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A Nonlocal Problem Arising in the Study of Magneto-Elastic Interactions

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Dedicated to the memory of Guido Stampacchia

Sunto. – *Si studia il funzionale non convesso che descrive l'energia di un materiale magneto-elastico. Sono considerati tre termini energetici: l'energia di scambio, l'energia elastica e l'energia magneto-elastica generalmente adottata per cristalli cubici. Si introduce un problema penalizzato monodimensionale e si studia il flusso di gradiente dell'associato funzionale del tipo Ginzburg-Landau. Si prova l'esistenza e unicità di una soluzione classica che tende asintoticamente, per sottosuccessione, a un punto stazionario del funzionale dell'energia.*

Abstract. – *The energy of magneto-elastic materials is described by a nonconvex functional. Three terms of the total free energy are taken into account: the exchange energy, the elastic energy and the magneto-elastic energy usually adopted for cubic crystals. We focus our attention to a one dimensional penalty problem and study the gradient flow of the associated type Ginzburg-Landau functional. We prove the existence and uniqueness of a classical solution which tends asymptotically for subsequences to a stationary point of the energy functional.*

1. – Introduction.

The paper deals with the analysis of the equation

$$\frac{d\boldsymbol{u}}{dt} = -\operatorname{grad} F(\boldsymbol{u}) \tag{1.1}$$

where $F(\boldsymbol{u})$ is a type Ginzburg-Landau functional, associated to the energy of a magneto-elastic material, which contains a nonlinear nonlocal term. The derivation of the energy functional $F(\boldsymbol{u})$ is detailed in the next section starting from a general 3D-model depending on the displacements and the magnetization and assuming some simplifications. In particular in one-dimensional case the energy functional can be expressed in terms of the magnetization variable alone, and the equation (1.1) reduces to the following one

$$\boldsymbol{u}_t = \boldsymbol{u}_{xx} - \varepsilon^{-1}(|\boldsymbol{u}|^2 - 1)\boldsymbol{u} + \mu\Lambda(\boldsymbol{u})[\Lambda(\boldsymbol{u}) \cdot \boldsymbol{u} - \int_0^1 \Lambda(\boldsymbol{u}) \cdot \boldsymbol{u}\, dx], \tag{1.2}$$

where $\boldsymbol{u} = (u_1, u_2)$ and $\Lambda(\boldsymbol{u}) = (u_2, u_1)$.

The parameter μ couples the elastic and magnetic processes and ε is a small positive parameter introduced to relax the constraint $|\boldsymbol{u}| = 1$.

We assume that the equation (1.2) is associated with the boundary and initial conditions

$$\boldsymbol{u}_x(0,t) = \boldsymbol{u}_x(1,t) = 0, \qquad \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}). \tag{1.3}$$

The paper is organized as follows. In Section 2 we introduce the general 3D model, and present the reduction to the simplified one dimensional model. In Section 3 we study the minimization problem involving the energy functional $F_{\mu,\varepsilon}(\boldsymbol{u})$ associated with (1.2), namely

$$F_{\mu,\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\int_0^1 |\boldsymbol{u}_x|^2\, dx + \frac{\varepsilon^{-1}}{4}\int_0^1 (|\boldsymbol{u}|^2 - 1)^2 dx \\ - \frac{\mu}{4}\left[\int_0^1 (\Lambda(\boldsymbol{u}) \cdot \boldsymbol{u})^2\, dx - \left(\int_0^1 \Lambda(\boldsymbol{u}) \cdot \boldsymbol{u}\, dx\right)^2\right].$$

We show that there exists a critical value of μ, explicitly given by $\mu^* = \pi/2$, such that:

(i) for $\mu < \mu^*$ and ε small enough the only minimizers for $F_{\mu,\varepsilon}$ are constant functions $\boldsymbol{u} \equiv \alpha \in S^1$.

(ii) for $\mu > \mu^*$ the minimizer for $F_{\mu,\varepsilon}$ is nontrivial.

A similar bifurcation phenomenon was observed by Bethuel, Brezis, Coleman and Hélein in [2] in their study of nematics between cylinders. Finally, Section 4 is devoted to the study of the gradient flow. We prove existence and uniqueness of the solution $\boldsymbol{u}$ to (1.2), (1.3). Then we show that $\lim_{t\to\infty} \boldsymbol{u}(t) = \boldsymbol{u}_\infty$ exists and that the function $\boldsymbol{u}_\infty$ is a stationary point of the energy functional.

2. – The model.

The behaviour of a magnetoelastic material is described by a system of differential equations in the two unknowns: the displacement vector and the magnetization vector. Let $\Omega \subset \mathbb{R}^3$ be the volume of the magnetoelastic material and $\partial\Omega$ its boundary, the unknown magnetization vector $\boldsymbol{m}$ is a map from Ω to S^2

(the unit sphere of $\mathbb{R}^3$). The magnetization distribution is well described by a free energy functional which we assume composed of three terms namely the *exchange* energy E_{ex}, the *elastic* energy E_{el} and the *elastic-magnetic* energy E_{em}. Let **v** be the displacement vector, then the total free energy E for a deformable magnetoelastic material is given by

$$E(\boldsymbol{m},\boldsymbol{v}) \;=\; E_{\mathrm{ex}}(\boldsymbol{m}) \,+\, E_{\mathrm{em}}(\boldsymbol{m},\boldsymbol{v}) \,+\, E_{\mathrm{el}}(\boldsymbol{v}).$$

We neglect here other contributions to the free energy due, for example, to anisotropy and demagnetization energy terms.

