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Degenerate Elliptic Equations and Morrey Spaces

F. BORRELLO

Sunto. – *In questo articolo viene studiata la regolarità locale per la soluzione generalizzata del problema di Dirichlet relativo all'equazione*

$$Lu \equiv X_i^*(a_{ij}X_ju) = f,$$

dove $X_1, X_2, \dots, X_m$ *sono campi vettoriali soddisfacenti la condizione di Hörmander e* $a_{ij} \in L^\infty$.
Viene data una formula di rappresentazione per la soluzione generalizzata in termini di funzione di Green. I risultati sono ottenuti grazie a opportune stime di quest'ultima. Nel caso in cui $f \geq 0$ *i teoremi provati sono invertibili.*

Summary. – *In this paper we study local regularity for the generalized solution to the Dirichlet problem related to the equation*

$$Lu \equiv X_i^*(a_{ij}X_ju) = f,$$

where $X_1, X_2, \dots, X_m$ *are vector fields satisfying Hörmander condition and* $a_{ij} \in L^\infty$. *We give a representation formula for the generalized solution in terms of the Green function and thanks to suitable estimates we achieve our goal. In the case* $f \geq 0$ *we are able to give necessary condition too.*

1. – Introduction.

Let $X_1, \dots, X_m$ be a given system of Hörmander vector fields in $\mathbb{R}^n$ $(m \leq n)$. We study local regularity for the generalized solution to the Dirichlet problem related to the equation

$$Lu \equiv X_i^*(a_{ij}X_ju) = f\,. \tag{1.1}$$

The case $a_{ij} = \delta_{ij}$ has been studied by several authors (see e.g. [18], [19], [11], [6]). Some regularity results are available when $a_{ij} \in L^\infty$, see [17], [3].
We assume very mild integrability conditions on f. In a so general setting, a weak solution not always exists, so we are forced to use a different concept of generalized solution.

The plan of the paper is the following. In Section 2 we introduce the Carnot-Carathéodory metric space and some related function spaces. In Section 3 the

weak and very weak solutions are compared.We also define the Green function and give a representation formula for the very weak solution. Fourth and fifth sections are devoted to the study of regularity.
We start proving regularity results for the solution u, assuming that f belongs to several classes which properly contains the Lebesgue classes. These classes are much more natural to use than the Lebesgue ones. In fact we will see that, at least in the case of non negative right hand side, all the conditions are necessary too.

We note that the technique used in the proof is very simple in nature. It relies on a representation formula in terms of the Green function and suitable estimates. Our results generalizes results already known in the case of uniformly elliptic equations (see [8], [9]). The case of degenerate equations with respect to a A_2 weight is discussed in [4].
We stress that all our results remain true if f is replaced by measure whose density is f, but for simplicity we treat the case in which f is a function.

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2. – Preliminaries.

Let $\mathrm{X} = (X_1, X_2, \dots, X_m)$ be a system of C^∞ vector fields on $\mathbb{R}^n$.
We say that $X_1, X_2, \dots, X_m$ satisfy Hörmander condition in a bounded domain Ω if

$$rank\ Lie\{X_1, X_2, \dots, X_m\} = n$$

at every point $x \in \Omega$.

A piecewise C^1 curve $\gamma : [0, T] \to \mathbb{R}^n$ is called X–sub-unit, if

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \qquad \forall\, \xi \in \mathbb{R}^n, \quad \text{a.e. } t \in [0, T]. \tag{2.2}$$

The X–sub-unit lenght of γ is by definition $l_S(\gamma) = T$. Given $x, y \in \mathbb{R}^n$, we denote by $\Phi(x, y)$ the collection of all X–sub-unit curves connecting x to y. As it is well known, $\Phi(x, y)$ is not empty by Chow Theorem ([5]). Setting

$$\rho(x, y) = \inf\{l_S(\gamma) \ : \ \gamma \in \Phi(x, y)\} \tag{2.3}$$

we may define a distance, the Carnot–Carathéodory distance generated by the system X. We denote by $B(x, r) = \{y \in \mathbb{R}^n \ : \ \rho(x, y) < r\}$ the metric ball centered at x of radius r and whenever x is not relevant we write B_r. The number Q indicates the homogenous dimension of Ω. We list the function spaces we need in the sequel.

DEFINITION 2.1. – *Let* $1 \leq p < \infty$ *we say that* $u \in L^p(\Omega)$ *if*

$$\|u\|_p^p \equiv \int_\Omega |u|^p \,\mathrm{d}x < \infty$$

and $u \in L^\infty(\Omega)$ *if it is bounded in* Ω.

DEFINITION 2.2. (Sobolev spaces). – *Let* $1 \leq p < +\infty$. *We say that* u *belongs to* $W^{1,p}(\Omega, \mathrm{X})$ *if* $u \in L^p(\Omega)$ *and* $X_j u \in L^p(\Omega)$, $j = 1, 2, \dots, m$. *We set*

$$\|u\|_{W^{1,p}(\Omega,\mathrm{X})} \equiv \|u\|_{L^p(\Omega)} + \sum_{j=1}^m \|X_j u\|_{L^p(\Omega)}. \tag{2.4}$$

We denote by $W_0^{1,p}(\Omega, \mathrm{X})$ *the completion of* $C_0^\infty(\Omega)$ *with respect to the above norm. As usual, when* $p = 2$ *we set* $H^1(\Omega, \mathrm{X})$ *and* $H_0^1(\Omega, \mathrm{X})$ *the spaces* $W^{1,2}(\Omega, \mathrm{X})$ *and* $W_0^{1,2}(\Omega, \mathrm{X})$ *respectively.*

REMARK 2.3. – $X_j u$ denotes the distributional derivative of u defined by

$$< X_j u, \phi > = \int_\Omega u X_j^* \phi \,\mathrm{d}x, \qquad \forall\, \phi \in C_0^\infty(\Omega)$$

where $X_j^* = -\sum_{i=1}^n \partial_i(c_{ij}\ \cdot\ )$ is the formal adjoint of $X_j = \sum_{i=1}^n c_{ij}\partial_i, j = 1, \dots, m$.

DEFINITION 2.4. (Schechter classes). – *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, $n \geq 3$ *and let* $1 \leq p \leq \infty$. *We say that* $u \in L^1(\Omega)$ *belongs to the Schechter class* $M_p(\Omega, \mathrm{X})$ *if*

$$M_p(u) \equiv \left(\int_\Omega \left(\int_{B_\delta(x)\cap\Omega} |u(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|} \,\mathrm{d}y \right)^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty$$

for some $\delta > 0$ *and* $p < \infty$.
When $p = \infty$

$$M_\infty(u) \equiv \sup_{x\in\Omega} \int_{B_\delta\cap\Omega} |u(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|} \,\mathrm{d}y < \infty\,.$$

DEFINITION 2.5. (Stummel-Kato class). – *Let* $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$. *If*

$$\eta(r) \equiv \sup_{x\in\Omega} \int_{\{y\in\Omega|\rho(x,y)<r\}} |u(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|} \,\mathrm{d}y < \infty$$

we say that $u \in \tilde{S}(\Omega, \mathrm{X})$.
If, in addition, $\eta(r) \to 0$ *we say that* $u \in S(\Omega, \mathrm{X})$.

If $X_j \equiv \partial_{x_j}$ we get back the classical Stummel-Kato class. In the sequel we will use some properties of the above defined classes.

