol{0}^{\dagger}(s) = \boldsymbol{0}\ \forall s > 0$. Any past history $\boldsymbol{g}^t$ is equivalent to it if

$$\int_{\tau}^{+\infty} k(s) \boldsymbol{g}(t + \tau - s) ds \equiv \int_0^{+\infty} k(\tau + \xi) \boldsymbol{g}^t(\xi) d\xi = \boldsymbol{0}. \tag{3.9}$$

Thus, we see that two past histories $\boldsymbol{g}_j^t$, $j = 1, 2$, are equivalent in the sense of Definition 3.1 if the past history $\boldsymbol{g}^t = \boldsymbol{g}_1^t - \boldsymbol{g}_2^t$ satisfies (3.9) and hence we have the same state with the fixed value of $\vartheta(t)$, according Noll's definition of state [18].

4. – Thermal work.

We are now concerned with the notion of work. This quantity for our rigid heat conductor, if associated to a process $P(\tau) = (\dot{\vartheta}_P(\tau), \boldsymbol{g}_P(\tau))$ $\forall \tau \in [0, d)$ applied at time t, when the state is $\sigma(t) = (\vartheta(t), \boldsymbol{g}^t)$, is given by the following functional

$$\tilde{W}(\boldsymbol{g}^t, \boldsymbol{g}_P) = -\int_0^d \tilde{\boldsymbol{q}}((\boldsymbol{g}_P * \boldsymbol{g})^{t+\tau}) \cdot \boldsymbol{g}_P(\tau) d\tau = -\int_t^{t+d} \tilde{\boldsymbol{q}}((\boldsymbol{g}_P * \boldsymbol{g})^{\xi}) \cdot \boldsymbol{g}(\xi) d\xi, \tag{4.1}$$

which can be also written as follows

$$\tilde{W}(\boldsymbol{g}^t, \boldsymbol{g}_P) = -\int_0^d \boldsymbol{q}(t+\tau) \cdot \boldsymbol{g}_P(\tau) d\tau, \tag{4.2}$$

where $\boldsymbol{g}_P(\tau)$ is given by (3.6) and, by virtue of (2.2),

$$\boldsymbol{q}(t+\tau) = -\int_0^{+\infty} k(s) \boldsymbol{g}^{t+\tau}(s)\, ds. \tag{4.3}$$

If we consider the state $\sigma(0) = (0, \mathbf{0}^\dagger)$ as the initial one, to which a process $P = (\dot{\vartheta}_P, \boldsymbol{g}_P)$ of duration d is applied at time $t = 0$, for simplicity, the ensuing fields $(\vartheta_0(t), \boldsymbol{g}_0^t)$ with $t \in (0, d]$ are given by (3.4)-(3.5) and have the form

$$\vartheta_0(t) = \int_0^t \dot{\vartheta}_P(\xi) d\xi, \qquad \boldsymbol{g}_0^t(s) = \begin{cases} \boldsymbol{g}_P(t-s) & 0 < s \leq t, \\ \mathbf{0} & s > t. \end{cases} \tag{4.4}$$

The work done on P is given by (4.1) or (4.2) and, taking account of (3.7), assumes the form

$$\tilde{W}(\mathbf{0}, \boldsymbol{g}_P) = -\int_0^d \tilde{\boldsymbol{q}}(\boldsymbol{g}_0^t) \cdot \boldsymbol{g}_0(t) dt, \tag{4.5}$$

where $\boldsymbol{g}_0(t) \equiv \boldsymbol{g}_P(t)$ is assigned by means of P.

This quantity allows us to introduce the notion of finite work process, as it has been defined by Gentili in [12].

DEFINITION 4.1. – *A process P of duration d is said to be a finite work process if the work* (4.5) *satisfies*

$$\tilde{W}(\mathbf{0}, \boldsymbol{g}_P) < +\infty. \tag{4.6}$$

Moreover, for such a work we have the following result.

LEMMA 4.1. – *The work done on any finite work process, in the sense of Definition* 4.1, *is positive.*

PROOF. – Let P a finite work process, then (4.6) is satisfied. Extending P on $\boldsymbol{R}^+$ by putting $\boldsymbol{g}_P(\tau) = \boldsymbol{0}\ \forall \tau \geq d$ and using (3.7), from (4.5) we get

$$\tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P) = \int_0^{+\infty} \int_0^{+\infty} k(s)\boldsymbol{g}_0^t(s)ds \cdot \boldsymbol{g}_0(t)dt \tag{4.7}$$

and hence, by virtue of Plancherel's theorem, we obtain

$$\tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} k_F(\omega)\boldsymbol{g}_{0_F}(\omega) \cdot \boldsymbol{g}^*_{0_F}(\omega)d\omega, \tag{4.8}$$

where * denotes the complex conjugate. In (4.8) we have the Fourier transforms of functions which are defined on $\boldsymbol{R}^+$ and can be considered equal to zero on $\boldsymbol{R}^{--}$. Therefore, taking into account (2.10), where the cosine and sine transforms are even and odd functions, (4.8) reduces to

$$\tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} k_c(\omega)[\boldsymbol{g}^2_{0_c}(\omega) + \boldsymbol{g}^2_{0_s}(\omega)]d\omega > 0, \tag{4.9}$$