We refer to the books [3], [4]; moreover among the papers on this subject we quote [5], [6], [7], [8]. In the sequel we detail the three energetic terms and derive the governing differential equations. Some drastic hypotheses allows us to reach a reduced one dimensional problem and to carry out the variational analysis for the associated energy functional.

2.1 – *The general 3D model.*

Let x_i, $i = 1, 2, 3$ be the position of a point $\boldsymbol{x}$ of Ω and denote by

$$v_i = v_i(\boldsymbol{x}), \qquad i = 1, 2, 3$$

the components of the displacement vector $\boldsymbol{v}$ and by

$$\varepsilon_{kl}(\boldsymbol{v}) = \frac{1}{2}(v_{k,l} + v_{l,k}), \qquad k, l = 1, 2, 3$$

the deformation tensor where, as a common praxis, $v_{k,l}$ stands for $\dfrac{\partial v_k}{\partial x_l}$.

Moreover we denote by

$$m_j = m_j(\boldsymbol{x}), \qquad j = 1, 2, 3$$

the component of the unit magnetization vector $\boldsymbol{m}$. In the sequel, where not specified, the Latin indices vary in the set $\{1, 2, 3\}$ and the summation of the repeated indices is assumed. We define

$$E_{\mathrm{ex}}(\boldsymbol{m}) = \frac{1}{2}\int_{\Omega} a_{ij} m_{k,i} m_{k,j}\, d\Omega\,, \tag{2.1}$$

where (a_{ij}) is a symmetric positive definite matrix which is supposed diagonal for most materials with all diagonal elements equal to a positive number a. The magneto-elastic energy for cubic crystals is assumed. This implies

$$E_{\mathrm{em}}(\boldsymbol{m},\boldsymbol{v}) = \frac{1}{2}\int_{\Omega} \lambda_{ijkl} m_i m_j \varepsilon_{kl}(\boldsymbol{v})\, d\Omega\,, \tag{2.2}$$

where $\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. Finally we introduce the elastic energy

$$E_{\mathrm{el}}(\boldsymbol{v}) = \frac{1}{2} \int_{\Omega} \sigma_{klmn} \varepsilon_{kl}(\boldsymbol{v}) \varepsilon_{mn}(\boldsymbol{v}) d\Omega \tag{2.3}$$

where σ_{klmn} is the elasticity tensor satisfying the following symmetry property

$$\sigma_{klmn} = \sigma_{mnkl} = \sigma_{lkmn}$$

and moreover the inequality

$$\sigma_{klmn} \varepsilon_{kl} \varepsilon_{mn} \geq \beta \varepsilon_{kl} \varepsilon_{kl}$$

holds for some $\beta > 0$.

We consider the energy functional E given by

$$E(\boldsymbol{m}, \boldsymbol{v}) \; = \; E_{\mathrm{ex}}(\boldsymbol{m}) \; + \; E_{\mathrm{em}}(\boldsymbol{m}, \boldsymbol{v}) \; + \; E_{\mathrm{el}}(\boldsymbol{v}) \tag{2.4}$$

We introduce two tensors $\mathcal{S} = (\sigma_{ijkl}\varepsilon_{ij})$ and $\mathcal{L} = (\lambda_{ijkl} m_i m_j)$, moreover we denote by $\boldsymbol{p}$ the vector $\boldsymbol{p} = (\lambda_{ijkl} m_j \varepsilon_{kl})$.

The system of differential equations associated to the functional (2.4) reads

$$\begin{cases} \operatorname{div}\left(\mathcal{S} + \dfrac{1}{2}\mathcal{L}\right) = 0 & \text{in } \Omega \\ a\Delta \boldsymbol{m} - \boldsymbol{p} + (a|\nabla \boldsymbol{m}|^2 + \boldsymbol{p}\cdot\boldsymbol{m})\boldsymbol{m} = 0, & \text{in } \Omega \end{cases} \tag{2.5}$$

with boundary conditions

$$\boldsymbol{v} = 0, \qquad \frac{\partial \mathbf{m}}{\partial \upsilon} = 0 \qquad \text{on} \quad \partial\Omega \tag{2.6}$$

where υ is the outer unit normal at the boundary $\partial\Omega$.

An alternative form for describing the magnetoelastic interactions (2.5) is

$$\begin{cases} \operatorname{div}\left(\mathcal{S} + \dfrac{1}{2}\mathcal{L}\right) = 0 & \text{in } \Omega \\ \boldsymbol{m} \times (a\Delta \boldsymbol{m} - \boldsymbol{p}) = 0, \;\; |\boldsymbol{m}| = 1 & \text{in } \Omega \end{cases} \tag{2.7}$$