LEMMA 2.6 [10]. – *Let $V \in S(\Omega, \mathrm{X})$. There exists $C \geq 0$ such that*

$$\int_{B_R} |V(x)||u(x)|^2 \,\mathrm{d}x \leq C\eta(2R) \int_{B_R} |\mathrm{X}u(x)|^2 \,\mathrm{d}x, \forall\, u \in C_0^\infty(\Omega)$$

Moreover, for any $\varepsilon > 0 \; \exists\, k(\varepsilon) > 0$ such that

$$\int_{\Omega} |V(x)||u(x)|^2 \,\mathrm{d}x \leq \varepsilon \int_{\Omega} |\mathrm{X}u(x)|^2 \,\mathrm{d}x + k(\varepsilon) \int_{\Omega} |u(x)|^2 \,\mathrm{d}x, \forall\, u \in C_0^\infty(\Omega).$$

PROPOSITION 2.7. – $\tilde{S}(\Omega, \mathrm{X}) \subset (H_0^1(\Omega, \mathrm{X}))^*$.

PROOF. – Let $\varphi \in C_0^\infty(\Omega)$ and B_r be a metric ball such that $\operatorname{supp} \varphi \subseteq B_r$. We have

$$| < f, \varphi > | \leq \int_{B_r} \sqrt{|f\varphi^2|} \sqrt{|f|} \,\mathrm{d}x \leq \left(\int_{B_r} |f\varphi^2| \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_r} |f| \,\mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq C\eta(2r) \left(\int_{B_r} |\mathrm{X}\varphi|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \|f\|_{L^1(\Omega)} \leq C \|\varphi\|_{H_0^1(\Omega,\mathrm{X})}.$$

□

DEFINITION 2.8. (Morrey classes). – *Let Ω be a bounded domain in $\mathbb{R}^n$, $1 \leq p < \infty$ and $\lambda > 0$. We say that $f \in L^p_{loc}(\Omega)$ belongs to the Morrey class $L^{p,\lambda}(\Omega, \mathrm{X})$ if*

$$|f|_{p,\lambda} \equiv \sup_B \left(\frac{r_B^\lambda}{|B|} \int_B |f(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}} < \infty$$

where the supremum is taken over the class of metric balls $B \equiv B(x, r_B)$.

REMARK 2.9. – If $\lambda = Q$ then $L^{p,\lambda}(\Omega, \mathrm{X}) \equiv L^p_{loc}(\Omega)$ and if $\lambda > Q$ then $L^{1,\lambda}(\Omega, \mathrm{X}) \equiv \{0\}$.

It is worth to compare the Morrey and the Lebesgue classes.

PROPOSITION 2.10. – *Let $q \geq p$ and $\frac{\mu}{q} \leq \frac{\lambda}{p}$. Then*

$$L^{q,\mu}(\Omega, \mathrm{X}) \subseteq L^{p,\lambda}(\Omega, \mathrm{X}).$$

PROOF. – Let $q > p$. We have

$$\left(\frac{r^{\lambda}}{|B|}\int_B |f|^p\right)^{\frac{1}{p}} = r^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B |f|^p\right)^{\frac{1}{p}} \leq r^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B |f|^q\right)^{\frac{1}{q}} = r^{\frac{\lambda}{p}-\frac{\mu}{q}}\left(\frac{r^{\mu}}{|B|}\int_B |f|^q\right)^{\frac{1}{q}}$$

$$\leq r^{\frac{\lambda}{p}-\frac{\mu}{q}}\|f\|_{q,\mu} \leq C(\lambda, p, \mu, q, \Omega)\|f\|_{q,\mu}.$$

□

DEFINITION 2.11. – *We say that $f \in L^{p,\lambda}_w(\Omega, \mathrm{X})$ if there exists $C > 0$, independent on r and x_0, such that*

$$\sup_{t>0} t^p |\{x \in \Omega \cap B_r(x_0) : |f(x)| > t\}| \leq C\frac{|B_r(x_0)|}{r^{\lambda}}.$$

PROPOSITION 2.12. – *Let $1 \leq q < p < \infty$ and $0 < \lambda < Q$, then*

$$L^{p,\lambda}_w(\Omega, \mathrm{X}) \subseteq L^{q,\mu}(\Omega, \mathrm{X})$$

where $\mu = \frac{\lambda}{p}\, q$.

PROOF. – We have

$$\int_{\Omega\cap B_r(x_0)} |f(x)|^q \,\mathrm{d}x = \int_0^{+\infty} qt^{q-1}|\{x \in \Omega \cap B_r(x_0)|\ |f(x)| > t\}|\,\mathrm{d}t$$

$$\leq Cq\int_0^{\varepsilon} t^{q-1} r^Q \,\mathrm{d}t + Cq\int_{\varepsilon}^{+\infty} t^{q-p-1} r^{Q-\lambda}\,\mathrm{d}t$$

$$= C\varepsilon^q r^Q - \frac{q}{q-p} C r^{Q-\lambda}\varepsilon^{q-p}, \quad \forall \varepsilon > 0.$$

Minimizing with respect to ε, we get:

$$\int_{\Omega\cap B_r(x_0)} |f(x)|^q \leq C\frac{p}{p-q} r^{\frac{Q-\lambda}{p}q} \leq C\frac{p}{p-q}\frac{|B_r(x_0)|}{r^{\frac{\lambda}{p}q}}.$$

□

PROPOSITION 2.13. – *Let $0 < \lambda < 2 \leq \mu < Q$. We have*

$$L^{1,\lambda}(\Omega, \mathrm{X}) \subseteq S(\Omega, \mathrm{X}) \subseteq \tilde{S}(\Omega, \mathrm{X}) \subseteq L^{1,\mu}(\Omega, \mathrm{X}).$$

PROOF. – Let $f \in L^{1,\lambda}(\Omega, \mathrm{X})$ and $x \in \Omega$. We have

$$\int\limits_{\Omega\cap B_r(x)} |f(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\,\mathrm{d}y$$

$$\leq \sum_{k=1}^{+\infty} \int\limits_{\{y\in\Omega | \frac{r}{2^k}\leq\rho(x,y)<\frac{r}{2^{k-1}}\}} |f(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\,\mathrm{d}y$$

$$\leq C\sum_{k=1}^{+\infty} \int\limits_{\{y\in\Omega | \frac{r}{2^k}\leq\rho(x,y)<\frac{r}{2^{k-1}}\}} |f(y)|\rho(x,y)^{2-Q}\,\mathrm{d}y$$

$$\leq C\sum_{k=1}^{+\infty}\left(\frac{r}{2^k}\right)^{2-Q} \int\limits_{\{y\in\Omega | \rho(x,y)<\frac{r}{2^{k-1}}\}} |f(y)|\,\mathrm{d}y$$

$$\leq C\sum_{k=1}^{+\infty}\left(\frac{r}{2^k}\right)^{2-Q} \frac{|B_{\frac{r}{2^{k-1}}}|}{\left(\frac{r}{2^{k-1}}\right)^{\lambda}}\|f\|_{1,\lambda}$$

$$\leq C\|f\|_{1,\lambda}\sum_{k=1}^{+\infty}\left(\frac{r}{2^k}\right)^{2-Q-\lambda}|B_{\frac{r}{2^{k-1}}}|$$

$$\leq C\|f\|_{1,\lambda}\sum_{k=1}^{+\infty}\left(\frac{r}{2^k}\right)^{2-\lambda}$$

and then $L^{1,\lambda}(\Omega, \mathrm{X}) \subseteq S(\Omega, \mathrm{X})$.
Now we prove that $\tilde{S}(\Omega, \mathrm{X}) \subseteq L^{1,\mu}(\Omega, \mathrm{X})$.

