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Asymptotics for Eigenvalues of a Non-Linear Integral System

D. E. EDMUNDS - J. LANG

Sunto. – Sia $I = [a, b]$ un sottinsieme di \mathbb{R} . Siano $1 < q, p < \infty$, e siano u e v funzioni positive, con $u \in L_{p'}(I)$ e $v \in L_q(I)$. Sia $T : L_p(I) \rightarrow L_q(I)$ un operatore di Hardy definito nel modo seguente:

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \quad x \in I.$$

Dimostreremo che il comportamento asintotico degli autovalori λ nel sistema integrale non lineare

$$g(x) = (Tf)(x) \quad (f(x))_{(p)} = \lambda(T^*(g_{(p)}))(x)$$

(dove, per esempio, $t_{(p)} = |t|^{p-1} \operatorname{sgn}(t)$), è dato da

$$\lim_{n \rightarrow \infty} n \hat{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r}, \quad \text{quando } 1 < q < p < \infty,$$

$$\lim_{n \rightarrow \infty} n \check{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r}, \quad \text{quando } 1 < p < q < \infty.$$

Qui $r = 1/p' + 1/p$, $c_{p,q}$ è una costante esplicita che dipende solo da p e q , $\hat{\lambda}_n(T) = \max(sp_n(T, p, q))$, $\check{\lambda}_n(T) = \min(sp_n(T, p, q))$, dove $sp_n(T, p, q)$ rappresenta l'insieme di tutti gli autovalori λ che corrispondono alle autofunzioni g con n zeri.

Abstract. – Let $I = [a, b] \subset \mathbb{R}$, let $1 < q, p < \infty$, let u and v be positive functions with $u \in L_{p'}(I)$, $v \in L_q(I)$ and let $T : L_p(I) \rightarrow L_q(I)$ be the Hardy-type operator given by

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \quad x \in I.$$

We show that the asymptotic behavior of the eigenvalues λ of the non-linear integral system

$$g(x) = (Tf)(x) \quad (f(x))_{(p)} = \lambda(T^*(g_{(p)}))(x)$$

(where, for example, $t_{(p)} = |t|^{p-1} \operatorname{sgn}(t)$) is given by

$$\lim_{n \rightarrow \infty} n \hat{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r}, \quad \text{for } 1 < q < p < \infty,$$

$$\lim_{n \rightarrow \infty} n \check{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r}, \text{ for } 1 < p < q < \infty.$$

Here $r = 1/p' + 1/p$, $c_{p,q}$ is an explicit constant depending only on p and q , $\check{\lambda}_n(T) = \max(sp_n(T, p, q))$, $\check{\lambda}_n(T) = \min(sp_n(T, p, q))$ where $sp_n(T, p, q)$ stands for the set of all eigenvalues λ corresponding to eigenfunctions g with n zeros.

1. – Introduction and preliminaries

Through this paper we shall assume $I = [a, b]$, where $-\infty < a < b < \infty$, and let $p, q \in (1, \infty)$, $(x)_{(p)} := |x|^{p-1} \operatorname{sgn}(x)$, $x \in \mathbb{R}$ and $1/p' = 1 - 1/p$.

Let u and v be positive functions on I , with $u \in L_{p'}(I)$, $v \in L_q(I)$.

Define the Hardy-type operator $T : L_p(I) \rightarrow L_q(I)$ by

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \quad x \in I.$$

Such maps have been intensively studied: see [4, Chapter 2].

Since $|I| = b - a < \infty$, $u \in L_{p'}(I)$ and $v \in L_q(I)$ then T is compact, see [5, chapter 2].

As more detailed information about the native of the compactness of a map is provided by its approximation, Kolmogorov and Bernstein numbers, much attention has been paid to the asymptotic behavior of these numbers for the map T . The analysis is decidedly easier when $p = q$, and an account of the situation in this case is given in [5]. For the case $p \neq q$ we refer to [6], [7] in which a key role is played by the non-linear integral system:

$$(1.1) \quad g(x) = (Tf)(x)$$

and

$$(1.2) \quad (f(x))_{(p)} = \lambda(T^*(g_{(q)}))(x),$$

where $g_{(q)}$ is the function with value $(g(x))_{(q)}$ at x and T^* is the map defined by $(T^*f)(x) = u(x) \int_x^b v(y)f(y)dy$.

The non-linear system (1.1) and (1.2) gives us the following non-linear equation:

$$(1.3) \quad (f(x))_{(p)} = \lambda T^*((Tf)_{(q)})(x).$$

This is equivalent to its dual equation:

$$(1.4) \quad (s(x))_{(q')} = \lambda^* T((T^*s)_{(p')})(x).$$

And we have this relation: For given f and λ satisfying (1.3) we have $s = (Tf)_{(q)}$ and $\lambda^* = \lambda_{(p')}$ satisfying (1.4), and for given s and λ^* satisfying (1.4) we have $f = (T^*s)_{(p')}$ and $\lambda = \lambda_{(q)}^*$ satisfying (1.3).