The dynamical systems associated to the problems (2.5), (2.7) are respectively

$$\begin{cases} \rho \boldsymbol{v}_{tt} = \operatorname{div}\left(\mathcal{S} + \dfrac{1}{2}\mathcal{L}\right) & \text{in } \Omega \times (0,T) \\ \boldsymbol{m}_t + \gamma(\boldsymbol{m}_t \times \boldsymbol{m}) = a\Delta \boldsymbol{m} - \boldsymbol{p} + (a|\nabla \boldsymbol{m}|^2 + \boldsymbol{p}\cdot\boldsymbol{m})\boldsymbol{m}, & \text{in } \Omega \times (0,T) \end{cases} \tag{2.8}$$

and

$$\begin{cases} \rho \boldsymbol{v}_{tt} - \operatorname{div}\left(\mathcal{S} + \dfrac{1}{2}\mathcal{L}\right) = 0 & \text{in } \Omega \times (0,T) \\ \gamma \boldsymbol{m}_t = \boldsymbol{m} \times (a\Delta \boldsymbol{m} - \boldsymbol{m}_t - \boldsymbol{p}), & \text{in } \Omega \times (0,T) \end{cases} \tag{2.9}$$

with γ and ρ two positive constants. For results concerning the existence of weak solutions to the dynamical problems related to (2.8), (2.9), we refer the reader to [1], [9].

2.2 – *The proposed 1D problem.*

A simplified model and a simplified energy functional can be obtained assuming that Ω is a subset of $\mathbb{R}$ and neglecting some components of the unknowns $\boldsymbol{v}$ and $\boldsymbol{m}$. More precisely we consider the single space variable x and assume $\Omega = (0,1)$, $\boldsymbol{v} = (0, w, 0)$ and $\boldsymbol{m} = (m_1, m_2, 0)$. Then one has

$$\varepsilon_{kl}(v) = \varepsilon_{12}(v) = \varepsilon_{21}(v) = \frac{1}{2} w_x, \tag{2.10}$$

$$\lambda_{ijkl} = \lambda_{ij12} = \lambda_3(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) = \lambda_{ij21}, \tag{2.11}$$

and the different energies are now

$$E_{\rm ex}(\boldsymbol{m}) = \frac{1}{2}\int_0^1 |\boldsymbol{m}_x|^2\, dx, \qquad ((a_{ij}) = a\, Id = Id), \tag{2.12}$$

$$E_{\rm em}(\boldsymbol{m}, \boldsymbol{v}) = \frac{\lambda}{2}\int_0^1 (m_1 m_2 + m_2 m_1) w_x\, dx \qquad (\lambda_3 = \lambda), \tag{2.13}$$

$$E_{\rm el}(\boldsymbol{v}) = \frac{1}{2}\int_0^1 w_x^2\, dx \qquad (\sigma_{1221} = 1). \tag{2.14}$$

To deal with the constraint $|\boldsymbol{m}| = 1$, especially when having in mind numerical computations, we introduce the penalization

$$\frac{1}{4\varepsilon}\int_0^1 (|\boldsymbol{m}|^2 - 1)^2\, dx. \tag{2.15}$$

If for $\boldsymbol{m} = (m_1, m_2)$ we define the linear operator Λ by $\Lambda(\boldsymbol{m}) = (m_2, m_1)$, then the problem of minimization of the energy reduces to minimize

$$\begin{aligned} &E_\varepsilon(\boldsymbol{m}, w) \\ &= \frac{1}{2}\int_0^1 |\boldsymbol{m}_x|^2\, dx + \frac{1}{4\varepsilon}\int_0^1 (|\boldsymbol{m}|^2 - 1)^2\, dx + \frac{\lambda}{2}\int_0^1 (\Lambda(\boldsymbol{m})\cdot \boldsymbol{m}) w_x\, dx + \frac{1}{2}\int_0^1 w_x^2\, dx, \end{aligned} \tag{2.16}$$

over functions satisfying the boundary conditions

$$\boldsymbol{m}_x = 0, \qquad w = 0, \qquad \text{on} \quad \partial\Omega = \{0, 1\}. \tag{2.17}$$

The corresponding Euler equation reads, for $\boldsymbol{m} = \boldsymbol{m}^\varepsilon$,

$$\begin{cases} \boldsymbol{m}^\varepsilon_{xx} - \lambda \Lambda(\boldsymbol{m}^\varepsilon) w_x - \varepsilon^{-1}(|\boldsymbol{m}^\varepsilon|^2 - 1)\boldsymbol{m}^\varepsilon = 0 \\ w^\varepsilon_{xx} + \dfrac{\lambda}{2}\,(\Lambda(\boldsymbol{m}^\varepsilon)\cdot \boldsymbol{m}^\varepsilon)_x = 0. \end{cases} \tag{2.18}$$

Integrating the second equation leads to

$$w_x = -\frac{\lambda}{2}\,(\Lambda(\boldsymbol{m}^\varepsilon)\cdot \boldsymbol{m}^\varepsilon) + C. \tag{2.19}$$

The constant C is obtained by integrating the above equation on $(0,1)$ and using the boundary condition, i.e.,

$$C = \frac{\lambda}{2}\int_0^1 (\Lambda(\boldsymbol{m}^\varepsilon)\cdot \boldsymbol{m}^\varepsilon)\,dx. \tag{2.20}$$

Then replacing w_x by its value in the first equation of (2.18) and setting $\mu = \lambda^2/2$ we obtain the following penalty nonlocal equation

$$\boldsymbol{m}^\varepsilon_{xx} - \varepsilon^{-1}(|\boldsymbol{m}^\varepsilon|^2 - 1)\boldsymbol{m}^\varepsilon + \mu\,\Lambda(\boldsymbol{m}^\varepsilon)[\Lambda(\boldsymbol{m}^\varepsilon)\cdot \boldsymbol{m}^\varepsilon - \int_0^1 \Lambda(\boldsymbol{m}^\varepsilon)\cdot \boldsymbol{m}^\varepsilon dx] = 0\,, \tag{2.21}$$

with boundary conditions

$$\boldsymbol{m}^\varepsilon_x(0) = \boldsymbol{m}^\varepsilon_x(1) = 0\,. \tag{2.22}$$