$$\frac{r^\mu}{|B_r|}\int\limits_{B_r\cap\Omega}|f(y)|\,\mathrm{d}y$$

$$= \frac{r^\mu}{|B_r|}\int\limits_{B_r\cap\Omega}|f(y)|\frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\frac{|B(x,\rho(x,y))|}{\rho^2(x,y)}\,\mathrm{d}y$$

$$\leq C\frac{r^\mu}{|B_r|}\int\limits_{B_r\cap\Omega}|f(y)|\frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\rho^{Q-2}(x,y)\,\mathrm{d}y$$

$$\leq C\frac{r^{\mu+Q-2}}{|B_r|}\,\eta(r) \leq C,$$

because $\mu \geq 2$. □

PROPOSITION 2.14. – *Let* $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$ *be a bounded domain and* $1 \leq p < q \leq \infty$.*Then*

$$M_\infty(\Omega, \mathrm{X}) \subset M_q(\Omega, \mathrm{X}) \subset M_p(\Omega, \mathrm{X}) \subset M_1(\Omega, \mathrm{X}).$$

PROOF. – The last inclusion is the only one we need to show.
Let $f \in M_p(\Omega, \mathrm{X})$

$$\begin{aligned}
&\int_\Omega \left(\int_{B_\delta(x)\cap\Omega} |f(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\, \mathrm{d}y \right) \mathrm{d}x \\
&\leq \left(\int_\Omega \left(\int_{B_\delta(x)\cap\Omega} |f(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\, \mathrm{d}y \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_\Omega \mathrm{d}x \right)^{\frac{1}{q}} \\
&\leq |\Omega|^{\frac{1}{q}} M_p(f) < \infty\,.
\end{aligned}$$

□

Let $0 < a < Q,\ x_0 \in \Omega$ and $B = B(x_0, r)$. Setting

$$I_a f(x) = \int_B |f(y)| \frac{\rho(x,y)^a}{|B(x,\rho(x,y))|}\, \mathrm{d}y,$$

we have

PROPOSITION 2.15. – *Let* $0 < \lambda < 2 < Q$. *Then*

$$L^{1,\lambda}(\Omega, \mathrm{X}) \subseteq M_\infty(\Omega, \mathrm{X}) \subseteq L^{1,2}(\Omega, \mathrm{X}) \subseteq \bigcap_{1 \leq p < \infty} M_p(\Omega, \mathrm{X}).$$

PROOF. – We only need to show the inclusion $L^{1,2}(\Omega, \mathrm{X}) \subseteq M^p(\Omega, \mathrm{X}),\ p \geq 1$. By proposition 2.10 $L^{1,2}(\Omega, \mathrm{X}) \subset L^{1,\lambda}(\Omega, \mathrm{X}),\ 2 < \lambda < Q$. Then, by Theorem 2.8 in [7], we get $I_2 f \in L_w^{q_\lambda,\lambda}(\Omega, \mathrm{X})$. By proposition 2.12, it follows that

$$I_2 f \in L^{p,\mu}(\Omega, \mathrm{X}),$$

where $1 \leq p < q_\lambda < \infty$ e $\mu = \lambda p / q_\lambda$. □

We will need the following definitions. They are well known but we recall them for reader's convenience.

DEFINITION 2.16. (BMO space). – *We say that* $f \in L^1_{loc}(\Omega)$ *belongs to the space* BMO(Ω, X) *if*

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, \mathrm{d}x < \infty$$

where $B = B(x, r)$ *ranges over the set of metric balls with* $x \in \Omega$ *and* $0 < r \leq R$.

DEFINITION 2.17. – *Let f a locally integrable function. The function*

$$Mf(x) = \sup_B \frac{1}{|B|}\int_B |f(y)|\,\mathrm{d}y,$$

is called the Hardy-Littlewood maximal function of f. The supremum is taken over all balls B centered at x.

3. – Generalized solutions and Green function.

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\mathrm{X} = (X_1, X_2, \ldots, X_m)$ be a system of C^∞, free vector fields, satisfying Hörmander condition on a neighborhood of $\overline{\Omega}$. Let us consider the operator

$$L = \sum_{i,j=1}^{m} X_i^*(a_{ij}X_j)$$

where $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ for $i,j = 1,2,\ldots,m$. We assume that there exist Λ, $\lambda > 0$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^m, \text{ a.e. } x \in \Omega.$$

First we recall what we mean by weak solution of the equation $Lu = f$.

DEFINITION 3.1. – *Let $f \in (H_0^1(\Omega, \mathrm{X})^*)$. We say that $u \in H^1(\Omega, \mathrm{X})$ is a weak solution of $Lu = f$ if*

$$\int_\Omega a_{ij}(x)X_ju(x)X_i\varphi(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x, \qquad \forall\, \varphi \in C_0^\infty(\Omega).$$

Now we are going to introduce a very general definition of generalized solution for the case of the Dirichlet problem. A crucial step to overcome is the presence of characteristic points that we could get on the boundary of our domain Ω.

DEFINITION 3.2. – *We say that $x \in \partial\Omega$ is a characteristic point of $\partial\Omega$ if $X_j(x)$ is tangent to $\partial\Omega$ at x for every $j = 1, \ldots, m$.*

In general we can get some subset of $\partial\Omega$, Σ which consists of characteristic points. This subset can be not empty but it cannot contain "too many points". We refer interested reader to [13] for discussions about the size of the set Σ.

In [15] the authors have proven regularity up to the boundary when the coefficients a_{ij} are smooth functions except for the set Σ. Since we are interested in the Dirichlet problem in the case of measurable coefficients a_{ij}, we cannot hope to get a better result than [15]. In the sequel we assume the following notion regarding the boundary of the domain Ω.

DEFINITION 3.3. – *We say that a domain Ω satisfies outer sphere condition if for all $x \in \partial\Omega$ there exists $x^c \in \mathbb{R}^n \setminus \Omega$ such that $B(x^c, \rho(x, x^c)) \subseteq \mathbb{R}^n \setminus \Omega$.*

Following [1] and getting inspiration from [16] we give the notion of generalized solution to the Dirichlet problem associated to the equation $Lu = f$ in Ω.

DEFINITION 3.4. – *Let μ be a measure of bounded variation on Ω. We say that $u \in L^1(\Omega)$ is a very weak solution of $Lu = \mu$ vanishing at the boundary $\partial\Omega$, if*

$$(3.5) \quad < Lv, u > = \int_\Omega v \, \mathrm{d}\mu, \; \forall \, v \in C^2(\overline{\Omega} \setminus \Sigma) \cap C^0(\overline{\Omega}) | Lv \in L^\infty(\Omega), \; v = 0 \text{ on } \partial\Omega.$$

REMARK 3.5. – In what follows when we omit the words “vanishing at the boundary”.

We have

THEOREM 3.6. – *Let Ω be a bounded domain in $\mathbb{R}^n$. Then very weak solution of $Lu = \mu$ exists. Moreover it is unique.*

In general if $\mu \notin (H_0^1(\Omega, \mathrm{X}))^*$ weak solutions do not exist. When both exist, the very weak solution is also a weak one.

DEFINITION 3.7. – *We call Green function for the operator L and the domain Ω with pole at $y \in \Omega$, the very weak solution of $LG^y = \delta_y$.*

In the sequel we use the representation formula

THEOREM 3.8. – *Let u be the very weak solution of $Lu = \mu$ in Ω. Then*

$$u(y) = \int_\Omega G^y(x) \, \mathrm{d}\mu(x), \qquad \text{a.e. } y \in \Omega.$$

DEFINITION 3.9. – *Let $y \in \Omega$ and $\eta > 0$ such that $B_\eta(y) \subset \Omega$. Let G_η^y be the very weak solution to*

$$LG_\eta^y = \frac{1}{|B_\eta(y)|} \chi_{B_\eta(y)}.$$

The function G_η^y will be called an approximate Green function.

We are going to prove estimates for the Green function. We will achieve this goal trough estimates for the approximate Green functions.