where the inequality follows by virtue of (2.13). □

Let us consider the work done on an assigned process P of duration d, given by (4.5) but also expressed by (4.7) when P is extended on $\boldsymbol{R}^+$ by putting $\boldsymbol{g}_P(\tau) = \boldsymbol{0}\ \forall \tau \geq d$. Thus, we obtain

$$\begin{aligned} \tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P) &= \int_0^{+\infty} \int_0^{\eta} k(s)\boldsymbol{g}_P(\eta - s)ds \cdot \boldsymbol{g}_P(\eta)d\eta \\ &= \int_0^{+\infty} \int_0^{\eta} k(\eta - \rho)\boldsymbol{g}_P(\rho)d\rho \cdot \boldsymbol{g}_P(\eta)d\eta, \end{aligned} \tag{4.10}$$

since we have the null histories for $s > \eta$. The expression now derived can be equivalently written as follows

$$\tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P) = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} k(|\eta - \rho|)\boldsymbol{g}_P(\rho) \cdot \boldsymbol{g}_P(\eta)d\rho d\eta \tag{4.11}$$

or, by applying Plancherel's theorem and recalling that the Fourier transform of an even function f is $f_F(\omega) = 2f_c(\omega)$, also in the form

$$\tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} k_c(\omega)\boldsymbol{g}_+(\omega) \cdot \boldsymbol{g}^*_+(\omega)d\omega, \tag{4.12}$$

where, using $(2.9)_3$, we have put $\boldsymbol{g}_+(\omega) = \int_0^{+\infty} \boldsymbol{g}_P(s)e^{-i\omega s}ds$.

This allows us to characterize the set of the finite work processes by means of

$$\tilde{H}(\mathbf{R}^+,\mathbf{R}^3) = \left\{ \mathbf{g} : \mathbf{R}^+ \to \mathbf{R}^3; \quad \int_{-\infty}^{+\infty} k_c(\omega)\,\mathbf{g}_+(\omega)\cdot\mathbf{g}_+^*(\omega)d\omega < +\infty \right\}, \tag{4.13}$$

which, by virtue of (2.13), can be endowed with the inner product $(\mathbf{g}_1,\mathbf{g}_2)_k = \int_{-\infty}^{+\infty} k_c(\omega)\,\mathbf{g}_{1+}(\omega)\cdot\mathbf{g}_{2+}^*(\omega)d\omega$ and the corresponding norm $\|\mathbf{g}\|_k^2 = (\mathbf{g},\mathbf{g})_k$. The completion with respect to such a norm of $\tilde{H}(\mathbf{R}^+,\mathbf{R})$ yields the Hilbert space $H(\mathbf{R}^+,\mathbf{R}^3)$ of the processes.

Thus, we can consider the set of the past histories as the set of all $\mathbf{g}^t$ such that the work done on any process, characterized by $\mathbf{g}_P \in H(\mathbf{R}^+,\mathbf{R}^3)$, starting from the state corresponding to them, is positive. This work, always supposing that the process $P(\tau)$ be zero for any $\tau \geq d$ ($d < +\infty$), is given by (4.1), which, by virtue of (3.7) and (3.5), becomes

$$\begin{aligned}
\tilde{W}(\mathbf{g}^t,\mathbf{g}_P) &= -\int_0^{+\infty} \tilde{\mathbf{q}}((\mathbf{g}_P * \mathbf{g})^{t+\tau})\cdot\mathbf{g}_P(\tau)d\tau \\
&= \int_0^{+\infty}\left[\int_0^{\tau} k(s)\,\mathbf{g}_P(\tau-s)ds + \int_{\tau}^{+\infty} k(s)\mathbf{g}(t+\tau-s)ds\right]\cdot\mathbf{g}_P(\tau)d\tau \\
&= \int_0^{+\infty}\left[\int_0^{\tau} k(\tau-\xi)\mathbf{g}_P(\xi)d\xi + \int_0^{+\infty} k(\tau+\eta)\mathbf{g}^t(\eta)d\eta\right]\cdot\mathbf{g}_P(\tau)d\tau.
\end{aligned} \tag{4.14}$$

This expression, putting

$$\mathbf{I}(\tau,\mathbf{g}^t) = -\int_0^{+\infty} k(\tau+\eta)\,\mathbf{g}^t(\eta)d\eta, \ \ \tau \geq 0, \tag{4.15}$$

can be written in the following form

$$\begin{aligned}
\tilde{W}(\mathbf{g}^t,\mathbf{g}_P) =& \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} k(|\tau-\xi|)\mathbf{g}_P(\xi)d\xi\cdot\mathbf{g}_P(\tau)d\tau \\
&- \int_0^{+\infty} \mathbf{I}(\tau,\mathbf{g}^t)\cdot\mathbf{g}_P(\tau)d\tau
\end{aligned} \tag{4.16}$$

or, using Plancherel's theorem, equivalently, by

$$\tilde{W}(\mathbf{g}^t,\mathbf{g}_P) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[k_c(\omega)\,\mathbf{g}_+(\omega) - \mathbf{I}_+(\omega,\mathbf{g}^t)\right]\cdot\mathbf{g}_+^*(\omega)d\omega, \tag{4.17}$$

where $\mathbf{I}_+(\omega,\mathbf{g}^t) = \int_0^{+\infty} \mathbf{I}(\tau,\mathbf{g}^t)e^{-i\omega\tau}d\tau$.