This is the problem we would like to address, as well as its parabolic analogue, i.e.,

$$\begin{cases} \boldsymbol{u}_t = \boldsymbol{u}_{xx} - \varepsilon^{-1}(|\boldsymbol{u}|^2 - 1)\boldsymbol{u} + \mu\,\Lambda(\boldsymbol{u})[\Lambda(\boldsymbol{u})\cdot \boldsymbol{u} - \int_0^1 \Lambda(\boldsymbol{u})\cdot \boldsymbol{u} dx] \quad \text{in} \quad \Omega\times(0,\infty) \\ \boldsymbol{u}_x = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty), \quad \boldsymbol{u}(x,0) = \boldsymbol{u}_0. \end{cases}$$

3. – The minimization problem.

The equation (2.21) is the Euler-Lagrange equation of the energy functional

$$\begin{aligned} F_{\mu,\varepsilon}(\boldsymbol{m}) = {} & \frac{1}{2}\int_0^1 |\boldsymbol{m}_x|^2 dx + \frac{\varepsilon^{-1}}{4}\int_0^1 (|\boldsymbol{m}|^2 - 1)^2\,dx \\ & - \frac{\mu}{4}\left[\int_0^1 (\Lambda(\boldsymbol{m})\cdot \boldsymbol{m})^2 dx - \left(\int_0^1 \Lambda(\boldsymbol{m})\cdot \boldsymbol{m}\,dx\right)^2\right] \end{aligned} \tag{3.1}$$

Let us consider the minimization problem

$$\mathcal{F}_{\mu,\varepsilon} = \inf_{\boldsymbol{m}\in \boldsymbol{H}^1(0,1)} F_{\mu,\varepsilon}(\boldsymbol{m})\,. \tag{3.2}$$

Above we used the notation $\boldsymbol{H}^1(0,1)$ for $H^1((0,1),\mathbb{R}^2)$.

THEOREM 3.1. – *For each μ and for each positive ε small enough, i.e., such that $\varepsilon^{-1}-\mu>0$, the minimum of the functional $F_{\mu,\varepsilon}(\boldsymbol{m})$ is achieved by a function $\boldsymbol{m}^\varepsilon = \boldsymbol{m}^{\mu,\varepsilon} \in \boldsymbol{H}^1(0,1)$. Furthermore, $\boldsymbol{m}^\varepsilon$ is a solution* (2.21)–(2.22) *and is therefore of class C^∞.*

PROOF. – First of all we observe that by the Cauchy-Young inequality it holds, for any $\delta>0$,

$$\left(\int_0^1 \Lambda(\boldsymbol{m})\cdot \boldsymbol{m}dx\right)^2 \le \int_0^1 (\Lambda(\boldsymbol{m})\cdot\boldsymbol{m})^2dx \le \int_0^1 |\boldsymbol{m}|^4\,dx \tag{3.3}$$

$$= \int_0^1 (|\boldsymbol{m}|^2-1+1)^2dx \le \left(1+\frac{1}{\delta}\right) + (1+\delta)\int_0^1 (|\boldsymbol{m}|^2-1)^2\,dx.$$

So we have:

(i) If $\varepsilon^{-1}-\mu>0$ then for δ small enough $\varepsilon^{-1}-(1+\delta)\mu\ge 0$ and the functional $F_{\mu,\varepsilon}(\boldsymbol{m})$ is bounded from below. Indeed,

$$F_{\mu,\varepsilon}(\boldsymbol{m}) \ge \frac{1}{2}\int_0^1 |\boldsymbol{m}_x|^2dx + \frac{\varepsilon^{-1}-(1+\delta)\mu}{4}\int_0^1 (|\boldsymbol{m}|^2-1)^2dx - \left(1+\frac{1}{\delta}\right)\frac{\mu}{4} \ge -\left(1+\frac{1}{\delta}\right)\frac{\mu}{4}$$

(ii) The functional $F_{\mu,\varepsilon}(\boldsymbol{m})$ is coercive, i.e.,

$$F_{\mu,\varepsilon}(\boldsymbol{m}) \to +\infty, \qquad \text{as} \quad \|\boldsymbol{m}\|_{\boldsymbol{H}^1(0,1)} \to \infty\,.$$

This follows easily from the inequality $(|\boldsymbol{m}|^2-1)^2 \ge |\boldsymbol{m}|^2 - 5/4$.

(iii) The functional is weakly lower semicontinuous, that is: if $\{\boldsymbol{m}_n\}$ is a sequence of functions in $\boldsymbol{H}^1(0,1)$ such that $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ weakly in $\boldsymbol{H}^1(0,1)$, then

$$\liminf_{n\to\infty} F_{\mu,\varepsilon}(\boldsymbol{m}_n) \ge F_{\mu,\varepsilon}(\boldsymbol{m})\,.$$

Indeed, for such a weakly convergent sequence we have

$$\int_0^1 |\boldsymbol{m}_x|^2\,dx \le \liminf_{n\to\infty}\int_0^1 |(\boldsymbol{m}_n)_x|^2\,dx\,,$$

$|\boldsymbol{m}_n|^2 \to |\boldsymbol{m}|^2$ and $\Lambda(\boldsymbol{m}_n)\cdot\boldsymbol{m}_n \to \Lambda(\boldsymbol{m})\cdot\boldsymbol{m}$ strongly in $L^2(0,1)$.