By a spectral triple will be meant a triple (g, f, λ) satisfying (1.1) and (1.2), where $\|f\|_p = 1$; (g, λ) will be called a spectral pair; the function g corresponding to λ is called a spectral function and the number λ occurring in a spectral pair will be called a spectral number.

For the system (1.1) and (1.2) we denote by $SP(T, p, q)$ the set of all spectral triples; $sp(T, p, q)$ will stand for the set of all spectral numbers λ from $SP(T, p, q)$.

It can be seen that this non-linear system is related to the isoperimetric problem of determining

$$(1.5) \quad \sup_{g \in T(B)} \|g\|_q,$$

where $B := \{f \in L_p(I) : \|f\|_p \leq 1\}$.

Moreover, this problem can be seen as a natural generalization of the p, q -Laplacian differential equation. For if u and v are identically equal to 1 on I , then (1.1) and (1.2) can be transformed into the p, q -Laplacian differential equation:

$$(1.6) \quad -((w')_{(p)})' = \lambda(w)_{(q)},$$

with the boundary condition

$$(1.7) \quad w(a) = 0.$$

If g, f and λ satisfy (1.1) and (1.2) then, the integrals being over I ,

$$\begin{aligned} \int |g(x)|^q dx &= \int g(g)_{(q)} dx = \int Tf(x)(g)_{(q)} dx \\ &= \int f(x)T^*(g)_{(q)} dx = \lambda^{-1} \int f(x)(f)_{(p)} \\ &= \lambda^{-1} \int |f(x)|^p dx. \end{aligned}$$

From this it follows that $\lambda^{-1} = \|g\|_q^q / \|f\|_p^p$ and then for $(g_1, \lambda_1) \in SP(T, p, q)$ we have $\lambda_1^{-1/q} = \|g_1\|_q$.

Given any continuous function f on I we denote by $Z(f)$ the number of distinct zeros of f on I and by $P(f)$ the number of sign changes of f on this interval. The set of all spectral triples (g, f, λ) with $Z(g) = n$ ($n \in \mathbb{N}_0$) will be denoted by $SP_n(T, p, q)$, and $sp_n(T, p, q)$ will represent the set of all corresponding numbers λ . We set $\hat{\lambda}_n = \max sp_n(T, p, q)$ and $\check{\lambda}_n = \min sp_n(T, p, q)$.

Our main result is that the asymptotic behavior of the $\hat{\lambda}_n$ can be determined when $1 < q < p < \infty$: we show that

$$\lim_{n \rightarrow \infty} n \hat{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r},$$

where $r = 1/p' + 1/q$ and c_{pq} is a constant whose dependence on p and q is given explicitly. A corresponding result holds for $\check{\lambda}_n$ when $1 < p < q < \infty$. Moreover, $sp_n(T, p, p)$ contains exactly one element, so that in this case $\hat{\lambda}_n = \check{\lambda}_n = \lambda_n$ say, and the asymptotic behavior of the λ_n is given by the formula above.

We now give some results to prepare for the major theorems in §2 and §3.

LEMMA 1.1. – *Let $f \neq 0$ be a function on $[a, b]$ such that $Tf(a) = Tf(b) = 0$. Then $P(f) \geq 1$.*

PROOF. – This follows from the positivity of T and Rolle's theorem. □

LEMMA 1.2. – *Let $(g_i, f_i, \lambda_i) \in SP(T, p, q)$, $i = 1, 2$, $1 < p, q < \infty$. Then for any $\varepsilon > 0$,*

$$(1.8) \quad P(Tf_1 - \varepsilon Tf_2) \leq P(Tf_1 - \varepsilon^{(p-1)/(q-1)} (\lambda_2/\lambda_1)^{1/(q-1)} Tf_2).$$

If the function $f_1 - \varepsilon f_2$ has a multiple zero and

$$P(Tf_1 - \varepsilon^{(p-1)/(q-1)} (\lambda_2/\lambda_1)^{q/(q-1)} Tf_2) < \infty,$$

then the inequality (1.8) is strict.