We observe that the quantity $\boldsymbol{I}(\tau, \boldsymbol{g}^t)$, now introduced in (4.15), is related to the heat flux obtained by means of a static continuation of the history $(\boldsymbol{g}(t), \boldsymbol{g}^t)$, already examined in Sect. 2. This heat flux is expressed by (2.5), which reduces to

$$\boldsymbol{q}(\boldsymbol{g}^{t_{(\tau)}}) + \nu(\tau)\boldsymbol{g}(t) = \boldsymbol{I}(\tau, \boldsymbol{g}^t) \tag{4.18}$$

when the duration of the continuation is equal to τ; moreover, we observe that $\boldsymbol{I}(\tau, \boldsymbol{g}^t)$ has the same regularities of the functions at the left hand-side of (4.18).

The comparison between the two expressions (4.16) and (4.11) shows that the boundedness of $\tilde{W}(\boldsymbol{0}, \boldsymbol{g}_P)$ yields that one of the first term of $\tilde{W}(\boldsymbol{g}^t, \boldsymbol{g}_P)$; therefore, it rests to consider the second term in (4.16). This quantity must be finite; thus, the possible past histories are those such that $\boldsymbol{I}(\cdot, \boldsymbol{g}^t)$, to which are related by means of (4.15), belong to $H'(\boldsymbol{R}^+, \boldsymbol{R}^3)$, the dual space of $H(\boldsymbol{R}^+, \boldsymbol{R}^3)$, whose functions satisfy

$$|<\boldsymbol{f}, \boldsymbol{g}>| = \left| \int_0^{+\infty} \boldsymbol{f}(t) \cdot \boldsymbol{g}(t) dt \right| = \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} \boldsymbol{f}_+(\omega) \cdot \boldsymbol{g}_+^*(\omega) d\omega \right| < +\infty. \tag{4.19}$$

The equivalence between past histories of $\boldsymbol{g}$ can be expressed in terms of work.

DEFINITION 4.2. – *Two past histories $\boldsymbol{g}_j^t$, $j = 1, 2$, corresponding to the same value of the temperature $\vartheta(t)$, are termed w-equivalent if, for any $P_{[0,\tau)}$ and for any $\tau > 0$,*

$$\tilde{W}(\boldsymbol{g}_1^t, \boldsymbol{g}_P) = \tilde{W}(\boldsymbol{g}_2^t, \boldsymbol{g}_P). \tag{4.20}$$

This equivalence coincides with the one introduced by Definition 3.1.

THEOREM 4.1. – *Two past histories of the temperature gradient are w-equivalent if and only if they are equivalent in the sense of Definition* 3.1.

PROOF. – If two past histories $\boldsymbol{g}_j^t$, $j = 1, 2$, are equivalent, then (3.8) holds for any process $P_{[0,\tau)}$ and hence also (4.20) is satisfied for any $\tau > 0$ and any $\boldsymbol{g}_P(\tau)$, since

$$\int_0^d \tilde{\boldsymbol{q}}((\boldsymbol{g}_P * \boldsymbol{g}_1)^{t+\tau}) \cdot \boldsymbol{g}_P(\tau) d\tau = \int_0^d \tilde{\boldsymbol{q}}((\boldsymbol{g}_P * \boldsymbol{g}_2)^{t+\tau}) \cdot \boldsymbol{g}_P(\tau) d\tau; \tag{4.21}$$

therefore, they are w-equivalent.

On the other hand, if $\boldsymbol{g}_j^t$, $j = 1, 2$, are w-equivalent, then (4.20) holds for any $P_{[0,\tau)}$ and $\tau > 0$. Taking into account (4.16), written for the past histories $\boldsymbol{g}_j^t$, $j = 1, 2$, we see that the integral with $k(|\tau - \xi|)$ coincides for the expressions of the two works and, therefore, these quantities eliminate each other. Thus, (4.20) reduces to

$$\int_0^{+\infty} [\boldsymbol{I}(\tau, \boldsymbol{g}_2^t) - \boldsymbol{I}(\tau, \boldsymbol{g}_1^t)] \cdot \boldsymbol{g}_P(\tau) d\tau = 0, \tag{4.22}$$

whence, the arbitrariness of $\boldsymbol{g}_P$ yields

$$\boldsymbol{I}(\tau, \boldsymbol{g}_1^t) = \boldsymbol{I}(\tau, \boldsymbol{g}_2^t) \qquad \forall \tau > 0, \tag{4.23}$$

or, equivalently, because of (4.15),

$$\int_0^{+\infty} k(\tau + \eta)[\boldsymbol{g}_2^t(\eta) - \boldsymbol{g}_1^t(\eta)] d\eta = \boldsymbol{0} \qquad \forall \tau > 0. \tag{4.24}$$

This last relation expresses that the difference $\boldsymbol{g}^t(\eta) = \boldsymbol{g}_2^t(\eta) - \boldsymbol{g}_1^t(\eta)$ satisfies $(3.9)_2$ and hence that $\boldsymbol{g}_1^t(\eta)$ is equivalent to $\boldsymbol{g}_2^t(\eta)$. □

5. – Maximum recoverable work.