Since the functional (3.1) is C^1, it follows that the stationary points of $F_{\mu,\varepsilon}$ are solutions to the Euler-Lagrange equations (2.21)–(2.22), and it is easily verified that any solution to this one-dimensional problem is of class C^∞. □

REMARK 3.1. – The result is sharp since for $\varepsilon > \frac{1}{\mu}$, $F_{\mu,\varepsilon}$ is unbounded from below. Indeed, suppose that $1 - \frac{1}{\mu\varepsilon} > 0$. Consider the function $f = (\delta - x)^+$. One has

$$\left(\int_0^1 f^2\right)^2 \Big/ \int_0^1 f^4 = \left(\int_0^\delta (\delta - x)^2\right)^2 \Big/ \int_0^\delta (\delta - x)^4 = \frac{\delta^6}{9} \Big/ \frac{\delta^5}{5} = \frac{5}{9}\delta < 1 - \frac{1}{\mu\varepsilon}$$

for δ small enough. So we may choose δ small enough such that

$$\left(\int_0^1 f^2\right)^2 < \left(1 - \frac{1}{\mu\varepsilon}\right)\int_0^1 f^4.$$

Next, consider $\boldsymbol{m}^{(\alpha)} = \alpha f(x)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. We have

$$\begin{aligned}
F_{\mu,\varepsilon}(\boldsymbol{m}^{(\alpha)}) &= \frac{1}{2}\alpha^2 \int_0^1 f'(x)^2 + \frac{1}{4\varepsilon}\int_0^1 (\alpha^2 f(x)^2 - 1)^2 - \frac{\mu}{4}\int_0^1 \alpha^4 f^4 + \frac{\mu}{4}\left(\int_0^1 \alpha^2 f^2\right)^2 \\
&= \frac{1}{2}\alpha^2 \int_0^1 f'(x)^2 + \frac{\alpha^4}{4}\mu\left\{\frac{1}{\varepsilon\mu}\int_0^1 \left(f^2 - \frac{1}{\alpha^2}\right)^2 - \int_0^1 f^4 + \left(\int_0^1 f^2\right)^2\right\}.
\end{aligned}$$

For α large enough the quantity

$$\frac{1}{\varepsilon\mu}\int_0^1 \left(f^2 - \frac{1}{\alpha^2}\right)^2 - \int_0^1 f^4 + \left(\int_0^1 f^2\right)^2$$

is close to $\left(\frac{1}{\varepsilon\mu} - 1\right)\int_0^1 f^4 + \left(\int_0^1 f^2\right)^2 < 0$ and thus $F_{\mu,\varepsilon}(\boldsymbol{m}^{(\alpha)}) \to -\infty$ when $\alpha \to +\infty$.

The functional $F_{\mu,\varepsilon}(\boldsymbol{m})$ has some obvious symmetry properties. We have clearly $F_{\mu,\varepsilon}(\mathcal{S}_i(\boldsymbol{m})) = F_{\mu,\varepsilon}(\boldsymbol{m})$ for each $\mathcal{S}_i$ in the group

$$\mathcal{G} = \{\mathcal{S}_0, \ldots, \mathcal{S}_7\} \tag{3.4}$$

generated by the rotation by $\pi/2$ and the complex conjugation.

LEMMA 3.1. – *Let* $\boldsymbol{m}$ *be a solution of the problem* (2.21)–(2.22) *satisfying* $F_{\mu,\varepsilon}(\boldsymbol{m}) \leq 0$, *for some* $\varepsilon < \dfrac{1}{\mu}$. *Then, the following a-priori estimate holds,*

$$|\boldsymbol{m}|^2 \leq K := \frac{\varepsilon^{-1} + \mu\sqrt{\dfrac{\varepsilon^{-1}}{\varepsilon^{-1}+\mu}}}{\varepsilon^{-1}-\mu}. \tag{3.5}$$

PROOF. – By the assumption on $\boldsymbol{m}$ we have

$$\frac{\varepsilon^{-1}}{4}\int_0^1 (|\boldsymbol{m}|^2-1)^2\,dx + \frac{\mu}{4}\left[\left(\int_0^1 \Lambda(\boldsymbol{m})\cdot\boldsymbol{m}\,dx\right)^2 - \int_0^1 (\Lambda(\boldsymbol{m})\cdot\boldsymbol{m})^2\,dx\right] \leq 0\,.$$

Combining this with (3.3) yields

$$\frac{\varepsilon^{-1}}{4}\int_0^1 (|\boldsymbol{m}|^2-1)^2\,dx + \frac{\mu}{4}\left(\int_0^1 \Lambda(\boldsymbol{m})\cdot\boldsymbol{m}\,dx\right)^2 - (1+\delta)\,\frac{\mu}{4}\int_0^1 (|\boldsymbol{m}|^2-1)^2\,dx \leq \left(1+\frac{1}{\delta}\right)\frac{\mu}{4}.$$