THEOREM 3.10. – *Let G^y be as in definition* 3.7. *Then*

i) there exists $C \geq 0$ independent on y, such that

$$G^y(x) \leq C\frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}, \qquad \forall\, x \in \Omega,\ x \neq y;$$

ii) there exists $c \geq 0$ independent on y, such that

$$c\frac{\rho^2(x,y)}{|B(x,\rho(x,y)|} \leq G^y(x),$$

$\forall\, x \in \Omega,\ x \neq y$ *such that* $\rho(x,y) \leq \frac{1}{2}\rho(y,\partial\Omega)$.

PROOF. – *i)* Let x_0 be fixed in $\Omega, \eta > 0$. For any $y \in \Omega, y \neq x_0$ let $R = \rho(x_0,y)$. Let R be such that $R \geq 2\eta$ and $B_{R/2}(x_0) \subseteq \Omega$, then $G^y_\eta(x)$ is a weak solution to $LG^y_\eta = 0$ in $\Omega \setminus B_\eta(y)$. Since $B_{\frac{R}{2}}(x_0) \subset \Omega \setminus B_\eta(y)$ we may apply Harnack inequality to get

$$\left(G^y_\eta(x)\right)^p \leq C \fint_{B_{\frac{R}{4}}(x_0)} \left(G^y_\eta\right)^p, \qquad \forall\, x \in B_{\frac{R}{8}}(x_0), \quad p > 1. \tag{3.6}$$

Arguing like the authors do in [14] and [2], we easily get

$$\|G^y_\eta\|^p_{L^p(B(x_0,\frac{R}{4}))} \leq C|B\left(x_0,\frac{R}{4}\right)|^{p(2/Q-1)}\|G^y_\eta\|^p_{L_w^{\frac{Q}{Q-2}}}$$

from which it follows

$$G^y_\eta(x_0) \leq C\frac{\rho^2(x_0,y)}{|B(x_0,\rho(x_0,y))|}. \tag{3.7}$$

Thanks to the Hölder continuity of the Green function we get the first estimate.

ii) We argue like in [14] and [2]. Let $x, y \in \Omega$ such that $r \equiv \rho(x,y) < 1/2\rho(x,\partial\Omega)$. Let ψ be a smooth function such that $0 \leq \psi \leq 1$, $|X\psi| \leq C/r$. Moreover, we choose ψ such that $\psi \equiv 1$ in $B_r(y) \setminus B_{r/2}(y)$ and $\psi \equiv 0$ outside $B_{\frac{3}{2}r}(y) \setminus B_{\frac{r}{4}}(y)$. We may take ψG_y as a test function in the definition of the weak solution and so we get

$$\int_{r/2 \leq \rho(z,y) \leq r} |XG(z,y)|^2\,\mathrm{d}z \leq cr^{Q/2} \sup_{\frac{r}{4} \leq \rho(z,y) \leq \frac{3}{2}r} G^2(z,y) \leq Cr^{Q-2}G^2(x,y).$$

Now let φ a cut-off function such that $\varphi \equiv 1$ on $B_{\frac{r}{2}}(y)$ and $\varphi \equiv 0$ outside $B_r(y)$,

$0 \leq \varphi \leq 1$ and $|X\varphi| \leq \frac{c}{r}$. We get

$$1 = \int\limits_{\frac{r}{2} \leq \rho(z,y) \leq r} a_{i,j} X_i G(z,y) X_j \varphi \, \mathrm{d}z \leq \frac{c}{r} \int\limits_{\frac{r}{2} \leq \rho(z,y) \leq r} |\mathrm{X}G(z,y)| \, \mathrm{d}z$$

$$\leq \frac{c}{r} \left(\int\limits_{\frac{r}{2} \leq \rho(z,y) \leq r} |\mathrm{X}G|^2 \right)^{\frac{1}{2}} |B_r|^{\frac{1}{2}} \leq \frac{c}{r} \cdot r^{\frac{Q}{2}} \cdot r^{\frac{Q-2}{2}} G(x,y)$$

$$= c[\rho(x,y)]^{Q-2} G(x,y).$$

□

4. – Regularity of the very weak solution.

In what follows the word smooth means C^∞ and Ω is a bounded domain with smooth boundary, satisfying outer sphere condition.

THEOREM 4.1. – *Let* $2 < \lambda \leq Q$, $f \in L^{1,\lambda}(\Omega, \mathrm{X})$. *Let* u *be the very weak solution of* $Lu = f$. *Then* $u \in L_w^{p_\lambda,\lambda}(\Omega, \mathrm{X})$, *where* $\frac{1}{p_\lambda} = 1 - \frac{2}{\lambda}$.

Moreover, there exists $C > 0$, *independent of* u *and* f, *such that*

$$\|u\|_q \leq C\|f\|_{1,\lambda}.$$

PROOF. – Using the representation formula and the estimate of the Green function in Theorem 3.10,we get

$$|u(y)| \leq C \int\limits_\Omega \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|} |f(x)| \, \mathrm{d}x \leq C \int\limits_\Omega \frac{|f(x)|}{[\rho(x,y)]^{Q-2}} \, \mathrm{d}x, \qquad \text{a.e. } y \in \Omega.$$

We get the result by estimating the last integral.

Let $R > 0$ be such that $B_{2R}(y) \subset \Omega$. We have

$$\int\limits_\Omega \frac{|f(x)|}{[\rho(x,y)]^{Q-2}} \, \mathrm{d}x = \int\limits_{B_{2R}(y)} \frac{|f(x)|}{[\rho(x,y)]^{Q-2}} \, \mathrm{d}x + \int\limits_{\Omega \setminus B_{2R}(y)} \frac{|f(x)|}{[\rho(x,y)]^{Q-2}} \, \mathrm{d}x = I + II.$$

We estimate the first integral using the maximal function.

$$I = \sum_{k=0}^{\infty} \int\limits_{\{x \in \Omega | R/2^k \leq \rho(x,y) < R/2^{k-1}\}} \frac{|f(x)|}{[\rho(x,y)]^{Q-2}} \, \mathrm{d}x$$

$$\leq \sum_{k=0}^{\infty} \left(\frac{R}{2^k}\right)^{2-Q} \left|B\left(y, R/2^{k-1}\right)\right| \quad ⨍_{\{\rho(x,y) < \frac{R}{2^{k-1}}\}} |f(x)| \, \mathrm{d}x$$

$$\leq CR^2 Mf(y).$$

On the other hand

$$II = \sum_{k=1}^{\infty} \int\limits_{\{2^k R \le \rho(x,y) < 2^{k+1}R\}} \frac{|f(x)|}{[\rho(x,y)]^{Q-2}}\, \mathrm{d}x$$

$$\le \sum_{k=1}^{\infty} \frac{1}{(2^k R)^{Q-2}} \frac{1}{(2^{k+1}R)^{\lambda}} \left|\big(B(y, 2^{k+1}R)\big)\right| \|f\|_{L^{1,\lambda}}$$

$$= \frac{C}{R^{\lambda-2}} \|f\|_{1,\lambda} \sum_{k=1}^{\infty} \frac{1}{2^{k(\lambda-2)}}\,.$$

Joining together the two estimates, we get

$$|u(y)| \le C[R^2 Mf(y) + R^{2-\lambda}\|f\|_{1,\lambda}], \qquad \forall\, R < \frac{1}{2}\, d(y, \partial\Omega)$$

and then,

$$|u(y)| \le C\|f\|^{\frac{1}{p'_\lambda}} (Mf(y))^{\frac{1}{p_\lambda}}, \quad \text{a.e. } y \in \Omega. \tag{4.8}$$

where

$$\frac{1}{p_\lambda} = 1 - \frac{2}{\lambda}.$$

The result follows immediately from the fact that the maximal operator is an operator of weak type (1,1) (see [7]). □

THEOREM 4.2. – *Let $f \in L^{1,2}(\Omega, \mathrm{X})$. Let u be the very weak solution of $Lu = f$. Then $u \in \mathrm{BMO}_{loc}(\Omega, \mathrm{X})$ in the sense that there exists $r_0 > 0$ such that $\forall\ \Omega' \Subset \Omega$ with $d := \rho(\Omega', \Omega) < r_0$, $x_0 \in \Omega'$ $\forall\, 0 < r < d/2$ we have*

$$⨍_{B_r(x)} |u(y) - u_{B_r(x)}|\, \mathrm{d}y \le C,$$

where C is independent on u, x, r.