PROOF. – We will use Lemma 1.1 and the fact that $\text{sgn}(a - b) = \text{sgn}((a)_{(p)} - (b)_{(p)})$.

$$\begin{aligned} P(Tf_1 - \varepsilon Tf_2) &\leq Z(Tf_1 - \varepsilon Tf_2) \leq P(f_1 - \varepsilon f_2) \\ &\leq P((f_1)_{(p)} - \varepsilon^{p-1} (f_2)_{(p)}) \\ &\text{(use (1.2) for } f_1 \text{ and } f_2), \\ &\leq P(\lambda_1 T^*((g_1)_{(q)}) - \varepsilon^{p-1} \lambda_2 T^*((g_2)_{(q)})) \\ &\leq Z(\lambda_1 T^*((g_1)_{(q)}) - \varepsilon^{p-1} \lambda_2 T^*((g_2)_{(q)})) \\ &\leq P((g_1)_{(q)} - \varepsilon^{p-1} (\lambda_2/\lambda_1) (g_2)_{(q)}) \\ &\leq P(g_1 - \varepsilon^{(p-1)/(q-1)} (\lambda_2/\lambda_1)^{1/(q-1)} g_2) \\ &\leq P(Tf_1 - \varepsilon^{(p-1)/(q-1)} (\lambda_2/\lambda_1)^{1/(q-1)} Tf_2). \end{aligned}$$

□

THEOREM 1.3. – For all $n \in \mathbb{N}$, $SP_n(T, p, q) \neq \emptyset$.

PROOF. – This essentially follows ideas from [3] (see also [8]), but we give the details for the convenience of the reader. For simplicity we suppose that I is the interval $[0, 1]$. A key idea in the proof is the introduction of an iterative procedure used in [3].

Let $n \in \mathbb{N}$ and define

$$\mathcal{O}_n = \left\{ z = (z_1, \dots, z_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} |z_i| = 1 \right\}$$

and

$$f_0(x, z) = \operatorname{sgn}(z_j) \text{ for } \sum_{i=0}^{j-1} |z_i| < x < \sum_{i=1}^j |z_i|, \quad j = 1, \dots, n+1, \text{ with } z_0 = 0.$$

With $g_0(x, z) = Tf_0(x, z)$ we construct the iterative process

$$g_k(x, z) = Tf_k(x, z), \quad f_{k+1}(x, z) = (\lambda_k(z)T^*(g_k(x, z))_{(q)})_{(p')},$$

where λ_k is a constant so chosen that

$$\|f_{k+1}\|_p = 1$$

and $1/p + 1/p' = 1$. Then, all integrals being over I ,

$$\begin{aligned} 1 &= \int |f_k(x, z)|^p dx = \int f_k(f_k)_{(p)} dx = \int f_k([\lambda_{k-1}T^*((g_{k-1})_{(q)})]_{(p')})_{(p)} dx \\ &= \int f_k \lambda_{k-1} T^*((g_{k-1})_{(q)}) dx \\ &= \lambda_{k-1} \int T(f_k)(g_{k-1})_{(q)} dx \leq \lambda_{k-1} \|g_k\|_q \|g_{k-1}\|_q^{q-1} \end{aligned}$$

and also

$$\begin{aligned} \|g_{k-1}\|_q^q &= \int |g_{k-1}(x, z)|^q dx = \int (g_{k-1})_{(q)} g_{k-1} dx \\ &= \int (g_{k-1})_{(q)} T(f_{k-1}) dx = \int T^*((g_{k-1})_{(q)}) f_{k-1} dx \\ &= \lambda_{k-1}^{-1} \int \lambda_{k-1} T^*((g_{k-1})_{(q)}) f_{k-1} dx \\ &\leq \lambda_{k-1}^{-1} \left(\int |(\lambda_{k-1} T^*((g_{k-1})_{(q)})_{(p')})|^{p'} dx \right)^{1/p'} \left(\int |f_{k-1}|^p dx \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&= \lambda_{k-1}^{-1} \left(\int |(\lambda_{k-1} T^*((g_{k-1})_{(q)})_{(p')})|^{p'} dx \right)^{1/p'} \\
&= \lambda_{k-1}^{-1} \left(\int |f_k|^p dx \right)^{1/p} = \lambda_{k-1}^{-1}.
\end{aligned}$$

From these inequalities it follows that

$$\|g_{k-1}(\cdot, z)\|_q \leq \lambda_{k-1}^{-1/q} \leq \|g_k(\cdot, z)\|_q.$$

This shows that the sequences $\{g_k(\cdot, z)\}$ and $\{\lambda_k^{-1/q}(z)\}$ are monotonic increasing. Put $\lambda(z) = \lim_{k \rightarrow \infty} \lambda_k(z)$; then $\|g_k(\cdot, z)\|_q \rightarrow \lambda^{-1/q}(z)$.

As the sequence $\{f_k(\cdot, z)\}$ is bounded in $L_p(I)$, there is a subsequence $\{f_{k_i}(\cdot, z)\}$ that is weakly convergent, to $f(\cdot, z)$, say. Since T is compact, $g_{k_i}(\cdot, z) \rightarrow Tf(\cdot, z) := g(\cdot, z)$ and we also have $f(\cdot, z) = (\lambda(z)T^*(g(\cdot, z))_{(q)})_{(p')}$. It follows that for each $z \in \mathcal{O}_n$, the sequence $\{g_{k_i}(\cdot, z)\}$ converges to a spectral function.

