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L^p Maximal Regularity for Second Order Cauchy Problems is Independent of p

RALPH CHILL - SACHI SRIVASTAVA (*)

Sunto. – *Si prova che se il problema del secondo ordine $\ddot{u} + B\dot{u} + Au = f$ ha regolarità massimale L^p per qualche $p \in (1, \infty)$ allora ha regolarità massimale L^p per ogni $p \in (1, \infty)$.*

Abstract. – *If the second order problem $\ddot{u} + B\dot{u} + Au = f$ has L^p maximal regularity for some $p \in (1, \infty)$, then it has L^p-maximal regularity for every $p \in (1, \infty)$.*

1. – Introduction.

The notion of L^p maximal regularity for the abstract linear second order problem

$$\ddot{u} + B\dot{u} + Au = f \text{ on } [0, T], \quad u(0) = \dot{u}(0) = 0, \tag{1}$$

was first introduced and studied in [5]. Here A and B are two closed linear operators on a Banach space $\mathcal{X}$, with dense domains $\mathcal{D}_A$ and $\mathcal{D}_B$, respectively. We say that the problem (1) has *L^p-maximal regularity*, if for every $f \in L^p(0, T; \mathcal{X})$ there exists a unique *(strong) solution*

$$u \in MR_{p,T} := \{ v \in W^{2,p}(0, T; \mathcal{X}) \cap L^p(0, T; \mathcal{D}_A) : \dot{v} \in L^p(0, T; \mathcal{D}_B) \}$$

of the inhomogeneous problem (1). Strong solution means that $u(0) = \dot{u}(0) = 0$ and the differential equation (1) is satisfied almost everywhere. The space $MR_{p,T}$ is called *maximal regularity space*.

The definition of L^p maximal regularity is similar to that of L^p maximal regularity for the first order problem $\dot{u} + Au = f$, and is closely related to the abstract notion of maximal regularity studied first by Da Prato and Grisvard [6],

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and then also by Acquistapace and Terreni [1], Dore and Venni [9], and Labbas and Terreni [11].

In this note, which may be considered as a sequel to [5], we show that L^p maximal regularity for the second order problem is independent of the choice of p, $1 < p < \infty$. For the examples studied in [5], L^p maximal regularity was known to hold for all $p \in (1, \infty)$, but at the end of this note we describe another example for which only L^2 maximal regularity was known. The fact that L^p maximal regularity is independent of p is well known for the first order problem; see for example, De Simon [8] in the case of Hilbert spaces, Sobolevskii [12], Cannarsa and Vespri [4], Hieber [10] in the general case.

2. – Initial value problem.

We first prove that L^p-maximal regularity of (1) implies existence and uniqueness of strong solutions of the initial value Cauchy problem

$$\ddot{u} + B\dot{u} + Au = 0 \text{ on } [0, T], \quad u(0) = u_0,\ \dot{u}(0) = u_1, \tag{2}$$

if the pair of initial values belongs to the *trace space* Tr_p associated with $\mathcal{X}$, A and B, defined as

$$Tr_p := \{(u(0), \dot{u}(0)) : u \in MR_{p,1}\}.$$

Subsequently, two consequences of this result are recalled.

The existence and uniqueness theorem was proven in [5] under the additional assumption $\mathcal{D}_A \hookrightarrow \mathcal{D}_B$. We present a proof which does not require this assumption.

THEOREM 2.1. – *(Initial value problem) Let* $p \in (1, \infty)$. *Suppose that* (1) *has* L^p *maximal regularity for some* $T > 0$. *Then, for every* $(u_0, u_1) \in Tr_p$ *and every* $T > 0$, *there exists a unique solution* $u \in MR_{p,T}$ *of the initial value problem* (2).