PROOF. – Let Ω', d, x_0 and r be as in the statement and r_0 the same number of Theorem 7.7 in [17].

Let $B = B_r(x_0)$, and consider the functions $f_1 = f\chi_{2B}$ and $f_2 = f(1 - \chi_{2B})$. We split the solution $u = u_1 + u_2$, where $Lu_i = f_i$. Representation formula gives $u_i(y) = \int_{\Omega} G^y(x) f_i(x)\, \mathrm{d}x$, $i = 1, 2$.

We have

$$\begin{aligned}
\fint_B |u_1(x) - u_{1_B}|\,\mathrm{d}x \le & 2\fint_B \Big|\int_\Omega G^y(x) f_1(x)\,\mathrm{d}x\Big|\,\mathrm{d}y \\
\le & \int_{2B} |f(x)| \fint_B \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\,\mathrm{d}y\,\mathrm{d}x \\
\le & C\int_{2B} |f(x)| \frac{1}{|B|} \int_{B_{3r}(x)\cap\Omega} \rho(x,y)^{2-Q}\,\mathrm{d}y\,\mathrm{d}x \\
\le & C\int_{2B} |f(x)| \frac{1}{|B|} \sum_{k=1}^{\infty} \int_{\{y\in\Omega | \frac{3r}{2^k}\le\rho(x,y)<\frac{3r}{2^{k-1}}\}} \left(\frac{3r}{2^k}\right)^{2-Q}\,\mathrm{d}y\,\mathrm{d}x \\
\le & C\|f\|_{L^{1,2}(\Omega,\mathrm{X})}.
\end{aligned}$$

$$\begin{aligned}
\fint_B |u_2(x) - u_{2_B}|\,\mathrm{d}x = & \fint_B \Big| \int_{\Omega\setminus 2B} G^x(y) f(y)\,\mathrm{d}y - \fint_B \int_{\Omega\setminus 2B} G^z(y) f(y)\,\mathrm{d}y\ \mathrm{d}z\Big|\,\mathrm{d}x \\
\le & \int_{\Omega\setminus 2B} |f(y)| \fint_B \Big|G^x(y) - \fint_B G^z(y)\,\mathrm{d}z\Big|\,\mathrm{d}x\ \mathrm{d}y \\
= & \int_{\Omega\setminus 2B} |f(y)| \fint_B |G^x(y) - G^{x^*}(y)|\,\mathrm{d}x\ \mathrm{d}y \\
= & \int_{\Omega\setminus 2B\cap\{y|\rho(x_0,y)\le d\}} \cdots + \int_{\Omega\setminus 2B\cap\{y|\rho(x_0,y)>d\}} \cdots \\
= & I + II
\end{aligned}$$

where x^* is an element of B according to mean value formula for the Green function.

Let us estimate I.

The function $G^x(y)$ is a weak solution outside of the pole and then it is Hölder continuous.

$$\begin{aligned}
\fint_B |G^x(y) - G^{x^*}(y)|\,\mathrm{d}x \le & C\fint_B \left(\frac{2\rho(x,x^*)}{\rho(x_0,y)}\right)^a \left(\fint_{B_{\frac{\rho(x_0,y)}{2}}} |G^z(y)|^p\,\mathrm{d}z\right)^{\frac{1}{p}} \mathrm{d}x \\
\le & C\left(\frac{2r}{\rho(x_0,y)}\right)^a \left(\fint_{B_{\frac{\rho(x_0,y)}{2}}} |G^z(y)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}
\end{aligned}$$

where $0 < a < 1$, C is independent on x^* and $p > 1$.

Harnack inequality yields

$$\fint_B |G^x(y) - G^{x^*}(y)|\,\mathrm{d}x \le Cr^a(\rho(x_0,y))^{2-Q-a}$$

and then

$$\begin{aligned} I &\le Cr^a \int\limits_{(\Omega\setminus 2B)\cap\{y|\rho(x_0,y)\le d\}} |f(y)|(\rho(x_0,y))^{2-a-Q}\,\mathrm{d}y \\ &\le C\|f\|_{1,2}. \end{aligned}$$

Now we estimate II.

Despite of the previous case, now we do not know if $B(x_0,\rho(x_0,y))$ is contained or not in Ω. So we have,

$$\begin{aligned} \fint_B \Big|G^x(y) - \fint_B G^z(y)\,\mathrm{d}z\Big|\,\mathrm{d}x &= \fint_B |G^x(y) - G^{x^*}(y)|\,\mathrm{d}x \\ &\le \fint_B \left(\frac{\rho^2(x,y)}{|B(x,\rho(x,y))|} + \frac{\rho^2(x^*,y)}{|B(x^*,\rho(x^*,y))|}\right)\mathrm{d}x \\ &\le C\fint_B [(\rho(x,y))^{2-Q} + (\rho(x^*,y))^{2-Q}]\,\mathrm{d}x \\ &\le Cd^{2-Q} \end{aligned}$$

since $\rho(z,y) \ge \rho(y,x_0) - \rho(x_0,z) > \frac{d}{2}$, $\forall\, z \in B$. Thus implies,

$$\begin{aligned} II &\le C\,d^{2-Q} \int\limits_{(\Omega\setminus 2B)\cap\{y|\rho(x_0,y)>d\}} |f(y)|\,\mathrm{d}y \\ &\le Cd^{2-Q}\sum_{k=1}^{\infty} \int\limits_{\{y\in\Omega|2^k r\le\rho(x_0,y)<2^{k+1}r\}\cap\{y\in\Omega|\rho(x_0,y)>d\}} |f(y)|\,\mathrm{d}y \le C\|f\|_{1,2}. \end{aligned}$$

□

We are going to get regularity for weak solutions. Indeed, from now on we assume that $f \in \tilde{S}(\Omega,\mathrm{X})$ or in some subspace of $f \in \tilde{S}(\Omega,\mathrm{X})$.

Our first result concerns the boundedness of the weak solution. This is easily achieved via representation formula.

THEOREM 4.3. – *Let $f \in \tilde{S}(\Omega)$ and u be the very weak solution of $Lu = f$. Then u is bounded in Ω.*

PROOF. – For a.e. $x \in \Omega$ we have:

$$|u(x)| = \Big|\int_\Omega G^x(y) f(y)\,\mathrm{d}y\Big| \le \int_\Omega \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}|f(y)|\,\mathrm{d}y$$

$$\le C\int_{\Omega\cap\bar{B}} \frac{\rho^2(x,y)}{|B(y,\rho(x,y))|}|f(y)|\,\mathrm{d}y \le \eta(\bar{r}) < \infty$$

where $\bar{B}$ is a ball of radius $\bar{r}$ containing Ω. □

The next step is to improve regularity of the weak solutions assuming more on the function f.