Therefore, for $\varepsilon^{-1} > \mu$ and any δ such that $\varepsilon^{-1} - (1+\delta)\mu \geq 0$, i.e., $\dfrac{1}{\delta} \geq \dfrac{\mu}{\varepsilon^{-1}-\mu}$, we have

$$\left(\int_0^1 \Lambda(\boldsymbol{m})\cdot\boldsymbol{m}\,dx\right)^2 \leq 1 + \frac{1}{\delta}. \tag{3.6}$$

Now we multiply the Euler equation (2.21) by $\boldsymbol{m}$ and write the equation for $|\boldsymbol{m}|^2$:

$$-\frac{1}{2}\frac{d^2}{dx^2}|\boldsymbol{m}|^2 + |\boldsymbol{m}_x|^2 + \varepsilon^{-1}(|\boldsymbol{m}|^2-1)|\boldsymbol{m}|^2 - \mu(\Lambda(\boldsymbol{m})\cdot\boldsymbol{m})^2 + \mu\Lambda(\boldsymbol{m})\cdot\boldsymbol{m}\int_0^1 \Lambda(\boldsymbol{m})\cdot\boldsymbol{m}\,dx = 0.$$

Using (3.6) we obtain

$$-\frac{1}{2}\frac{d^2}{dx^2}|\boldsymbol{m}|^2 + \varepsilon^{-1}(|\boldsymbol{m}|^2-1)|\boldsymbol{m}|^2 - \mu|\boldsymbol{m}|^4 - \mu\sqrt{1+\frac{1}{\delta}}\,|\boldsymbol{m}|^2 \leq 0\,,$$

that is

$$-\frac{1}{2}\frac{d^2}{dx^2}|\boldsymbol{m}|^2 + (\varepsilon^{-1}-\mu)\,|\boldsymbol{m}|^2\left(|\boldsymbol{m}|^2 - \frac{\varepsilon^{-1}+\mu\sqrt{1+\dfrac{1}{\delta}}}{\varepsilon^{-1}-\mu}\right) \leq 0\,.$$

Choosing $\frac{1}{\delta} = \frac{\mu}{\varepsilon^{-1} - \mu}$ and setting $K = \left(\varepsilon^{-1} + \mu\sqrt{\frac{\varepsilon^{-1}}{\varepsilon^{-1} - \mu}}\right)/(\varepsilon^{-1} - \mu)$ gives

$$-\frac{1}{2}\frac{d^2}{dx^2}(|\boldsymbol{m}|^2 - K) + (\varepsilon^{-1} - \mu)\,|\boldsymbol{m}|^2\,(|\boldsymbol{m}|^2 - K) \le 0\,.$$

By the maximum principle, applied to the function $h = |\boldsymbol{m}|^2 - K$, we get that $h \le 0$, i.e., $|\boldsymbol{m}|^2 \le K$. □

Let us denote by λ_2 the first nontrivial eigenvalue for the Neumann problem:

$$\begin{cases} -f_{xx} = \lambda f & \text{in } (0,1)\,, \\ f_x(0) = f_x(1) = 0\,. \end{cases} \tag{3.7}$$

It is well known that $\lambda_2 = \pi^2$ and that it yields the optimal constant in the following Poincaré inequality,

$$\int_0^1 |g_x|^2\,dx \ge \lambda_2 \int_0^1 \Big(g(x) - \int_0^1 g(t)\,dt\Big)^2\,dx\,, \quad \forall g \in H^1(0,1)\,. \tag{3.8}$$

Next, we analyze the minimization problem (3.2) restricted to S^1-valued maps. When applied to maps $\boldsymbol{m} \in H^1((0,1);S^1)$, all the functionals $\{F_{\mu,\varepsilon}\}_{\varepsilon>0}$ take the same value, that we shall now use to define a new functional on $H^1((0,1);S^1)$:

$$E_\mu(\boldsymbol{m}) = \frac{1}{2}\int_0^1 |\boldsymbol{m}_x|^2\,dx - \frac{\mu}{4}\left[\int_0^1 (\Lambda(\boldsymbol{m})\cdot\boldsymbol{m})^2\,dx - \left(\int_0^1 \Lambda(\boldsymbol{m})\cdot\boldsymbol{m}\,dx\right)^2\right].$$

In the next proposition we shall apply a bifurcation analysis similar to the one used in [2] in a study of minimizing harmonic maps on an annulus.

PROPOSITION 3.1. – *Put*

$$I(\mu) = \inf_{\boldsymbol{m}\in H^1((0,1);S^1)} E_\mu(\boldsymbol{m})\,. \tag{3.9}$$

Then:

(i) *For* $\mu \le \lambda_2/2$ *we have* $I(\mu) = 0$ *and the minimum is attained only by constant functions,* $\boldsymbol{m} \equiv \alpha \in S^1$.