PROOF. – Suppose that (1) has L^p maximal regularity for $T > 0$. Let $(u_0, u_1) \in Tr_p$ be given. By [5, Lemma 6.3 (i), (ii)], for every $t \in [0, T]$,

$$\{(u(t), \dot{u}(t)) : u \in MR_{p,T}\} = Tr_p.$$

Hence, there exists $v \in MR_{p,T}$ such that $u_0 = v(0)$ while $u_1 = \dot{v}(0)$. Let $f := \ddot{v} + B\dot{v} + Av \in L^p(0, T; \mathcal{X})$. By L^p maximal regularity, there exists a solution $w \in MR_{p,T}$ of (1) with f as chosen above. Then $w(0) = 0 = \dot{w}(0)$. Let $u := v - w \in MR_{p,T}$. Clearly u is then a solution of (2) with initial values $u(0) = u_0$ and $\dot{u}(0) = u_1$. Uniqueness of this solution on $[0, T]$ follows from linearity and unique solvability of the problem (1).

This solution of (2) can be extended to a solution on $[0, 2T]$. Indeed, by the same argument as above, for $(u(T), \dot{u}(T)) \in Tr_p$, there exists a unique solution $z \in MR_{p,T}$ of the initial value problem (2) satisfying $z(0) = u(T)$, $\dot{z}(0) = \dot{u}(T)$. Then setting $\tilde{u}(t) = u(t)$ if $0 \le t \le T$ and equal to $z(t - T)$ if $T \le t \le 2T$, we obtain a solution of (2) in $MR_{p,2T}$. Iterating this argument we see that the solution u can be extended or restricted to a solution in $MR_{p,T'}$ for any $T' > 0$.

In order to show uniqueness on $[0, T']$ for any $T' > 0$, let u and v be two solutions of (2) on $[0, T']$. Extending both solutions, if necessary, we can assume that $T' = kT$ for some $k \in \mathbb{N}$. Then it suffices to note that $u = v$ on $[0, T]$ by uniqueness on the intervall $[0, T]$, and iterate this argument. □

The same arguments as in the proof of [5, Corollary 2.4] can now be used to show that if the problem (1) has L^p maximal regularity for some $T > 0$ then it has L^p maximal regularity for all $T > 0$. This strengthens the earlier result as the assumption that $\mathcal{D}_A \hookrightarrow \mathcal{D}_B$ is no longer required. We record this statement formally as

COROLLARY 2.2 [Independence of $T > 0$]. – *Let* $p \in (1, \infty)$. *Suppose that the problem* (1) *has* L^p *maximal regularity on* $[0, T]$ *for some* $0 < T < \infty$. *Then the problem* (1) *has* L^p *maximal regularity on* $[0, T]$ *for every* T, $0 < T < \infty$.

Recall from [5, Lemma 6.1] that the trace space Tr_p is the product $V_0 \times V_1$ of two Banach spaces which continuously embed into $\mathcal{X}$. Hence, Theorem 2.1 implies that for every $x \in V_1$ and every $T > 0$ the initial value problem

(3) $$\ddot{u} + B\dot{u} + Au = 0 \text{ on } [0, T], \quad u(0) = 0,\ \dot{u}(0) = x,$$

admits a unique solution $u \in MR_{p,T}$. Setting $S(t)x := u(t)$, where u is this solution of the preceeding problem, we thus obtain a solution family $(S(t))_{t \ge 0}$ of operators in $\mathcal{L}(V_1, V_0)$ which plays the role of a *sine family*. In a similar way one could define the *cosine family* associated with (2), but unlike in [2] (where $B = 0$), the sine family is in general not the primitive of the cosine family.

As in [5, Proposition 2.2], one can show that the $S(t)$ extend to operators in $\mathcal{L}(\mathcal{X})$, and that for every $f \in L^p(0, T; \mathcal{X})$ the convolution $S * f$ is the unique solution of the inhomogeneous problem (1). Infact, the following is true.

COROLLARY 2.3 [Sine family and inhomogeneous problem]. – *Suppose that* (1) *has* L^p *maximal regularity and define the sine family* $(S(t))$ *as above. Then*

$$S \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X})) \cap C^\infty((0, \infty); \mathcal{L}(\mathcal{X}; \mathcal{D}_A \cap \mathcal{D}_B))$$

and for every $f \in L^p(0, T; \mathcal{X})$ *the solution* u *of the inhomogeneous problem* (1) *is given by*

$$u(t) = (S * f)(t) := \int_{\mathbb{R}_+} S(t - s) f(s) ds.$$

3. – Regularity of solutions.