LEMMA 4.4 (Caccioppoli inequality). – *Let $f \in S(\Omega, \mathrm{X})$ and u be the solution of $Lu = f$.*

Then $\forall \varphi \in C_0^\infty(\Omega)$ we have:

$$\int_\Omega \varphi^2 |\mathrm{X}u|^2\,\mathrm{d}x \le C\int_\Omega |u|^2 |\mathrm{X}\varphi|^2\,\mathrm{d}x + C\eta(2R)\int_\Omega |\mathrm{X}\varphi|^2\,\mathrm{d}x$$

PROOF. – The proof is standard and we used Proposition 2.6. For the existence of test functions we refer to [12]. □

THEOREM 4.5. – *Let $f \in S(\Omega, \mathrm{X})$. Then the very weak solution of $Lu = f$ is continuous in Ω.*

PROOF. – Let $x_0 \in \Omega$ and $r > 0$ such that $B \equiv B(x_0, r) \Subset \Omega$ and consider a cut-off function $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ on $B_{\frac{3}{2}r}$, $\phi \equiv 0$ outside B_{2r} and $|\mathrm{X}\phi| \le C/r$. For all $\psi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega a_{ij} X_i(u\phi) X_j\psi\,\mathrm{d}x = \int_\Omega a_{ij} u X_i\phi X_j\psi\,\mathrm{d}x + \int_\Omega a_{ij}\phi X_i u X_j\psi\,\mathrm{d}x$$

$$= \int_\Omega f\phi\psi\,\mathrm{d}x + \int_\Omega a_{ij} u X_i\phi X_j\psi\,\mathrm{d}x - \int_\Omega a_{ij} X_i u \psi X_j\phi\,\mathrm{d}x.$$

So $u\phi$ satisfies the equation

$$L(u\phi) = f\phi + X_j^*(a_{ij} u X_i\phi) - a_{ij} X_i u X_j\phi.$$

By representation formula,

$$(u\phi)(x) = \int_\Omega f(y)\phi(y) G^y(x)\,\mathrm{d}y + \int_\Omega a_{ij}(y) u(y) X_i\phi(y) X_j G^y(x)\,\mathrm{d}x$$

$$-\int_\Omega a_{ij}(y) X_i u(y) X_j\phi(y) G^y(x)\,\mathrm{d}y, \qquad \text{a.e. } x \in \Omega.$$

For a.e. $x \in B_{\frac{r}{2}}$ we get

$$
\begin{aligned}
u(x) - u(x_0) = & \int_{\Omega} f(y)\phi(y)(G^y(x) - G^y(x_0))\,\mathrm{d}y \\
& - \int_{\Omega} a_{ij}(y) X_i u(y) X_j \phi(y)(G^y(x) - G^y(x_0))\,\mathrm{d}y \\
& + \int_{\Omega} a_{ij}(y) u(y) X_i \phi(y) X_j (G^y(x) - G^y(x_0))\,\mathrm{d}y \\
\equiv & I + II + III.
\end{aligned}
$$

We will get the result giving estimates of I, II and III by a modulus of continuity.

Let us start estimating I.

Let $N > 1$ to be chosen later. We have

$$
\begin{aligned}
|I| \leq & \int_{\{y \in \Omega | \rho(x_0, y) > N\rho(x, x_0)\}} |f(y)||G^y(x) - G^y(x_0)|\phi(y)\,\mathrm{d}y \\
+ & \int_{\{y \in \Omega | \rho(x_0, y) \leq N\rho(x, x_0)\}} |f(y)||G^y(x) - G^y(x_0)|\phi(y)\,\mathrm{d}y \\
\equiv & I_A + I_B.
\end{aligned}
$$

Using the regularity properties of the Green function outside the pole,we get

$$
\begin{aligned}
I_A \leq & C\left(\frac{\rho(x, x_o)}{r}\right)^a \int_{\{y \in \Omega | \rho(x_0, y) > N\rho(x, x_0)\}} |f(y)||\phi(y)|\left(\frac{\rho(x, x_o)}{r}\right)^a \frac{(\rho(y, x_0))^2}{|B(x_0, \rho(y, x_0))|}\,\mathrm{d}y \\
\leq & C(Nr)^{-a} \int_{\Omega \cap B_{2r}} (\rho(x_0, y))^a |f(y)| \frac{(\rho(x_0, y))^2}{|B(x_0, \rho(x_0, y))|}\,\mathrm{d}y \leq CN^{-a}\eta(2r).
\end{aligned}
$$

Let us now estimate I_B.

$$
\begin{aligned}
I_B \leq & C\Bigg(\int_{\{y \in \Omega | \rho(x_0, y) \leq N\rho(x, x_0)\}} |f(y)|\phi(y) \frac{(\rho(x, y))^2}{|B(y, \rho(x, y))|}\,\mathrm{d}y. \\
& + \int_{\{y \in \Omega | \rho(x_0, y) \leq N\rho(x, x_0)\}} |f(y)|\phi(y) \frac{(\rho(x_0, y))^2}{|B(y, \rho(x_0, y))|}\,\mathrm{d}y\Bigg) \\
\leq & C\Bigg(\int_{\Omega \cap B(x, (N+1)\rho(x, x_0))} |f(y)| \frac{(\rho(x, y)^2)}{|B(x, \rho(x, y))|}\,\mathrm{d}y + \eta(N\rho(x_0, y))\Bigg) \\
\leq & C\{\eta[(N+1)\rho(x, x_0)] + \eta(N\rho(x, x_0))\}.
\end{aligned}
$$

Now, choose $N = \left(\frac{r}{\rho(x,x_0)}\right)^{\frac{1}{2}}$.

$$|I| \le C\left[\left(\frac{\rho(x,x_0)}{r}\right)^{\frac{a}{2}}\eta(2r) + \eta(\sqrt{r\rho(x,x_0)} + \rho(x,x_0)) + \eta(\sqrt{r\rho(x,x_0)})\right].$$

Caccioppoli inequality yields

$$|II| \le C\left(\frac{\rho(x,x_0)}{r}\right)^{a} M \int\limits_{B_{2r}\setminus B_{\frac{3}{2}r}} |Xu||X\phi|\,(\rho(x_0,y))^{2-Q}\,\mathrm{d}y$$

$$\le \frac{C}{r^{Q/2-1}}\left(\frac{\rho(x,x_0)}{r}\right)^{a}\left(\int\limits_{B_{2r}} |Xu|^2\,\mathrm{d}y\right)^{\frac{1}{2}}$$

$$\le \frac{C}{r^{Q/2}}\left(\frac{\rho(x,x_0)}{r}\right)^{a}\left(\int\limits_{\Omega} |u|^2\,\mathrm{d}y\right)^{\frac{1}{2}}.$$

To complete the proof we estimate III.

$$|III| \le C\frac{M}{r}\int\limits_{B_{2r}\setminus B_{\frac{3}{2}r}} |X[G^y(x) - G^y(x_0)]||u|\,\mathrm{d}y$$

$$\le C\frac{M}{r}\left(\int\limits_{B_{2r}\setminus B_{\frac{3}{2}r}} |X[G^y(x) - G^y(x_0)]|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int\limits_{B_{2r}} |u|^2\,\mathrm{d}y\right)^{\frac{1}{2}}.$$

Finally we get

$$|u(x) - u(x_0)| \le C\Bigg\{\left(\frac{\rho(x,x_0)}{r}\right)^{a}\eta(2r) + \eta(\sqrt{r\rho(x,x_0)} + \rho(x,x_0))$$

$$+\eta(\sqrt{r\rho(x,x_0)}) + \frac{(\rho(x,x_0))^a}{r^{a+\frac{Q}{2}}}\left(\int\limits_{B_{2r}} |u|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\Bigg\}$$

and the result follows. □

The last result in this section is the Hölder continuity of u.The proof is similar to the one of the previous Theorem. We also use Proposition 2.13.