Let σ be a given state of the material, the maximum recoverable work we can obtain starting from σ is so defined

$$W_R(\sigma) = \sup\{-W(\sigma, P) : \ P \in \Pi\}, \tag{5.1}$$

where Π denotes the set of finite work processes [9].

We note that $W_R(\sigma)$ gives the amount of the available energy; moreover, since the null process belongs to Π and its corresponding work is null, $W_R(\sigma)$ is a non-negative function of the state and it is bounded from above, $W_R(\sigma) < +\infty$, by virtue of thermodynamic considerations.

Many authors [9, 11, 14] have shown that such a work coincides with the minimum free energy $\psi_m(\sigma)$, that is

$$\psi_m(\sigma) = W_R(\sigma). \tag{5.2}$$

Let $P \in \Pi$ be a process of duration $d < +\infty$ and applied to the initial state $\sigma(t) = (\vartheta(t), \boldsymbol{g}^t)$ at time t. We can extend P on $\boldsymbol{R}^+$ by means of its trivial extension on $[d, +\infty)$, where, therefore, $P = 0$. The work done on P is expressed by (4.16), that is

$$W(\sigma(t), P) = \int_0^{+\infty} \left[\frac{1}{2} \int_0^{+\infty} k(|\tau - \xi|) \boldsymbol{g}_P(\xi) d\xi - \boldsymbol{I}(\tau, \boldsymbol{g}^t) \right] \cdot \boldsymbol{g}_P(\tau) d\tau. \tag{5.3}$$

From this relation we must derive the maximum recoverable work, which, on the ground of (5.1), will correspond to an"optimal" process denoted by $P^{(m)}$. For this purpose, in any process P we put

$$\boldsymbol{g}_P(\tau) = \boldsymbol{g}^{(m)}(\tau) + \delta \boldsymbol{v}(\tau) \qquad \forall \tau \in \boldsymbol{R}^+, \tag{5.4}$$

where $\boldsymbol{g}^{(m)}(\tau)$ denotes the temperature gradient related to the required process, δ is a real parameter and $\boldsymbol{v}$ is an arbitrary smooth functions such that $\boldsymbol{v}(0) = \boldsymbol{0}$. Substituting (5.4) into (5.3), we get an expression which yields

$$\frac{\partial}{\partial \delta}[-\tilde{W}(\sigma, P)]|_{\delta=0} = -\int_0^{+\infty} \left[\int_0^{+\infty} k(|\tau-\xi|)\boldsymbol{g}^{(m)}(\xi)d\xi - \boldsymbol{I}(\tau, \boldsymbol{g}^t) \right] \cdot \boldsymbol{v}(\tau)d\tau = 0 \tag{5.5}$$

and hence for the arbitrariness of $\boldsymbol{v}$ we obtain

$$\int_0^{+\infty} k(|\tau-\xi|)\boldsymbol{g}^{(m)}(\xi)d\xi = \boldsymbol{I}(\tau, \boldsymbol{g}^t) \qquad \forall \tau \in \boldsymbol{R}^+. \tag{5.6}$$

This relation is an integral equation of the Wiener-Hopf type of the first kind, whose solution $\boldsymbol{g}^{(m)}$ yields the required maximum recoverable work. Using (5.2)-(5.3), we have

$$\psi_m(\sigma) = W_R(\sigma) = \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} k(|\tau-\xi|)\boldsymbol{g}^{(m)}(\xi)\cdot \boldsymbol{g}^{(m)}(\tau)d\xi d\tau \tag{5.7}$$

or, equivalently, by virtue of Plancherel's theorem,

$$\psi_m(\sigma) = W_R(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} k_c(\omega)\boldsymbol{g}_+^{(m)}(\omega)\cdot \left(\boldsymbol{g}_+^{(m)}(\omega)\right)^* d\omega. \tag{5.8}$$

It remains to solve (5.6). For this purpose we introduce the function

$$\boldsymbol{r}(\tau) = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} k(|\tau-s|)\boldsymbol{g}^{(m)}(s)ds & \forall \tau \in \boldsymbol{R}^- \\ \boldsymbol{0} & \forall \tau \in \boldsymbol{R}^{++} \end{cases} \tag{5.9}$$

and observe that $supp(\boldsymbol{g}^{(m)}) \subseteq \boldsymbol{R}^+$, $supp(\boldsymbol{I}(\cdot, \boldsymbol{g}^t)) \subseteq \boldsymbol{R}^+$ and $supp(\boldsymbol{r}) \subseteq \boldsymbol{R}^-$; thus, (5.6) can be written as follows

$$\int_{-\infty}^{+\infty} k(|\tau-\xi|)\boldsymbol{g}^{(m)}(\xi)d\xi = \boldsymbol{I}(\tau, \boldsymbol{g}^t) + \boldsymbol{r}(\tau) \qquad \forall \tau \in \boldsymbol{R}. \tag{5.10}$$

Applying the Fourier transform to the last relation, we have

$$2k_c(\omega)\boldsymbol{g}_+^{(m)}(\omega) = \boldsymbol{I}_+^t(\omega, \boldsymbol{g}^t) + \boldsymbol{r}_-(\omega), \tag{5.11}$$

where, in particular, $\boldsymbol{g}_+^{(m)}$ is analytic in $\boldsymbol{C}^-$.