The following result exhibits the regularity of the solutions of the initial value problem (2) by invoking an idea used in the proof of [5, Proposition 2.2].

THEOREM 3.1. – *Let* $p \in (1, \infty)$ *and assume that* (1) *has* L^p *maximal regularity. Let* $(u_0, u_1) \in Tr_p$, $T > 0$, *and let* $u \in Mr_{p,T}$ *be the unique solution of the initial value problem* (2)*. Then* $u \in C^\infty((0,T); \mathcal{D}_A)$ *and* $\dot{u} \in C^\infty((0,T); \mathcal{D}_B)$. *Moreover, if for* $k \in \mathbb{N}$ *one defines*

$$u_k(t) := t^k u^{(k)}(t), \quad t \in [0,T],$$

then $u_k \in MR_{p,T}$.

REMARK 3.2. – The proof below will actually show that the solution extends to an analytic function in a sector around the positive real axis.

PROOF. -- By Theorem 2.1 we know that the solution extends to a solution on $\mathbb{R}_+$. We consider the operator

$$G : (-1,1) \times MR_{p,T} \to L^p(0,T;\mathcal{X}) \times Tr_p,$$
$$(\lambda, v) \mapsto (\ddot{v} + (1+\lambda)B\dot{v} + (1+\lambda)^2 Av, (v(0) - u_0, \dot{v}(0) - u_1)).$$

The operator G is clearly analytic (see [13] for the definition of an analytic function between two Banach spaces). For $\lambda \in (-1,1)$ we put $u^\lambda(t) := u(t + \lambda t)$. Then $u^\lambda \in MR_{p,T}$, $u^0 = u$, and $G(\lambda, u^\lambda) = 0$. Moreover, the partial derivative

$$\frac{\partial G}{\partial v}(0,u) : MR_{p,T} \to L^p(0,T;\mathcal{X}) \times Tr_p,$$
$$v \mapsto (\ddot{v} + B\dot{v} + Av, v(0), \dot{v}(0))$$

is boundedly invertible by L^p-maximal regularity and Theorem 2.1. Hence, by the implicit function theorem, [13, Theorem 4.B, p. 150], there exists $\varepsilon > 0$, a neighbourhood U of u in $MR_{p,T}$, and an analytic function $g : (-\varepsilon, \varepsilon) \to U$ such that

$$\{(\lambda, v) \in (-\varepsilon, \varepsilon) \times U : G(\lambda, v) = 0\} = \{(\lambda, g(\lambda)) : \lambda \in (-\varepsilon, \varepsilon)\}.$$

From this we obtain $g(\lambda) = u^\lambda$, that is, the function $\lambda \to u^\lambda$ is analytic in $(-\varepsilon, \varepsilon)$. In particular, the derivatives $\frac{d^k}{d\lambda^k} u^\lambda|_{\lambda=0}$ exist in $MR_{p,T}$ for every $k \in \mathbb{N}$. One easily checks that $\frac{d^k}{d\lambda^k} u^\lambda|_{\lambda=0}$ coincides with the function u_k defined in the statement, so that one part of the claim is proved. The regularity of u and $\dot{u}$ is an easy consequence of this part. □

Our next Lemma is of a technical nature. We define, for $0 < T < \infty$, $k \in \mathbb{N}_0$, and $1 < p < \infty$, the spaces $D^{k,p}(0,T;\mathcal{X})$ as

$$D^{k,p}(0,T;\mathcal{X}) := \Big\{ f \in W^{k,p}_{loc}((0,T);X) : f_j \in L^p(0,T;\mathcal{X}) \text{ for every } 0 \le j \le k, \\ \text{where } f_j(t) = t^j f^{(j)}(t) \big\}.$$

Equipped with the norm $\|\cdot\|_{D^{k,p}}$ given by

$$\|f\|_{D^{k,p}} := \sum_{j=0}^{k} \|f_j\|_{L^p}$$

these spaces are Banach spaces.