(ii) *For* $\mu > \lambda_2/2$ *we have* $I(\mu) < 0$ *and the minimum is attained by* $\boldsymbol{m}^0 = e^{i\phi^0}$ *where* ϕ^0 *is a nontrivial solution of the problem*

$$\begin{cases} -\phi^0_{xx} = \mu\left(\sin 2\phi^0 - \int_0^1 \sin 2\phi^0\,dt\right)\cos 2\phi^0 & \textit{in } (0,1)\,, \\ \phi^0_x(0) = \phi^0_x(1) = 0\,. \end{cases} \tag{3.10}$$

PROOF. – Each $\boldsymbol{m} \in H^1((0,1);S^1)$ can be written as $\boldsymbol{m} = e^{i\phi}$ for some ϕ in $H^1((0,1);\mathbb{R})$. For such $\boldsymbol{m}$ we may rewrite the energy in (3.1) as

$$E_\mu(\boldsymbol{m}) = \frac{1}{2}\int_0^1 |\phi_x|^2\,dx - \frac{\mu}{4}\int_0^1 \left(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\right)^2 dx\,. \tag{3.11}$$

The function $f = \sin 2\phi$ satisfies $f_x = 2(\cos 2\phi)\phi_x$, so that

$$|\phi_x| = \frac{|f_x|}{2|\cos 2\phi|} \geq \frac{|f_x|}{2}\,. \tag{3.12}$$

Write the r.h.s. of (3.11) as a sum of two integrals to obtain

$$\begin{aligned} E_\mu(\boldsymbol{m}) = &\int_0^1 \left(\frac{1}{2}\phi_x^2 - \frac{1}{8}f_x^2\right) dx \\ &+\int_0^1 \left(\frac{1}{8}f_x^2 - \frac{\mu}{4}\left(\sin 2\phi - \int_0^1 \sin 2\phi\,dt\right)^2\right) dx := I_1 + I_2\,. \end{aligned} \tag{3.13}$$

Clearly, for $\mu < \lambda_2/2$ and any $f \not\equiv \text{const}$ we have by (3.12) and (3.8) that $I_1 > 0$ and $I_2 > 0$. For $\mu = \lambda_2/2$ and $f \not\equiv \text{const}$ we have still $I_1 > 0$ while I_2 is non-negative. This yields assertion (i) of the proposition.

Assume next that $\mu > \lambda_2/2$. From the optimality of λ_2 in (3.8) follows the existence of $\tilde{f} \in H^1((0,1);\mathbb{R})$ with

$$\int_0^1 \left(\frac{1}{8}|\tilde{f}_x|^2 - \frac{\mu}{4}\tilde{f}^2\right) dx = -c < 0 \quad \textit{and} \quad \int_0^1 \tilde{f}\,dx = 0\,.$$

For $t > 0$ small enough set $\psi^{(t)} = \dfrac{1}{2}\arcsin(t\tilde{f})$ and then $\boldsymbol{m}^{(t)} = e^{i\psi^{(t)}}$. Using (3.13) we get

$$E_\mu(\boldsymbol{m}^{(t)}) = -ct^2 + O(t^4) < 0\,, \qquad \text{for } t \text{ small enough.}$$

This yields $I(\mu) < 0$, and the existence of a minimizer, $\boldsymbol{m}^0 = e^{i\phi^0}$ with ϕ^0 a non-trivial solution of (3.10) is obvious. □

A more precise description of the minimizers in the case $\mu > \lambda_2/2 = \pi^2/2$ is given by the next proposition.

PROPOSITION 3.2. – *In the case $\mu > \lambda_2/2$ the minimizer $\boldsymbol{m}^0 = e^{i\phi^0}$ is unique modulo the operation of the symmetry group $\mathcal{G}$ (see (3.4)), namely, up to performing the operations:*

$$\phi^0 \leftarrow \phi^0 + k\pi/2 \quad \text{or} \quad \phi^0 \leftarrow -\phi^0 + k\pi/2,\ k \in \mathbb{Z}\,. \tag{3.14}$$

Such a unique representative of the minimizers can be chosen which is a strictly *increasing function on* $[0,1]$ *that satisfies*

$$\phi_0(x) = -\phi_0(1-x) \quad x \in [0,1]. \tag{3.15}$$

PROOF. – Setting

$$a = \int_0^1 \sin 2\phi^0 \, dx ,$$

we can rewrite (3.10) as

$$\begin{cases} -\phi^0_{xx} = \mu\big(\sin 2\phi^0 - a\big)\cos 2\phi^0 & \text{in } (0,1), \\ \phi^0_x(0) = \phi^0_x(1) = 0 . \end{cases} \tag{3.16}$$

The rest of the proof is divided to several steps.

STEP 1: ϕ^0 is strictly monotone.

Replacing ϕ^0 by its increasing rearrangement $(\phi^0)^*$ will decrease the first term on the r.h.s. of (3.11) (strictly, if ϕ^0 is not a monotone function), without changing the second term on the r.h.s. of (3.11). Since we may replace ϕ^0 by $-\phi^0$ we can assume in the sequel that $\phi^0_x \geq 0$ in $[0,1]$. We next claim that actually we have:

$$\phi^0_x > 0 \quad \text{on } (0,1). \tag{3.17}$$

Indeed, the function $\psi = \phi^0_x$ satisfies

$$\begin{cases} -\psi_{xx} = 2\mu\big(\cos 4\phi^0 + a\sin 2\phi^0\big)\psi & \text{in } (0,1), \\ \psi \geq 0 \text{ in } (0,1),\ \psi(0) = \psi(1) = 0 . \end{cases} \tag{3.18}$$

Since $\psi \not\equiv 0$ we deduce (3.17) from the maximum principle.

STEP 2: $|\sin 2\phi^0| < 1$ in (0,1) and $\sin 2\phi^0$ is strictly monotone increasing on [0,1].