We now consider the function

$$K(\omega) = (1+\omega^2)k_c(\omega) \tag{5.12}$$

and observe that it has no zero for any real ω and also at infinity, by virtue of (2.16). Thus, we can factorize both $K(\omega)$ and $k_c(\omega)$, i.e.

$$K(\omega) = K_{(+)}(\omega)K_{(-)}(\omega), \qquad k_c(\omega) = k_{(+)}(\omega)k_{(-)}(\omega), \tag{5.13}$$

and hence we get

$$k_{(\pm)}(\omega) = \frac{K_{(\pm)}(\omega)}{1 \pm i\omega}. \tag{5.14}$$

Using (5.14), from (5.11) it follows that

$$k_{(+)}(\omega)\boldsymbol{g}_+^{(m)}(\omega) = \frac{1}{2k_{(-)}(\omega)}[\boldsymbol{I}_+^t(\omega,\boldsymbol{g}^t) + \boldsymbol{r}_-(\omega)]. \tag{5.15}$$

The Plemelj formulae [17] give for the first term at the right hand-side of (5.15)

$$\frac{1}{2}\frac{\boldsymbol{I}_+^t(\omega,\boldsymbol{g}^t)}{k_{(-)}(\omega)} = \boldsymbol{P}_{(-)}^t(\omega) - \boldsymbol{P}_{(+)}^t(\omega), \tag{5.16}$$

where

$$\boldsymbol{P}^t(z) = \frac{1}{4\pi i}\int_{-\infty}^{+\infty} \frac{\boldsymbol{I}_+^t(\omega,\boldsymbol{g}^t)/k_{(-)}(\omega)}{\omega - z}d\omega, \quad \boldsymbol{P}_{(\pm)}^t(\omega) = \lim_{\beta\to 0^\mp} \boldsymbol{P}^t(\omega + i\beta). \tag{5.17}$$

We observe that $\boldsymbol{P}_{(\pm)}^t(z)$ has zeros and singularities in $z \in \boldsymbol{C}^\pm$. Therefore, it follows that $\boldsymbol{P}_{(\pm)}^t(z)$ is analytic in $\boldsymbol{C}^{(\mp)}$ while the analyticity on $\boldsymbol{R}$ is assured by virtue of the assumption in Sect. 2 for the Fourier transforms [14].

By substituting (5.16) into (5.15), we get

$$k_{(+)}(\omega)\boldsymbol{g}_+^{(m)}(\omega) + \boldsymbol{P}_{(+)}^t(\omega) = \boldsymbol{P}_{(-)}^t(\omega) + \frac{1}{2}\frac{\boldsymbol{r}_-(\omega)}{k_{(-)}(\omega)}, \tag{5.18}$$

which expresses the equality of two quantities, each of which has an analytic extension on the complex plane and vanishes at infinity and, consequently, it must be equal to zero. Thus, we have

$$\boldsymbol{g}_+^{(m)}(\omega) = -\frac{\boldsymbol{P}_{(+)}^t(\omega)}{k_{(+)}(\omega)}, \qquad \boldsymbol{P}_{(-)}^t(\omega) = -\frac{1}{2}\frac{\boldsymbol{r}_-(\omega)}{k_{(-)}(\omega)}. \tag{5.19}$$

The solution $(5.19)_1$ allows us to write the expression of the minimum free energy (5.8) as follows

$$\psi_m(\sigma(t)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\boldsymbol{P}_{(+)}^t(\omega)|^2 d\omega. \tag{5.20}$$

6. – A different formulation for ψ_m .

The relation (5.20) now derived expresses the minimum free energy in terms of $\boldsymbol{P}^t_{(+)}(\omega)$. Another equivalent form can be deduced for ψ_m by means of the relation between $\boldsymbol{P}^t_{(+)}(\omega)$ and $\boldsymbol{g}^t$. For this purpose we identify $\boldsymbol{g}^t(s)$ with its causal extension to $\boldsymbol{R}^{--}$, where we thus assume $\boldsymbol{g}^t(s) = \boldsymbol{0}\ \forall s \in (-\infty, 0)$, and the kernel $k(s)$ with the even function $k^{(e)}(s)$, whose Fourier transform is $k^{(e)}_F(\omega) = 2k_c(\omega)$.

With this assumptions, the quantity $\boldsymbol{I}(\tau, \boldsymbol{g}^t)$, defined in (4.15), can be written in this form

$$\boldsymbol{I}(\tau, \boldsymbol{g}^t) = -\int_{-\infty}^{+\infty} k^{(e)}(\eta + \tau)\boldsymbol{g}^t(\eta)d\eta \qquad \forall \tau \geq 0, \tag{6.1}$$

and can be extended on $\boldsymbol{R}$ by means of

$$\boldsymbol{I}^{(\boldsymbol{R})}(\tau, \boldsymbol{g}^t) = -\int_{-\infty}^{+\infty} k^{(e)}(\eta + \tau)\boldsymbol{g}^t(\eta)d\eta = \begin{cases} \boldsymbol{I}(\tau, \boldsymbol{g}^t) & \forall \tau \geq 0, \\ \boldsymbol{I}^{(n)}(\tau, \boldsymbol{g}^t) & \forall \tau < 0, \end{cases} \tag{6.2}$$

where

$$\boldsymbol{I}^{(n)}(\tau, \boldsymbol{g}^t) = -\int_{-\infty}^{+\infty} k^{(e)}(\eta + \tau)\boldsymbol{g}^t(\eta)d\eta \quad \forall \tau < 0. \tag{6.3}$$