LEMMA 3.3. – *Suppose the problem* (1) *has L^p maximal regularity for some $p \in (1,\infty)$. Then for every $k \in \mathbb{N}_0$ the map $\psi_{k,T} : D^{k,p}(0,2T;\mathcal{X}) \to MR_{p,T}$ given by $(\psi_{k,T} f)(t) = t^k u^{(k)}(t)$, where u is the unique solution of* (1) *corresponding to f, is bounded. Moreover, there is a constant $C_k > 0$ such that $\|\psi_{k,T}\| \le C_k$ for all $0 < T \le 1$.*

PROOF. – In the following, we will write, for convenience $\psi_{k,T} = \psi$, and only specify the indices when there is a chance of confusion. Let $f \in D^{k,p}(0,2T;X)$ and let u denote the unique solution of (1) for this f. Let $MR^0_{p,T}$ be the subspace of $MR_{p,T}$ given by

$$MR^0_{p,T} = \big\{ v \in MR_{p,T} : v(0) = \dot{v}(0) = 0 \big\}.$$

By considering the C^k map

$$G : (-1,1) \times MR^0_{p,T} \to L^p(0,T;\mathcal{X})$$
$$G(\lambda, v)(t) = \ddot{v}(t) + (1+\lambda)B\dot{v}(t) + (1+\lambda)^2 A v(t) - (1+\lambda)^2 f((1+\lambda)t)$$

and following the same strategy as in the proof of Theorem 3.1 we get an $\varepsilon > 0$, such that the function $\lambda \to u^\lambda$ is C^k from $(-\varepsilon,\varepsilon)$ into $MR_{p,T}$; $u^\lambda(t) = u(t+\lambda t)$. In particular, the derivatives $\dfrac{d^j}{d\lambda^j} u^\lambda|_{\lambda=0}$ exist in $MR_{p,T}$ for every $j \in \{0,1,\ldots,k\}$. But $\dfrac{d^j}{d\lambda^j} u^\lambda|_{\lambda=0}(t) = t^j u^{(j)}(t)$. Therefore ψ maps $D^{k,p}(0,2T;\mathcal{X})$ into $MR_{p,T}$. Recall here that the operator that maps $f \in L^p(0,T;\mathcal{X})$ to the unique solution $u \in MR_{p,T}$ of problem (1) is bounded. It is straightforward to check then that ψ is closed. By the closed graph theorem it follows that ψ is bounded.

Let E be a bounded operator that maps any $f \in D^{k,p}(0,2;\mathcal{X})$ to an extension $Ef \in D^{k,p}(0,4;\mathcal{X})$ in such a way that $(Ef)(t) = 0$ for $t \in (3,4)$.

For $a > 0$ and $\tau > 0$, let $D_a : D^{k,p}(0,\tau;\mathcal{X}) \to D^{k,p}(0,\frac{\tau}{a};\mathcal{X})$ be the dilation given by $(D_a f)(t) = f(at)$. Then, for every $\tau > 0$,

$$\|D_a f\|_{D^{k,p}(0,\frac{\tau}{a};\mathcal{X})} = \frac{1}{a}\|f\|_{D^{k,p}(0,\tau;\mathcal{X})}. \tag{4}$$

Let $0 < T \le 1$. Then for any $f \in D^{k,p}(0,2T;\mathcal{X})$ and $t \in [0,2]$ set

$$(E_T f)(t) := \begin{cases} (D_{\frac{1}{T}} \circ E \circ D_T f)(t) & \text{for } t \in [0,4T] \cap [0,2], \\ 0 & \text{otherwise.} \end{cases}$$

Then E_T is bounded extension operator from $D^{k,p}(0,2T;\mathcal{X})$ into $D^{k,p}(0,2;\mathcal{X})$, and we have, on using (4),

$$\begin{aligned} \|E_T f\|_{D^{k,p}(0,2;\mathcal{X})} &\le \|D_{\frac{1}{T}} \circ E \circ D_T f\|_{D^{k,p}(0,4T;\mathcal{X})} \\ &= T\,\|E \circ D_T f\|_{D^{k,p}(0,4;\mathcal{X})} \\ &\le T\,\|E\|\,\|D_T f\|_{D^{k,p}(0,2;\mathcal{X})} \\ &= \|E\|\,\|f\|_{D^{k,p}(0,2T;\mathcal{X})}. \end{aligned}$$