Now set $z = (0, 0, \dots, 0, 1) \in \mathcal{O}_n$. Then $f_0(\cdot, z) = 1$, and as the operators T and T^* are positive, $g_k(\cdot, z) \geq 0$ for all k , so that $g(\cdot, z) \geq 0$. Thus $(g(\cdot, z), f(\cdot, z), \lambda(z)) \in SP_0(T, p, q) : SP_0(T, p, q) \neq \emptyset$.

Next we show that for all $n \in \mathbb{N}$, $SP_n(T, p, q) \neq \emptyset$. Given $n, k \in \mathbb{N}$, set

$$E_k^n = \{z \in \mathcal{O}_n : Z(g_k(\cdot, z)) \leq n - 1\}.$$

From the definition of T it follows that $g_k(\cdot, z)$ depends continuously on z ; thus E_k^n is an open subset of \mathcal{O}_n and $F_k^n := \mathcal{O}_n \setminus E_k^n$ is a closed subset of \mathcal{O}_n . Let $0 < t_1 < \dots < t_n < 1$ and put

$$F_k(a) = (g_k(t_1, a), \dots, g_k(t_n, a)), \quad a \in \mathcal{O}_n.$$

Then F_k is a continuous, odd mapping from \mathcal{O}_n to \mathbb{R}^n . By Borsuk's theorem, there is a point $a_k \in \mathcal{O}_n$ such that $F_k(a_k) = 0$; that is, $a_k \in F_k^n$. From the definition of g_k and f_{k+1} , together with the positivity of T and T^* , we have

$$Z(g_{k+1}) \leq P(f_{k+1}) \leq Z(f_{k+1}) \leq P(g_k) \leq Z(g_k),$$

so that $E_k^n \subset E_{k+1}^n$, which implies that $F_k^n \supset F_{k+1}^n$. Hence there exists $\tilde{a} \in \bigcap_{k \geq 1} F_k^n$, and as above we see that $g_k(\cdot, \tilde{a})$ converges, as $k \rightarrow \infty$, to a spectral function $g(\cdot, \tilde{a}) \in SP_n(T, p, q)$. Thus $SP_n(T, p, q) \neq \emptyset$ and the proof is complete. \square

We note that the previous theorem is true for much more general integral operators (i.e. integral operators with totally positive kernel, see [8]).

We now define Kolmogorov widths $d_n(T)$ for T as a map from $L_p(I)$ to $L_q(I)$ when $1 < q, p < \infty$. These numbers are defined by:

$$d_n(T) = d_n = \inf_{X_n} \sup_{\|f\|_{p,I} \leq 1} \inf_{g \in X_n} \|Tf - g\|_{q,I} / \|f\|_{p,I}, \quad n \in \mathbb{N}$$

where the infimum is taken over all n -dimensional subspaces X_n of $L_q(I)$.

To get an upper estimate for eigenvalues via the Kolmogorov numbers, we start by recalling the Makovoz lemma (see 3.11 in [3]).

LEMMA 1.4. – *Let $U_n \subset \{Tf; \|f\|_{p,I} \leq 1\}$ be a continuous and odd image of the sphere S^n in \mathbb{R}^n endowed with the l_1 norm. Then*

$$d_n(T) \geq \inf\{\|x\|_{q,I}, x \in U_n\}.$$

LEMMA 1.5. – *If $n > 1$, then $d_n(T) \geq \hat{\lambda}^{-1/q}$ where $\hat{\lambda} = \max\left\{\lambda \in \bigcup_{i=0}^n sp_i(p, q)\right\}$.*

PROOF. – Let us denote $\hat{\lambda} = \max\left\{\lambda \in \bigcup_{i=0}^n sp_i(p, q)\right\}$. The iteration process from the proof of Theorem 1.3 gives us for each $k \in \mathbb{N}$ and $z \in \mathcal{O}_n$ a function $g_k(\cdot, z)$. By the Makavoz lemma we have

$$(1.9) \quad d_n(T) \geq \max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_{q,I}.$$

Let us suppose that we have

$$(1.10) \quad \min_{z \in \mathcal{O}_n} \lim_{k \rightarrow \infty} \|g_k(\cdot, z)\|_q = \max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_q.$$

Then from (1.9) and (1.10) it follows that

$$d_n(T) \geq \min_{z \in \mathcal{O}_n} \lim_{k \rightarrow \infty} \|g_k(\cdot, z)\|_q \geq \hat{\lambda}^{-1/q}$$

since $\lim_{k \rightarrow \infty} g_k(\cdot, z) \in SP(T, p, q)$. We have to prove (1.10). From the monotonicity of $\|g_k(\cdot, z)\|_{q,I}$ we have

$$\max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_q = \lim_{k \rightarrow \infty} \min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_q.$$

From $\max \min \leq \min \max$ it follows that

$$\begin{aligned} l &:= \lim_{k \rightarrow \infty} \min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_q = \max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_q \\ &\leq \min_{z \in \mathcal{O}_n} \max_{k \in \mathbb{N}} \|g_k(\cdot, z)\|_q = \min_{z \in \mathcal{O}_n} \lim_{k \rightarrow \infty} \|g_k(\cdot, z)\|_q =: h \end{aligned}$$

Denote $H_k(\varepsilon) = \{z \in \mathcal{O}_n : \|g_k(\cdot, z)\|_q \leq h - \varepsilon\}$ where $0 < \varepsilon \leq h$.