From (6.2) we have

$$\boldsymbol{I}^{(\boldsymbol{R})}_F(\omega, \boldsymbol{g}^t) = \int_{-\infty}^{+\infty} \boldsymbol{I}^{(\boldsymbol{R})}(\tau, \boldsymbol{g}^t)e^{-i\omega\tau}d\tau = \boldsymbol{I}^{(n)}_-(\omega, \boldsymbol{g}^t) + \boldsymbol{I}_+(\omega, \boldsymbol{g}^t), \tag{6.4}$$

whence, using (5.16), we get this relation

$$\frac{1}{2k_{(-)}(\omega)}\boldsymbol{I}^{(\boldsymbol{R})}_F(\omega, \boldsymbol{g}^t) = \frac{1}{2k_{(-)}(\omega)}\boldsymbol{I}^{(n)}_-(\omega, \boldsymbol{g}^t) + \boldsymbol{P}^t_{(-)}(\omega) - \boldsymbol{P}^t_{(+)}(\omega). \tag{6.5}$$

Applying the Plemelj formulae to the first term of (6.5), we have

$$\frac{1}{2k_{(-)}(\omega)}\boldsymbol{I}^{(\boldsymbol{R})}_F(\omega, \boldsymbol{g}^t) = \boldsymbol{P}'^t_{(-)}(\omega) - \boldsymbol{P}'^t_{(+)}(\omega), \tag{6.6}$$

where $\boldsymbol{P}'^t(z)$ is defined by a relation similar to $(5.17)_1$ and is such that $\boldsymbol{P}'^t_{(\pm)}(z)$ has zeros and singularities in $\boldsymbol{C}^{\pm}$.

From (6.5) with (6.6) we get

$$\boldsymbol{P}^t_{(+)}(\omega) - \boldsymbol{P}'^t_{(+)}(\omega) = \boldsymbol{P}^t_{(-)}(\omega) - \boldsymbol{P}'^t_{(-)}(\omega) + \frac{1}{2k_{(-)}(\omega)}\boldsymbol{I}^{(n)}_-(\omega, \boldsymbol{g}^t) = \boldsymbol{0}, \tag{6.7}$$

since, while the first two terms are analytic in $\boldsymbol{C}^-$, the others are analytic in $\boldsymbol{C}^+$ and, moreover, both of them vanish at infinity.

Thus, we obtain

$$\boldsymbol{P}^t_{(+)}(\omega) = \boldsymbol{P}'^t_{(+)}(\omega), \qquad \boldsymbol{P}^t_{(-)}(\omega) = \boldsymbol{P}'^t_{(-)}(\omega) - \frac{1}{2k_{(-)}(\omega)}\boldsymbol{I}^{(n)}_{-}(\omega, \boldsymbol{g}^t). \tag{6.8}$$

Now, we let $\boldsymbol{g}^t_n(s) = \boldsymbol{g}^t(-s)$ $\forall s \leq 0$ with its extension $\boldsymbol{g}^t_n(s) = \boldsymbol{0}$ $\forall s > 0$ and consider its Fourier transform

$$\boldsymbol{g}^t_{n_F}(\omega) = \boldsymbol{g}^t_{n_-}(\omega) = \big(\boldsymbol{g}^t_{+}(\omega)\big)^*, \tag{6.9}$$

which allows us to write (6.2) as follows

$$\boldsymbol{I}^{(\boldsymbol{R})}(\tau, \boldsymbol{g}^t) = -\int_{-\infty}^{+\infty} k^{(e)}(\tau - s)\,\boldsymbol{g}^t_n(s)ds, \tag{6.10}$$

so that

$$\boldsymbol{I}^{(\boldsymbol{R})}_F(\omega, \boldsymbol{g}^t) = -2k_c(\omega)\big(\boldsymbol{g}^t_{+}(\omega)\big)^* \tag{6.11}$$

and, using $(5.13)_2$,

$$\frac{1}{2k_{(-)}(\omega)}\boldsymbol{I}^{(\boldsymbol{R})}_F(\omega, \boldsymbol{g}^t) = -k_{(+)}(\omega)\big(\boldsymbol{g}^t_{+}(\omega)\big)^*. \tag{6.12}$$

Then, from $(6.8)_1$ with (6.6), (6.12) and the relation analogous to (5.17) but written for $\boldsymbol{P}'^t_{(+)}(\omega)$ yield

$$\begin{aligned}\boldsymbol{P}^t_{(+)}(\omega) =& \boldsymbol{P}'^t_{(+)}(\omega) = \lim_{z\to\omega^-}\boldsymbol{P}^t_{(+)}(z) = \lim_{z\to\omega^-}\frac{1}{4\pi i}\int_{-\infty}^{+\infty}\frac{\boldsymbol{I}^{(\boldsymbol{R})}_F(\omega', \boldsymbol{g}^t)/k_{(-)}(\omega')}{\omega' - z}d\omega' \\ &= -\lim_{z\to\omega^-}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{k_{(+)}(\omega')\big(\boldsymbol{g}^t_{+}(\omega')\big)^*}{\omega' - z}d\omega',\end{aligned} \tag{6.13}$$

from which it follows that

$$\big(\boldsymbol{P}^t_{(+)}(\omega)\big)^* = \lim_{\xi\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{k_{(-)}(\omega')\,\boldsymbol{g}^t_{+}(\omega')}{\omega' - \xi}d\omega'. \tag{6.14}$$