Hence, for all $T \in (0,1]$,

$$\begin{aligned} \left\| \psi_{k,T} f\right\|_{MR_{p,T}} &\le \left\| \psi_{k,1}(E_T f)\right\|_{MR_{p,1}} \\ &\le \|\psi_{k,1}\|\,\|E_T f\|_{D^{k,p}(0,2;\mathcal{X})} \\ &\le \|\psi_{k,1}\|\,\|E\|\,\|f\|_{D^{k,p}(0,2T;\mathcal{X})}. \end{aligned}$$

Therefore $\left\|\psi_{k,T}\right\| \le C_k := \|\psi_{k,1}\|\,\|E\|$ for all T, $0 < T \le 1$. ☐

4. – *p* independence.

In this section we will establish that if the problem (1) has L^p maximal regularity for some $p \in (1,\infty)$, then it has L^p maximal regularity for all $p \in (1,\infty)$. The scheme followed will be similar to that in [10], where the corresponding result for the first order problem is proven. We will make use of the following theorem, which is a vector valued version of a result due to Benedek, Calderón and Panzone [3].

THEOREM 4.1 ([10], Theorem 4.3). – *Suppose that T is a bounded operator on $L^p(R;\mathcal{X})$ for some $p \in (1,\infty)$ and is represented by*

$$Tf(t) := \int_{\mathbb{R}} K(t-s) f(s)\,ds \tag{5}$$

for $f \in L^\infty(\mathbb{R};\mathcal{X})$ *with compact support and* $t \notin \operatorname{supp} f$, *and the kernel* $K \in L^1_{loc}(\mathbb{R}\setminus\{0\},\mathcal{X})$ *satisfies*

$$\int_{|t|>c|s|} \|K(t-s)-K(t)\|dt \quad \le \quad C \quad \text{for every } s \neq 0 \text{ and some } c>1. \tag{6}$$

Then T *admits a bounded extension to* $L^r(\mathbb{R},\mathcal{X})$ *for every* $r\in(1,\infty)$, *that is, there exists a constant* $\tilde{C}_r$ *such that*

$$\| Tf \|_{L^r(\mathbb{R},\mathcal{X})} \le \tilde{C}_r \| f \|_{L^r(\mathbb{R},\mathcal{X})} .$$

We now state our main result concerning L^p maximal regularity for problem (1) and the choice of $p\in(1,\infty)$.

THEOREM 4.2. – *Suppose that for some* $p\in(1,\infty)$ *the problem* (1) *has* L^p *maximal regularity. Then* (1) *has* L^p *maximal regularity for every* $p\in(1,\infty)$.

PROOF. – Fix $p\in(1,\infty)$. Suppose that problem (1) has L^p maximal regularity and let $(S(t))$ be the sine family from Corollary 2.3. By L^p maximal regularity and Corollary 2.3, the convolution operator $f\mapsto S*f$ is a bounded linear operator from $L^p(0,T;\mathcal{X})$ into $MR_{p,T}$.

It is enough for our purposes to show that the convolution operator $f\mapsto S*f$ extends to a bounded linear operator from L^r to MR_r for every $r\in(1,\infty)$.

Fix $x\in X$ and $1\ge T>0$. Set $f(s)=x$, $s\ge 0$. Then $f\in D^{k,p}(0,2T,\mathcal{X})$ for all $k\in\mathbb{N}_0$. Therefore, applying Lemma 3.3 to f and the corresponding unique solution $u:=S*f$ of the problem (1), with $k=2$ we have that $t\mapsto t^2u^{(2)}(t)\in MR_{p,T}$ and

$$\begin{aligned}\|\psi_{2,T}f\|_{MR_{p,T}} &= \|t^2u^{(2)}(t)\|_{MR_{p,T}}\\ &\le C_2\,\|f\|_{D^{2,p}(0,2T,\mathcal{X})}\\ &= C_2\,(2T)^{\frac{1}{p}}\,\|x\|,\end{aligned}$$

where C_2 is the constant independent of T obtained in Lemma 3.3. Noting that $u(t)=\int_0^t S(t-s)f(s)ds=\int_0^t S(s)xds$, we therefore obtain from the above inequality

$$\|t^2A\dot{S}(t)x\|_{L^p(0,T;\mathcal{X})}\le C_2\,(2T)^{\frac{1}{p}}\,\|x\|. \tag{7}$$