Since the mapping $z \mapsto g_k(\cdot, z)$ is continuous, $H_k(\varepsilon)$ is a closed subset of \mathcal{O}_n , and from the construction of the sequence g_k we see that $H_0(\varepsilon) \supset H_1(\varepsilon) \supset \dots$

If $y_0 \in \bigcap_{k \in \mathbb{N}} H_k(\varepsilon) \neq \emptyset$ then $h = \min_{z \in \mathcal{O}_n} \lim_{k \rightarrow \infty} \|g_k(\cdot, z)\|_q \leq \lim_{k \rightarrow \infty} \|g_k(\cdot, y_0)\|_q \leq h - \varepsilon$ is a contradiction. Then there exist $k_0 \in \mathbb{N}$ such that $H_k(\varepsilon) = \emptyset$ for $k \geq k_0$ and $\min_{z \in \mathcal{O}_n} \|g_k(\cdot, z)\|_q \geq h - \varepsilon$ for $k \geq k_0$. Then we have that $h = l$ and (1.10) is proved. \square

Next we define Bernstein widths which will help us in section 3. The Bernstein widths $b_n(T)$ for $T: L_p(I) \rightarrow L_q(I)$ when $1 < p, q < \infty$ are defined by:

$$b_n(T) := \sup_{X_{n+1}} \inf_{Tf \in X_{n+1} \setminus \{0\}} \|Tf\|_{q,I} / \|f\|_{p,I},$$

where the supremum is taken over all subspaces X_{n+1} of $T(L_p(I))$ with dimension $n + 1$. Since u and v are positive functions, the Bernstein widths can be expressed as

$$b_n(T) = \sup_{X_{n+1}} \inf_{\alpha \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\left\| T \left(\sum_{i=1}^{n+1} \alpha_i f_i \right) \right\|_{q,I}}{\left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_{p,I}},$$

where the supremum is taken over all $(n + 1)$ -dimensional subspaces $X_{n+1} = \text{span} \{f_1, \dots, f_{n+1}\} \subset L_p(I)$.

Now we use techniques from Theorem 1.3 to obtain an upper estimate for the Bernstein widths.

LEMMA 1.6. – *If $n > 1$ then $b_n(T) \leq \check{\lambda}^{-1/q}$, where $\check{\lambda} = \min(sp_n(p, q))$.*

PROOF. – Suppose there exists a linearly independent system of functions $\{f_1, \dots, f_{n+1}\}$ on I , such that:

$$\min_{\alpha \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\left\| T \left(\sum_{i=1}^{n+1} \alpha_i f_i \right) \right\|_{q,I}}{\left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_{p,I}} > \check{\lambda}^{-1/q}.$$

Let us define the n -dimensional sphere

$$O_n = \left\{ T \left(\sum_{i=1}^{n+1} \alpha_i f_i \right) : \left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_{p,I} = 1 \right\}.$$

Let $g_0(\cdot) \in O_n$ and define a sequence of functions $h_k(\cdot), g_k(\cdot) = g_k(\cdot, g_0), k \in \mathbb{N}$, according to the following rule:

$$g_k(x) = Th_k(x), \quad h_{k+1}(x) = (\lambda_k T^*(g_k(x))_{(q)})_{(p')},$$

where $\lambda_k > 0$ is a constant chosen so that $\|h_{k+1}\|_{p,I} = 1$.

We denote $O_n(k) = \{g_k(\cdot, g_0) : g_0(\cdot) \in O_n\}$. As in the proof of Theorem 1.3 we have:

$\|g_k\|_{q,I}$ is a nondecreasing as $k \nearrow \infty$. For each $k \in \mathbb{N}$ there exists $g_k \in O_n(k)$ with n zeros inside I ; $\lim_{k \rightarrow \infty} g_k(\cdot, g_0)$ is an eigenfunction and there exists $g_0(\cdot)$ such

that $\lim_{k \rightarrow \infty} g_k(\cdot, g_0)$ is an eigenfunction with n zeros. Moreover λ_k is monotonically decreasing as $k \nearrow \infty$.

Let $\bar{a} \in \mathbb{R}^{n+1}$ be such that: $\bar{g}_0(\cdot) = \left(\sum_{i=1}^{n+1} \bar{a}_i f_i \right)$ is a function for which $\lim_{k \rightarrow \infty} \bar{g}_k(\cdot, g_0)$ is an eigenfunction with n zeros.