THEOREM 4.6. – *Let $f \in L^{1,\lambda}(\Omega, X)$ with $0 < \lambda < 2$. Then the very weak solution of $Lu = f$ is hölder continuos in Ω.*

PROOF. – There exist $\beta = \beta(\lambda, \mathrm{X})$ and C such that

$$|u(x) - u(y)| \leq C(\rho(x,y))^{\beta}, \qquad x, y \in \Omega.$$

As in the previous theorem we consider I, II, and III. Let us estimate I.

$$\begin{aligned}|I_A| \leq & \int\limits_{\{y\in\Omega|\rho(x_0,y)>N\rho(x_0,x)\}\cap B_{2r}} |f(y)|\left(\frac{\rho(x,x_0)}{r}\right)^{a}\frac{(\rho(x_0,y))^2}{|B(x_0,\rho(x_0,y))|}\,\mathrm{d}y \\ \leq & C\left(\frac{\rho(x,x_0)}{r}\right)^{a}\frac{r^2}{r^{\lambda}}\|f\|_{1,\lambda} \\ \leq & CN^{-a}r^{2-\lambda+a}\|f\|_{1,\lambda}.\end{aligned}$$

Taking $N = \sqrt{r/\rho(x,x_0)}$, we get

$$|I_B| \leq Cr^{\frac{2-\lambda}{2}}(\rho(x,x_0))^{\frac{2-\lambda}{2}}\|f\|_{1,\lambda}$$

hence

$$|I| \leq C[(\rho(x,x_0))^{\frac{a}{2}}r^{2-\lambda-\frac{a}{2}} + (\rho(x,x_0))^{\frac{2-\lambda}{2}}r^{\frac{2-\lambda}{2}}]\|f\|_{1,\lambda}.$$

Integrals II and III are estimated asin the previous Theorem. □

5. – Necessary Conditions.

In this section we study what are the natural assumptions on the data in order to get a given degree of regularity for the solution.

THEOREM 5.1. – *Let $u \in L^1(\Omega)$ the very weak solution of $Lu = f$, and $f \geq 0$. The very weak solution $u \in L^q_{loc}(\Omega)$, $q > 1$ iff $f \in M^q_{loc}(\Omega, \mathrm{X})$.*

PROOF. – Let $u \in L^q_{loc}(\Omega)$ and K be a compact set in Ω. Then

$$\begin{aligned}\int\limits_K |u(x)|^q\,\mathrm{d}x =& \int\limits_K \Big|\int\limits_\Omega G^y(x)f(y)\,\mathrm{d}y\Big|^q\,\mathrm{d}x \\ \geq & \int\limits_K \left(\int\limits_{\{y\in\Omega|\rho(x,y)\geq\delta\}} G^y(x)f(y)\,\mathrm{d}y\right)^q\,\mathrm{d}x \\ =& \int\limits_K |u_\delta(x)|^q\,\mathrm{d}x\end{aligned}$$

where

$$u_\delta(x) := \int\limits_{\{y\in\Omega|\rho(x,y)\geq\delta\}} G^y(x)f(y)\,\mathrm{d}y \tag{5.9}$$

with $\delta > 0$.

We have

$$\lim_{\delta\downarrow 0}\int\limits_K |u_\delta(x) - u(x)|^q\,\mathrm{d}x = 0$$

that implies

$$\lim_{\delta\downarrow 0}\int\limits_K \Bigg(\int\limits_{\{y\in\Omega|\rho(x,y)<\delta\}} G^y(x)f(y)\,\mathrm{d}y\Bigg)^q \mathrm{d}x \to 0.$$

Using the estimate from below for the Green function (see Theorem3.10) we get $f\in M^q_{loc}(\Omega,\mathrm{X})$.

Now let $f\in M^q_{loc}(\Omega,\mathrm{X})$ and we show that $u\in L^q_{loc}(\Omega)$ with $q>1$.

$$\begin{aligned}|u(x)| \leq &\int\limits_{\{y\in\Omega|\rho(x,y)<\delta\}} G^y(x)f(y)\,\mathrm{d}y\\ &+\int\limits_{\{y\in\Omega|\rho(x,y)\geq\delta\}} G^y(x)f(y)\,\mathrm{d}y\\ &\leq C\int\limits_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y)\frac{(\rho(x,y))^2}{|B(x,\rho(x,y))|}\,\mathrm{d}y.\end{aligned}$$

So, we have:

$$\begin{aligned}\int\limits_K |u(x)|^q\,\mathrm{d}x &\leq C\int\limits_K\Bigg(\int\limits_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y)\frac{(\rho(x,y))^2}{|B(x,\rho(x,y))|}\,\mathrm{d}y)^q\,\mathrm{d}x\\ &\leq C|K|.\end{aligned}$$

□

THEOREM 5.2. – *Let $u\in L^1(\Omega)$ the very weak solution of $Lu=f$, and $f\geq 0$. The very weak solution $u\in L^\infty(\Omega)$ iff $f\in\tilde{S}_{loc}(\Omega,\mathrm{X})$.*

PROOF. – We only need to show that $f\geq 0$ and $u\in L^\infty(\Omega)$ imply $f\in\tilde{S}_{loc}(\Omega,\mathrm{X})$. Let K be a compact set of Ω. We prove that there exists $\delta>0$ such that $\eta_K(\delta)<\infty$.

Let $0 < \delta \leq \rho(K, \partial\Omega)/2$. We have

$$\int_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y) \frac{(\rho(x,y))^2}{|B(x,\rho(x,y))|}\,\mathrm{d}y \leq \int_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y) G^y(x)\,\mathrm{d}y$$

a.e. $x \in K$. Then

$$\eta_K(\delta) \equiv \int_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y) \frac{(\rho(x,y))^2}{|B(x,\rho(x,y))|}\,\mathrm{d}y < \infty.$$

$\square$

THEOREM 5.3. – *Let $u \in L^1(\Omega)$ be the very weak solution of $Lu = f$, and $f \geq 0$. Then $u \in C^0(\Omega)$ iff $f \in S_{loc}(\Omega, \mathrm{X})$.*

PROOF. – Let $\delta > 0$. We shall prove that the function u_δ defined in (5.9) is continuous at any $x_0 \in \Omega$.

We have:

$$|u_\delta(x) - u_\delta(x_0)| = \Big| \int_\Omega f(y)[G^y(x)\chi(y) - G^y(x_0)\chi_{_0}(y)]\,\mathrm{d}y \Big|, \qquad x \in \Omega,$$

where χ and $\chi_{_0}$ are, respectively, the characteristic functions of sets $\{y \in \Omega \,|\, \rho(x,y) \geq \delta\}$ and $\{y \in \Omega \,|\, \rho(x_0,y) \geq \delta\}$.

Hence we have

$$\begin{aligned} |u_\delta(x) - u_\delta(x_0)| &\leq \int_{\Omega\setminus[B_\delta(x)\cup B_\delta(x_0)]} f(y)|G^y(x) - G^y(x_0)|\,\mathrm{d}y \\ &+ \int_{[B_\delta(x)\cup B_\delta(x_0)]\setminus[B_\delta(x)\cap B_\delta(x_0)]} f(y)|G^y(x)\chi(y) - G^y(x_0)\chi_{_0}(y)|\,\mathrm{d}y \\ &= I + II \end{aligned}$$

For the first integral we have

$$I \leq \int_{\Omega\setminus B_\delta(x_0)} f(y)|G^y(x) - G^y(x_0)|\,\mathrm{d}y \to 0$$

as $\rho(x,y) \to 0$, by the uniform continuity of the function $G^y(\cdot)$.