Define the operator-valued kernel K as

$$K(t)=\begin{cases} AS(t) & \text{if } t\in(0,T)\ ,\\ 0 & \text{otherwise.}\end{cases}$$

Due to the L^p maximal regularity of (1), the convolution operator T_A given by $T_Af(t) = \int_{\mathbb{R}_+} K(t-s)f(s)$ extends to a bounded linear operator on $L^p(0,T;\mathcal{X})$. Thus there is a constant $\tilde{C}_p$ such that for all $f \in L^p(\mathbb{R},\mathcal{X})$,

$$\| Tf \|_{L^p(\mathbb{R},\mathcal{X})} \leq \tilde{C}_p \| f \|_{L^p(\mathbb{R},\mathcal{X})} .$$

We claim that the kernel K further satisfies condition (6) for some $c > 1$. Indeed, for $T \geq t > s > 0$, and $x \in \mathcal{X}$, we have, on using (7) and Hölder's inequality

$$\begin{aligned}
\|K(t-s)x - K(t)x\| &= \|AS(t-s)x - AS(t)x\| \\
&= \left\| \int_{t-s}^{t} A\dot{S}(r)x\,dr \right\| \\
&\leq \left\| \int_{t-s}^{t} r^{-2}r^2 A\dot{S}(r)x\,dr \right\| \\
&\leq \left(\int_{t-s}^{t} r^{-2q}\,dr \right)^{\frac{1}{q}} \left(\int_{0}^{t} \|r^2 A\dot{S}(r)x\|^p\,dr \right)^{\frac{1}{p}} \\
&\leq \left(\int_{t-s}^{t} r^{-2q}\,dr \right)^{\frac{1}{q}} C_2\,(2t)^{\frac{1}{p}}\,\|x\|.
\end{aligned}$$

Therefore, for $c > 1$, and $s > 0$ we have

$$\begin{aligned}
\int_{t>cs} \|K(t-s) - K(t)\|dt &\leq C \int_{t>cs} (2t)^{\frac{1}{p}} \left| t^{(1-2q)} - (t-s)^{(1-2q)} \right|^{\frac{1}{q}} dt \\
&\leq C \int_{t>cs} (2t)^{\frac{1}{p}} t^{(\frac{1}{q}-2)} \left| 1 - (1 - st^{-1})^{(1-2q)} \right|^{\frac{1}{q}} dt \\
&\leq \tilde{C} \int_{t>cs} t^{\frac{1}{p}} s^{\frac{1}{q}} t^{-2} dt \\
&\leq C' s^{\frac{1}{q}} \int_{T>t>cs} t^{(\frac{1}{p}-2)} dt \\
&\leq M,
\end{aligned}$$

where $\tilde{C}, C'$, and M are constants depending only on $\tilde{C}_p$, q, c and T.

Therefore, from Theorem 4.1 it follows that T_A is a bounded operator from $L^r(0,T;\mathcal{X})$ to $L^r(0,T;\mathcal{X})$, for every $r \in (1,\infty)$. Thus, for each $r \in (1,\infty)$ there exists a constant $\tilde{C}_r$ such that for all $f \in L^r(0,T;\mathcal{X})$,

$$\|AS * f\|_{L^r(0,T;\mathcal{X})} = \|T_A f\|_{L^r(0,T;\mathcal{X})} \leq \tilde{C}_r \|f\|_{L^r(0,T;\mathcal{X})}. \tag{8}$$

Now define another operator valued kernel K_1 as follows.

$$K_1(t) = \begin{cases} B\dot{S}(t) & \text{if } t \in (0,T), \\ 0 & \text{otherwise.} \end{cases}$$