Then we have the following contradiction:

$$\begin{aligned} \min_{a \in \mathbb{R}^n \setminus \{0\}} \frac{\left\| T \left(\sum_{i=1}^{n+1} \bar{a}_i f_i \right) \right\|_{q,I}}{\left\| \sum_{i=1}^{n+1} \bar{a}_i f_i \right\|_{p,I}} &\leq \|\bar{g}_0(\cdot)\|_{q,I} \\ &\leq \lim_{k \rightarrow \infty} \|g_k(\cdot, \bar{g}_0(\cdot))\|_{q,I} \leq \check{\lambda}^{-1/q}. \end{aligned} \quad \square$$

In the next two sections we obtain an upper estimate for Kolmogorov numbers and a lower estimate for Bernstein numbers. We shall need the approximation numbers $a_n(T)$ of T , defined by $a_n(T) = \inf \|T - F\|$, where the infimum is taken over all linear operators F with rank at most $n - 1$.

2. – The case $q \leq p$

We recall Jensen's inequality (see, for example [9], p.133) which will be of help in the next lemma.

THEOREM 2.1. – *If F is a convex function, and $h(\cdot) \geq 0$ is a function such that $\int_I h(t) dt = 1$, then for every non-negative function g ,*

$$F\left(\int_I h(t)g(t)dt\right) \leq \int_I h(t)F(g(t))dt.$$

The following lemma give us a lower estimate for eigenvalues.

LEMMA 2.2. – *If $n > 1$ then $a_n(T) \leq \hat{\lambda}^{-1/q}$, where $\hat{\lambda} = \max(sp_n(p, q))$.*

PROOF. – For the sake of simplicity we suppose that $|I| = 1$.

Let $(\hat{g}, \hat{f}, \hat{\lambda}) \in SP_n(T, p, q)$. Denote by $\{a_i\}_{i=0}^n$ the set of zeros of \hat{g} (with $a_0 = a$) and by $\{b_i\}_{i=1}^{n+1}$ (with $b_{n+1} = b$) the set of zeros of \hat{f} . Set $I_i = (b_i, b_{i+1})$ for $i = 1, \dots, n$ and $I_0 = (a_0, b_1)$, and define

$$T_n f(x) := \sum_{i=0}^n \chi_{I_i}(\cdot) v(\cdot) \int_a^{a_i} u(t) f(t) dt.$$

Then the rank of T_n is at most n .

We have (see [4, Chapter 2]) $d_n(T) \leq a_n(T) \leq \sup_{\|f\|_p \leq 1} \|Tf - T_n f\|_q$.

Let us consider the extremal problem:

$$(2.1) \quad \sup_{\|f\|_p \leq 1} \|Tf - T_n f\|_q.$$

We can see that this problem is equivalent to

$$(2.2) \quad \sup\{\|Tf\|_q : \|f\|_p \leq 1, (Tf)(a_i) = 0 \text{ for } i = 0, \dots, n\}$$

Since T and T_n are compact then there is a solution of this problem, that is, the supremum is attained. Let \bar{f} be one such solution and denote $\bar{g} = T\bar{f}$. We can choose \bar{f} such that $\bar{g}(t)\hat{g}(t) \geq 0$, for all $t \in I$. We have $\|\bar{g}\|_{q,I} \geq \|\hat{g}\|_{q,I}$.

Note that for any $f \in L^p(I)$ such that $Tf(a_i) = 0$ for every $i = 0, \dots, n$ we have $Tf(x) = T^+f(x)$ for each $x \in I$, where

$$T^+f(x) := \int_I K(x, t)f(t)dt = \sum_{i=0}^n \chi_{I_i}(\cdot)v(\cdot) \int_a^x u(t)f(t)dt$$

and

$$K(x, t) := \sum_{i=0}^n \chi_{I_i}(x)v(x)u(t)\chi_{(a_i, x)} \operatorname{sgn}(x - a_i).$$

Set $s(t) = |\hat{g}(t)|^q \hat{\lambda}^q$, where $\hat{\lambda} = \|\hat{g}\|_{q,I}$. Then, all integrals being over I , we have

$$\left(\int |\bar{g}(t)|^q dt \right)^{1/q} = \hat{\lambda}^{-1/q} \left(\int s(t) \left| \frac{\bar{g}(t)}{\hat{g}(t)} \right|^q dt \right)^{1/q}$$