The quantity $k_{(-)}(\omega)\boldsymbol{g}^t_{+}(\omega)$, using the Plemelj formulae, can be put in the form

$$k_{(-)}(\omega)\boldsymbol{g}^t_{+}(\omega) = \boldsymbol{Q}^t_{(-)}(\omega) - \boldsymbol{Q}^t_{(+)}(\omega), \tag{6.15}$$

where

$$\boldsymbol{Q}^t_{(\pm)}(\omega) = \lim_{\xi\to\omega^\mp}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{k_{(-)}(\omega')\,\boldsymbol{g}^t_{+}(\omega')}{\omega' - \xi}d\omega'. \tag{6.16}$$

Hence, the quantity $\boldsymbol{Q}^t_{(-)}(\omega)$ now introduced is such that

$$\boldsymbol{Q}^t_{(-)}(\omega) = \left(\boldsymbol{P}^t_{(+)}(\omega)\right)^*; \tag{6.17}$$

moreover, we observe that, applying Plancherel's theorem and using $(5.13)_2$ and (6.15), it allows us to transform (3.7) as follows

$$\begin{aligned}\tilde{\boldsymbol{q}}(\boldsymbol{g}^t) &= -\frac{1}{\pi}\int_{-\infty}^{+\infty} k_c(\omega)\boldsymbol{g}^t_+(\omega)d\omega = -\frac{1}{\pi}\int_{-\infty}^{+\infty} k_{(+)}(\omega)\left[\boldsymbol{Q}^t_{(-)}(\omega) - \boldsymbol{Q}^t_{(+)}(\omega)\right]d\omega \\ &= -\frac{1}{\pi}\int_{-\infty}^{+\infty} k_{(+)}(\omega)\boldsymbol{Q}^t_{(-)}(\omega)d\omega,\end{aligned} \tag{6.18}$$

since the integral of $k_{(+)}(\omega)\,\boldsymbol{Q}^t_{(+)}(\omega)$ is zero because of the analyticity of this integrand in $\boldsymbol{C}^{(-)}$, where the same integral can be extended to an infinite contour without altering its value.

Finally, the expression (5.20) of the minimum free energy, by virtue of (6.17), assumes the equivalent form

$$\psi_m(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\boldsymbol{Q}^t_{(-)}(\omega)|^2 d\omega, \tag{6.19}$$

which is the new required formulation.

7. – The discrete spectrum model.

We now consider the results of the previous section for a particular class of relaxation functions, that is a discrete spectrum model.

Let the kernel $k(t)$ be expressed by the following linear combination of decaying exponentials

$$k(t) = \begin{cases} \sum_{i=1}^{n} h_i e^{-k_i t} & \forall t \geq 0 \\ 0 & \forall t < 0 \end{cases}, \qquad n \in \boldsymbol{N}, \tag{7.1}$$

where the inverse decay times $k_i \in \boldsymbol{R}^{++}$ $(i = 1, 2, ..., n)$ are such that $k_1 < k_2 < ... < k_n$ and also the coefficients h_i $(i = 1, 2, ..., n)$ are positive.

We recall some results we have already obtained for such a model in [1].

In particular, (7.1) satisfies $(2.17)_2$ since

$$k(0) = \sum_{i=1}^{n} h_i > 0. \tag{7.2}$$

The Fourier transform of (7.1) is given by

$$k_F(\omega) = \sum_{i=1}^{n} \frac{h_i}{k_i + i\omega} \quad \omega \in \boldsymbol{R}, \tag{7.3}$$

whence it follows that

$$k_c(\omega) = \sum_{i=1}^{n} \frac{h_i k_i}{k_i^2 + \omega^2} \quad \omega \in \boldsymbol{R}. \tag{7.4}$$

Thus, for $K(\omega)$, defined in (5.12), we have

$$K(\omega) = (1+\omega^2)\sum_{i=1}^{n} \frac{h_i k_i}{k_i^2 + \omega^2}, \qquad K_\infty = \lim_{\omega\to\infty} K(\omega) = \sum_{i=1}^{n} h_i k_i > 0. \tag{7.5}$$

We observe that $f(z) = K(\omega)$ with $z = -\omega^2$, when $n \neq 1$ and $k_i^2 \neq 1$, has n simple poles at k_i^2 $(i = 1, 2, ..., n)$ and n simple zeros denoted by $\delta_1^2 = 1$ and δ_j^2 $(j = 2, 3, ..., n)$ so ordered

$$k_1^2 < \delta_2^2 < k_2^2 < ... < k_p^2 < \delta_{p+1}^2 < k_{p+1}^2 < ... < k_{n-1}^2 < \delta_n^2 < k_n^2. \tag{7.6}$$

If there exists p such that $k_p^2 < 1 < k_{p+1}^2$, with p equal to only one of the values $1, 2, ..., n-1$, the number of zeros can reduce to $n-1$ since δ_{p+1}^2 can coincide with $\delta_1^2 = 1$, which thus becomes a zero of multiplicity 2.