Let T_B be the operator given by $T_Bf(t) := K_1 * f$. Due to L^p maximal regularity, T_B is a bounded operator on $L^p(0,T;\mathcal{X})$. Applying Lemma 3.3 to the constant function $f(t) = x, t \in (0,T)$, successively for $k = 0, 1$ and 2 we obtain for $u = S * f$,

$$\begin{aligned} \|t^2 B\ddot{S}(t)x\|_{L^p(0,T;X)} &\leq \|t^2\dot{S}(t)x\|_{MR_{p,T}} + 2\|tS(t)x\|_{MR_{p,T}} + 2\|(S*f)(t)x\|_{MR_{p,T}} \\ &\leq C_2\|f\|_{D^{2,p}} + 2C_1\|f\|_{D^{1,p}} + 2C_0\|f\|_{D^{0,p}} \\ &= C(2T)^{\frac{1}{p}}\|x\|, \end{aligned}$$

where the constant C is independent of T for $0 < T \leq 1$. Using this inequality and the same arguments as before, we can show that the kernel K_1 also satisfies (6) above. Thus, from Theorem 4.1 it follows that $f \mapsto K_1 * f$ is a bounded operator on $L^r(0,T;\mathcal{X})$ for every $r \in (1,\infty)$. Therefore, we have for every $f \in L^r(0,T;\mathcal{X})$,

$$\|B\dot{S} * f\|_{L^r(0,T;\mathcal{X})} = \|T_Bf\|_{L^r(0,T;\mathcal{X})} \leq C_r^1\|f\|_{L^r(0,T;\mathcal{X})}. \tag{9}$$

Since $u := S * f$ is a solution of the equation $\ddot{u}(t) + B\dot{u}(t) + Au(t) = f(t)$ a.e , it follows that $\ddot{u} \in L^r(0,T;\mathcal{X})$ and there exists a constant C_r' such that $\|u\|_{W^{2,r}} \leq C_r'\|f\|$. This statement, together with (8) and (9) implies that the problem (1) has L^r maximal regularity for every $r \in (1,\infty)$. □

There are, as of now, only a few results that ensure that the problem (1) has L^p maximal regularity for some $p \in (1,\infty)$. The main ingredient for showing L^p maximal regularity in [5] is a characterization of L^p maximal regularity in terms of Fourier multipliers. In a particular model problem L^p maximal regularity was shown by using the Mikhlin-Weis Fourier multiplier theorem [5, Theorem 4.1]. This then implied L^p maximal regularity for every $p \in (1,\infty)$, so that in the examples considered in [5], L^p maximal regularity is independent of $p \in (1,\infty)$.

However, consider the following variational setting, not covered by the results in [5]. Let V and H be two separable Hilbert spaces such that V embeds densely and continuously into H. We identify H with its dual H' so that H is also densely and continuously embedded into V'.

COROLLARY 4.3. – *Let A and B be two linear, maximal monotone, symmetric, not necessarily commuting operators from V to V'. Then for every $p \in (1,\infty)$ and every $f \in L^p(0,T;V')$, every $u_0 \in V$ and every $u_1 \in (V',V)_{\frac{1}{p'},p}$ there exists a unique solution*

$$u \in W^{2,p}(0,T;V') \cap W^{1,p}(0,T;V)$$

of the problem

$$\ddot{u} + B\dot{u} + Au = f \text{ on } [0,T], \quad u(0) = u_0,\ \dot{u}(0) = u_1. \tag{10}$$

In other words, the above problem has L^p maximal regularity for every $p \in (1, \infty)$.

PROOF. -- By a result of J.-L. Lions [7, Théorème 1, p. 670] the problem (10) has L^2 maximal regularity in V'. By Theorem 4.2, the problem (10) has L^p maximal regularity for every $p \in (1, \infty)$. Solvability of the initial value problem follows from Theorem 2.1. The fact that the associated trace space equals $V \times (V', V)_{\frac{1}{p'},p}$ follows from [5, Lemma 6.2]. □

The fact that in the variational setting above the problem (1) has L^2 maximal regularity was proved in [7] by the Faedo-Galerkin method and a priori estimates. The proof thus heavily depends on the Hilbert space setting and it does not imply L^p maximal regularity for p different from 2. We point out that the conditions on A and B can be considerably relaxed; for the precise assumptions, see [7].

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