(use Jensen's inequality, noting that $\int s(t)dt = 1$)

$$\begin{aligned} &\leq \hat{\lambda}^{-1/q} \left(\int s(t) \left| \frac{\bar{g}(t)}{\hat{g}(t)} \right|^p dt \right)^{1/p} \\ &= \hat{\lambda}^{-1/q} \left(\int s(t) \left| \frac{T^+\bar{f}(t)}{\hat{g}(t)} \right|^p dt \right)^{1/p} \\ &= \hat{\lambda}^{-1/q} \left(\int s(t) \left| \frac{\int K(t, \tau)\bar{f}(\tau)d\tau}{\hat{g}(t)} \right|^p dt \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&= \hat{\lambda}^{-1/q} \left(\int s(t) \left| \int \frac{K(t, \tau) \hat{f}(\tau) \bar{f}(\tau)}{\hat{g}(t) \hat{f}(\tau)} d\tau \right|^p dt \right)^{1/p} \\
&\quad \text{(use Jensen's inequality, noting that} \\
&\quad \frac{K(t, \tau) \hat{f}(\tau)}{\hat{g}(t)} \geq 0 \text{ and } \int \frac{K(t, \tau) \hat{f}(\tau)}{\hat{g}(t)} d\tau = 1) \\
&\leq \hat{\lambda}^{-1/q} \left(\int s(t) \int \frac{K(t, \tau) \hat{f}(\tau)}{\hat{g}(t)} \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p d\tau dt \right)^{1/p} \\
&= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \bar{f}(\tau) \int \frac{K(t, \tau) s(t)}{\hat{g}(t)} dt d\tau \right)^{1/p} \\
&= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \bar{f}(\tau) \int \frac{K(t, \tau) |\hat{g}(t)|^q}{\hat{g}(t)} \hat{\lambda} dt d\tau \right)^{1/p} \\
&= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \bar{f}(\tau) \int K(t, \tau) \hat{g}_{(q)}(t) dt \hat{\lambda} d\tau \right)^{1/p} \\
&\quad \left(\text{use } \int K(t, \tau) \hat{g}_{(q)}(t) dt \hat{\lambda}^q = \hat{\lambda} T^*(\hat{g}_{(q)})(t) = \hat{f}_{(p)}(t) \right) \\
&= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \hat{f}(\tau) \hat{f}_{(p)}(\tau) d\tau \right)^{1/p} \\
&\quad \text{(use } \hat{f}(t) \hat{f}_{(p)}(t) = |\hat{f}(t)|^p) \\
&= \hat{\lambda}^{-1/q} \left(\int |\bar{f}(\tau)|^p d\tau \right)^{1/p} = \hat{\lambda}^{-1/q}.
\end{aligned}$$

From this it follows that $a_n(T) \leq \hat{\lambda}^{-1/q}$. □

THEOREM 2.3. – *If $1 < q \leq p < \infty$, then*

$$\lim_{n \rightarrow \infty} n \hat{\lambda}_n^{-1/q} = c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r$$

where $r = 1/p' + 1/q$, $\hat{\lambda}_n = \max(sp_n(p, q))$ and

$$(2.3) \quad c_{pq} = \frac{(p')^{1/q} q^{1/p'} (p' + q)^{1/p-1/q}}{2B(1/q, 1/p')}$$

(B denotes the Beta function).

PROOF. – From [7] we have

$$\lim_{n \rightarrow \infty} na_n(T) = \lim_{n \rightarrow \infty} nd_n(T) = c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r$$

and since $d_n(T) \leq a_n(T)$, $a_n(T) \searrow 0$ and $d_n(T) \searrow 0$ then from Lemma 2.2 follows:

$$c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r \leq \liminf_{n \rightarrow \infty} n \hat{\lambda}_n^{-1/q},$$

and from Lemma 1.5 we have

$$\limsup_{n \rightarrow \infty} n \hat{\lambda}_n^{-1/q} \leq c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r$$

which finishes the proof. \square

3. – The case $p \leq q$

LEMMA 3.1. – *Let $1 < p \leq q < \infty$ and $n > 1$. Then $b_n(T) \geq \tilde{\lambda}^{-1/q}$, where $\tilde{\lambda} = \min(sp_n(p, q))$.*

PROOF. – We use the construction of Buslaev [2] Take $(\check{g}, \check{f}, \check{\lambda})$ from $SP_n(T, p, q)$ and denote by $a = x_0 < x_1 < \dots < x_i < \dots < x_n < x_{n+1} = b$ the zeros of \check{g} . Set $I_i = (x_{i-1}, x_i)$ for $1 \leq i \leq n+1$, $f_i(\cdot) = \check{f}(\cdot)\chi_{I_i}(\cdot)$ and $g_i(\cdot) = \check{g}(\cdot)\chi_{I_i}(\cdot)$. Then $Tf_i = g_i(\cdot)$ for $1 \leq i \leq n+1$.