Therefore, $(7.5)_1$ can be rewritten as

$$K(\omega) = K_\infty \prod_{i=1}^{n} \left\{ \frac{\delta_i^2 + \omega^2}{k_i^2 + \omega^2} \right\}, \tag{7.7}$$

whence we have its factorization $(5.13)_1$ by means of

$$K_{(-)}(\omega) = k_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\delta_i}{\omega + ik_i} \right\}, \quad K_{(+)}(\omega) = k_\infty \prod_{i=1}^{n} \left\{ \frac{\omega - i\delta_i}{\omega - ik_i} \right\}, \quad k_\infty = (K_\infty)^{\frac{1}{2}}. \tag{7.8}$$

Using (5.14) and $(7.8)_1$, we obtain

$$k_{(-)}(\omega) = ik_\infty \frac{\prod_{j=2}^{n} (\omega + i\delta_j)}{\prod_{i=1}^{n} (\omega + ik_i)} = ik_\infty \sum_{r=1}^{n} \frac{U_r}{\omega + ik_r}, \tag{7.9}$$

where

$$U_r = \frac{\prod_{j=2}^{n} (\delta_j - k_r)}{\prod_{i=1, i\neq r}^{n} (k_i - k_r)} \quad (r = 1, 2, ..., n). \tag{7.10}$$

If there exists $p' \in \{1, 2, ..., n\}$ such that $k_{p'}^2 \equiv \delta_1^2 = 1$, the two expressions (7.9)-(7.10) hold again, since in such a case the number both of the poles and of the zeros of $f(z)$ reduces to $n-1$, however in $k_{(-)}(\omega)$, given by (5.14), we have the factor $\frac{i}{\omega+i}$ which introduces the pole $k_{p'}^2 = 1$.

It rests to examine the particular case $n = 1$. In this case (5.14) reduces to

$$k_{(-)}(\omega) = ik_\infty \frac{1}{\omega + ik_1}, \qquad U_1 = 1. \tag{7.11}$$

In the general case when $n \neq 1$, (6.16) and (7.9) yield

$$\boldsymbol{Q}_{(-)}^t(\omega) = \frac{1}{2\pi i} \sum_{r=1}^n ik_\infty U_r \int_{-\infty}^{+\infty} \frac{\boldsymbol{g}_+^t(\omega')/(\omega' - \omega^+)}{\omega' - (-ik_r)} d\omega' = ik_\infty \sum_{r=1}^n \frac{U_r}{\omega + ik_r} \boldsymbol{g}_+^t(-ik_r), \tag{7.12}$$

where the integrals have been evaluated by closing in $\boldsymbol{C}^{(-)}$, where we have the imaginary numbers $-ik_r$ $(r = 1, 2, ..., n)$, and taking account of the sense of the integrations.

Since $(2.9)_3$ gives

$$\boldsymbol{g}_+^t(-ik_r) = \int_0^{+\infty} \boldsymbol{g}^t(s) e^{-k_r s} ds = \left(\boldsymbol{g}_+^t(-ik_r)\right)^*, \tag{7.13}$$

then (7.12) yields

$$\left(\boldsymbol{Q}_{(-)}^t(\omega)\right)^* = -ik_\infty \sum_{r=1}^n \frac{U_r}{\omega - ik_r} \boldsymbol{g}_+^t(-ik_r). \tag{7.14}$$

Thus, (6.19) becomes

$$\psi_m(t) = k_\infty^2 \sum_{r,l=1}^n U_r U_l \boldsymbol{g}_+^t(-ik_r) \cdot \boldsymbol{g}_+^t(-ik_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i/(\omega + ik_r)}{\omega - ik_l} d\omega, \tag{7.15}$$

where the integrals over $\boldsymbol{R}$ can be evaluated by closing in $\boldsymbol{C}^{(+)}$, where we have ik_l $(l = 1, 2, ..., n)$, and taking account of the sense of the integrations.

Therefore, using $(7.13)_1$, we get

$$\psi_m(t) = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} 2K_\infty \sum_{r,l=1}^n \frac{U_r U_l}{k_r + k_l} e^{-(k_r s_1 + k_l s_2)} \boldsymbol{g}^t(s_1) \cdot \boldsymbol{g}^t(s_2) ds_1 ds_2. \tag{7.16}$$

In the particular case when $n = 1$, as we have done for (7.16), we obtain

$$\begin{aligned} \psi_m(t) &= \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} 2K_\infty \frac{U_1^2}{2k_1} e^{-k_1(s_1+s_2)} \boldsymbol{g}^t(s_1) \cdot \boldsymbol{g}^t(s_2) ds_1 ds_2 \\ &= \frac{1}{2} h_1 \left[\int_0^{+\infty} \boldsymbol{g}^t(s) e^{-k_1 s} ds \right]^2 \end{aligned} \tag{7.17}$$

since now $K_\infty = h_1 k_1$ from $(7.5)_2$ and $U_1 = 1$ from $(7.11)_2$.

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