Moreover, $\lim_{x\to x_0} II = 0$ by the absolute continuity of the Lebesgue measure. So $(u_\delta)_{\delta>0}$ is a family of continuous functions and $u_\delta \uparrow u$, then $u_\delta \rightrightarrows u$ as $\delta \to 0$, that means

$$\sup_{x\in K} |u_\delta(x) - u(x)| = \sup_{x\in K} \int_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y)\, G^y(x)\,\mathrm{d}y \to 0$$

This obviously implies

$$\sup_{x\in K} \int\limits_{\{y\in\Omega|\rho(x,y)<\delta\}} f(y)\,\frac{\rho^2(x,y)}{|B(x,\rho(x,y))|}\,\mathrm{d}y \to 0$$

and the result follows. □

The next result concerns the necessary condition to get hölder continuity. By our previous results the solution is the weak one.

LEMMA 5.4. – *Let $f\in L^1(\Omega)$, $f\geq 0$ and let u be a local weak solution of $Lu=f$. Assume that $u\in C^{0,a}(\Omega,\mathrm{X})$. Then*

$$\int\limits_{B_r} |\mathrm{X}u(x)|^2\,\mathrm{d}x \leq C\left(r^a \int\limits_{B_{2r}} f(x)\,\mathrm{d}x \,+\, r^{Q-2+2a}\right),$$

for every ball B_r such that $B_{2r}\Subset\Omega$.

PROOF. – Let $\eta(x)\in C_0^\infty(B_{2r})$ such that

$$\eta(x)=1 \text{ in } B_r,\quad 0\leq\eta\leq 1 \quad\text{and}\quad |\mathrm{X}u|\leq\frac{C}{r}\quad\text{for some constant } C.$$

Consider the test function $\varphi\equiv\eta^2(x)(u(x)-u_{2r})$. We get

$$\int\limits_{B_r}|\mathrm{X}u|^2\,\mathrm{d}x \leq C\left(\int\limits_{B_{2r}}|\mathrm{X}u|^2|u(x)-u_{2r}|^2\,\mathrm{d}x + \int\limits_{B_{2r}} f(x)\eta^2(x)|u(x)-u_{2r}|\,\mathrm{d}x\right).$$

The Hölder continuity of u gives the result. □

Before proving our last result, let us recall a lemma due to Stampacchia (e.g. [20]).

LEMMA 5.5. – *Let $\omega(x)\geq 0$ non-decreasing in $(0,R)$ with $R<1$. Let us suppose that there exist $\tau\in(0,1)$, $H>0$ and $a>0$ such that*

$$\omega(\rho)\leq\tau\omega(4\rho)+H\rho^a,\qquad \forall\,\rho\in(0,R).$$

Then there exist $K>0$ and $\lambda\in(0,1)$ with $\lambda=\min(\log_4\frac{\tau+1}{2\tau},a))$, such that

$$\omega(\rho)\leq K\rho^\lambda.$$

We are ready to show our last result. It is the converse of the Theorem 4.6 when $f\geq 0$.

THEOREM 5.6. – *Let* $f \in L^1(\Omega)\, f \geq 0$. *and let* u *be the very weak solution of* $Lu = f$.

If $u \in C^{0,a}(\Omega, \mathrm{X})$ *with* $0 < a < 2$ *then* $f \in L^{1,a}_{loc}(\Omega, \mathrm{X})$.

PROOF. – Let $\varepsilon > 0$. Consider a ball B_r such that $B_{4r} \Subset \Omega$. Let

$$\varphi \in C_0^\infty(B_{2r}), \quad 0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi \equiv 1 \ \text{ in } \ B_r.$$

By Young inequality we get

$$\int_{B_r} f(x)\,\mathrm{d}x \leq \int_{B_{2r}} a_{ij}(x)\mathrm{X}_i u \mathrm{X}_j \varphi(x)\,\mathrm{d}x$$

$$\leq C\left(\varepsilon r^{-a} \int_{B_{2r}} |\mathrm{X}u|^2\,\mathrm{d}x + \frac{1}{\varepsilon} r^a \int_{B_{4r}} |\mathrm{X}\varphi|^2\,\mathrm{d}x\right).$$

Lemma 5.4 then yields

$$\int_{B_r} f(x)\,\mathrm{d}x \leq C\varepsilon \int_{B_{4r}} f(x)\,\mathrm{d}x + C_\varepsilon r^{Q-2+a}, \qquad \forall\, \varepsilon > 0$$

Now choose $\varepsilon : C\varepsilon < 1$. By Lemma 5.5, there exists $K > 0$ such that

$$\int_{B_r} f(x)\,\mathrm{d}x \leq K r^{Q-2+a}$$

hence

$$\frac{r^{2-a}}{|B_r|} \int_{B_r} f(x)\,\mathrm{d}x < K.$$

□

REFERENCES

[1] A. BALDI, *A non-existence problem for degenerate elliptic PDE's*, Comm. Partial Differential Equations, Communications in Partial Differential Equations, **25**, 7-8 (2000), 1371-1398.

[2] C. CANCELIER - C-J., XU, *Remarques sur les fonctions de Green associées aux opérateurs de Hörmander*, C. R. Acad. Sci. Paris Sér. I Math., **330**, 6 (2000), 433-436.

[3] L. CAPOGNA - D. DANIELLI - N. GAROFALO, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations, **18**, 9-10 (1993), 1765-1794.

[4] Y. CHEN, *Regularity of solutions to the Dirichlet problem for degenerate elliptic equation*, Chinese Ann. Math. Ser. B, **24**, 4 (2003), 529-540.

[5] W.-L. CHOW, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann., **117** (1939), 98-105.

[6] G. CITTI - N. GAROFALO - E. LANCONELLI, *Harnack's inequality for sum of squares of vector fields plus a potential*, Amer. J. Math., **115**, 3 (1993), 699-734.

[7] D. DANIELLI, *A Fefferman-Phong type inequality and applications to quasilinear subelliptic equations*, Potential Analysis, **11** (1999), 387-413.

[8] G. DI FAZIO, *Poisson equations and Morrey spaces*, J. Math. Anal. Appl., **163**, 1 (1992), 157-167.

[9] G. DI FAZIO, *Dirichlet problem characterization of regularity*, Manuscripta Math., **84**, 1 (1994), 47-56.

[10] G. DI FAZIO - P. ZAMBONI, *Hölder continuity for quasilinear subelliptic equations in Carnot Carathéodory spaces*, Math. Nachr., **272** (2004), 3-10.

[11] B. FRANCHI - E. LANCONELLI, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **10**, 4 (1983), 523-541.

[12] B. FRANCHI - R. SERAPIONI - F. SERRA CASSANO, *Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields*, Boll. Un. Mat. Ital. B (7), **11**, 1 (1997), 83-117.

[13] B. FRANCHI - R. L. WHEEDEN, *Compensation couples and isoperimetric estimates for vector fields*, Colloq. Math., **74**, 1 (1997), 9-27.

[14] M. GRÜTER - K.-O. WIDMAN, *The Green function for uniformly elliptic equations*, Manuscripta Math., **37**, 3 (1982), 303-342.

[15] J. J. KOHN - L. NIRENBERG, *Non-coercive boundary value problems*, Comm. Pure Appl. Math., **18** (1965), 443-492.

[16] W. LITTMAN - G. STAMPACCHIA - H. F. WEINBERGER, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3), **17** (1963), 43-77.

[17] G. LU, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoamericana, 8, 3 (1992), 367-439.

[18] A. NAGEL - E. M. STEIN - S. WAINGER, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math., **155**, 1-2 (1985), 103-147.

[19] A. SÁNCHEZ-CALLE, *Fundamental solutions and geometry of the sum of squares of vector fields*, Invent. Math., **78**, 1 (1984), 143-160.

[20] G. STAMPACCHIA, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble), **15**, 1 (1965), 189-258.

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