Define $X_{n+1} = \text{span}\{f_1, \dots, f_{n+1}\}$. Since the supports of $\{f_i\}$ and $\{g_i\}$ are disjoint, then we have

$$b_n(T) \geq \inf_{a \in \mathbb{R}^n \setminus \{0\}} \frac{\left\| T \left(\sum_{i=1}^{n+1} a_i f_i \right) \right\|_{q,I}}{\left\| \sum_{i=1}^{n+1} a_i f_i \right\|_{p,I}} = \inf_{a \in \mathbb{R}^n \setminus \{0\}} \frac{\left\| \sum_{i=1}^{n+1} a_i g_i \right\|_{q,I}}{\left\| \sum_{i=1}^{n+1} a_i f_i \right\|_{p,I}}.$$

We shall study the extremal problem of finding

$$\inf_{a \in \mathbb{R}^n \setminus \{0\}} \frac{\left\| \sum_{i=1}^{n+1} a_i g_i \right\|_{q,I}}{\left\| \sum_{i=1}^{n+1} a_i f_i \right\|_{p,I}}.$$

It is obvious that the extremal problem has a solution. Denote that solution by $\bar{a} = (\bar{a}_1, \bar{a}_2, \dots)$. Since $p \leq q$, a short computation shows us that $\bar{a}_i \neq 0$ for every i , moreover we can suppose that the \bar{a}_i alternate in sign. Label

$$\bar{\gamma} := \frac{\left\| \sum_{i=1}^{n+1} \bar{a}_i g_i \right\|_{q,I}^q}{\left\| \sum_{i=1}^{n+1} \bar{a}_i f_i \right\|_{p,I}^p};$$

then the solution of the extremal problem is given by $\bar{g} = \sum_{i=1}^{n+1} \bar{a}_i g_i$, $\bar{f} = \sum_{i=1}^{n+1} \bar{a}_i f_i$ where $\|\bar{f}\|_p = 1$.

Let us take the vector $\beta = (1, -1, \dots)$. Define the functions $\tilde{g} = \sum_{i=1}^{n+1} \beta_i g_i$, $\tilde{f} = \sum_{i=1}^{n+1} \beta_i f_i$. Then

$$\lambda_n^{-1} := \frac{\left\| \sum_{i=1}^{n+1} \beta_i g_i \right\|_{q,I}^q}{\left\| \sum_{i=1}^{n+1} \beta_i f_i \right\|_{p,I}^p}.$$

It is obvious that $\bar{\gamma} \leq \lambda_n^{-1}$. Suppose that $\bar{\gamma} < \lambda_n^{-1}$.

Since $\bar{a}_i \neq 0$, $|\beta_i| = 1$ and $\bar{\gamma} < \lambda_n^{-1}$ then $0 < \varepsilon^* := \min_{1 \leq i \leq n+1} (\beta_i / \bar{a}_i) < 1$. From Lemma 1.2 follows

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \leq P(T(\tilde{f}) - \varepsilon^{*(p-1)/(q-1)} (\bar{\gamma} / \lambda_n^{-1})^{1/(q-1)} T(\bar{f})).$$

By repeated use of Lemma 1.2 with the help of $(\varepsilon^*)^{(p-1)/(q-1)} \leq \varepsilon^* < 1$ and $\bar{\gamma} / \lambda_n^{-1} < 1$ we get

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \leq P(T(\tilde{f})) = n.$$

On the other hand we have from Lemma 1.1 and the definition of ε^* that

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \leq P(\tilde{f} - \varepsilon^* \bar{f}) = P\left(\sum_{i=1}^{n+1} \beta_i f_i - \varepsilon^* \sum_{i=1}^{n+1} \bar{a}_i f_i\right) \leq n-1,$$

which contradicts $\bar{\gamma} < \lambda_n^{-1}$. □

THEOREM 3.2. – *If $1 < p \leq q < \infty$ then*

$$\lim_{n \rightarrow \infty} n \tilde{\lambda}_n^{-1/q} = c_{pq} \left(\int_I |uv|^r dt \right)^{1/r},$$

where $r = 1/p' + 1/q$, $\tilde{\lambda}_n = \min(sp_n(p, q))$ and c_{pq} as in (2.3).

PROOF. – From [6] we have

$$\lim_{n \rightarrow \infty} nb_n(T) = c_{pq} \left(\int_I |uv|^r dt \right)^{1/r}$$

and since $b_n(T) \searrow 0$ then from Lemma 1.6 it follows that

$$c_{pq} \left(\int_I |uv|^r dt \right)^{1/r} \leq \liminf_{n \rightarrow \infty} n\check{\lambda}_n^{-1/q}.$$

Moreover, from Lemma 3.1 we have

$$\limsup_{n \rightarrow \infty} n\check{\lambda}_n^{-1/q} \leq c_{pq} \left(\int_I |uv|^r dt \right)^{1/r}$$

which finishes the proof. \square

When $p = q$ the following lemma follows from Theorem 2.3 and Theorem 3.2 (we can find this result in a sharper form in [1]).

REMARK 3.3. – When $p = q$ then

$$\lim_{n \rightarrow \infty} n\lambda_n^{-1/q} = c_{pq} \left(\int_I |uv|^r dt \right)^{1/r}$$

where $r = 1/p' + 1/q$, c_{pq} as in (2.3) and λ_n is the single point in $sp_n(p, q)$.

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