Looking for contradiction, assume for example that $\sin 2\phi^0(x_0) = 1$ for some x_0 in $(0,1)$. By (3.14) we may assume that $\phi^0(x_0) = \pi/4$. Set $\tilde\phi(x) = \pi/2 - \phi^0(2x_0 - x)$. It is easy to verify that $\tilde\phi$ satisfies the equation in (3.16), and also $\tilde\phi(x_0) = \phi^0(x_0)$, $\tilde\phi_x(x_0) = \phi^0_x(x_0)$. By the uniqueness theory for ODE we deduce that $\tilde\phi = \phi^0$, i.e., $\phi^0(x) = \pi/2 - \phi^0(2x_0 - x)$. For the boundary conditions in (3.16) to hold, the only possibility is that $x_0 = 1/2$. We thus conclude that

$$\phi^0(x) = \pi/2 - \phi^0(1-x), \quad x \in (0,1). \tag{3.19}$$

The relation (3.19) implies that

$$a = \int_0^1 \sin 2\phi^0 \, dx = 2\int_0^{1/2} \sin 2\phi^0 \, dx = \fint_0^{1/2} \sin 2\phi^0 \, dx .$$

Defining the following functional on $H^1((0,1/2);S^1)$,

$$E_\mu^{(1/2)}(e^{i\phi}) = \frac{1}{2}\int_0^{1/2} |\phi_x|^2\,dx - \frac{\mu}{4}\int_0^{1/2}\Big(\sin 2\phi - \fint_0^{1/2} \sin 2\phi\,dt\Big)^2\,dx\,,$$

we conclude that

$$E_\mu(e^{i\phi^0}) = 2E_\mu^{(1/2)}(e^{i\phi^0})\,. \tag{3.20}$$

Set, analogously to (3.9),

$$I_{1/2}(\mu) = \inf_{\boldsymbol{m}\in H^1((0,1/2);S^1)} E_\mu^{(1/2)}(\boldsymbol{m})\,. \tag{3.21}$$

The minimum in (3.21) is achieved by some function $\phi^1 \in H^1(0,1/2)$. Since $\phi_x^0(1/2) > 0$, the restriction of ϕ^0 to $(0,1/2)$ is not a minimizer and therefore,

$$E_\mu^{(1/2)}(e^{i\phi^1}) < E_\mu^{(1/2)}(e^{i\phi^0})\,. \tag{3.22}$$

We can extend ϕ^1 to a function $\tilde\phi^1 \in H^1(0,1)$ by setting

$$\tilde\phi^1(x) = \phi^1(1-x) \quad \text{ for } x \in [1/2,1)\,.$$

Combining it with (3.22) and (3.20) we deduce that $E_\mu(e^{i\tilde\phi^1}) < E_\mu(e^{i\phi^0})$. This contradiction completes the proof of the assertion $|\sin 2\phi^0| < 1$ in $(0,1)$.

In view of the above and Step 1 we conclude that the function $\sin 2\phi^0$ is strictly increasing on $[0,1]$. By adding an integer multiple of $\pi/4$, see (3.14), we may assume that the image of the interval $(0,1)$ by ϕ^0 is contained in $(-\pi/4,\pi/4)$. The uniqueness for that representative of the phase of the minimizer will be established in the sequel.

STEP 3: $a = 0$.

Multiplying the equation in (3.16) by ϕ_x^0 and integrating yields

$$(\phi_x^0)^2 = c^2 - \frac{\mu}{2}(\sin 2\phi^0 - a)^2 \quad \text{on } [0,1]\,, \tag{3.23}$$

for some constant $c > 0$. Write the roots of the polynomial $p(t) = c^2 - (\mu/2)(t-a)^2$ as $a-b$ and $a+b$ for some $b > 0$, i.e., $p(t) = (\mu/2)(a+b-t)(t-a+b)$. By Steps 1 and 2, (3.23), and the boundary condition in (3.16) it follows that

$$\sin 2\phi^0(0) = a - b \quad \text{ and } \quad \sin 2\phi^0(1) = a + b\,. \tag{3.24}$$

Assume by negation that $a \neq 0$. Next, we exploit the following two identities. First,

$$(3.25)\quad 1=\int_0^1 dx=\int_{\frac12\sin^{-1}(a-b)}^{\frac12\sin^{-1}(a+b)}\frac{d\phi}{p^{\frac12}(\sin 2\phi)}=\int_{a-b}^{a+b}\frac{dt}{\sqrt{2\mu(a+b-t)(t-a+b)(1-t^2)}}$$

$$=\int_{-b}^{b}\frac{ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}.$$

Similarly,

$$(3.26)\quad a=\int_0^1 \sin 2\phi^0(x)dx=\int_{\frac12\sin^{-1}(a-b)}^{\frac12\sin^{-1}(a+b)}\frac{\sin 2\phi d\phi}{p^{\frac12}(\sin 2\phi)}$$

$$=\int_{-b}^{b}\frac{(s+a)ds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}.$$

From (3.25) and (3.26) we deduce that

$$(3.27)\quad 0=\int_{-b}^{b}\frac{sds}{\sqrt{2\mu(b-s)(b+s)(1-(a+s)^2)}}$$

$$=\int_0^b\frac{s}{\sqrt{2\mu(b-s)(b+s)}}\left(\frac{1}{\sqrt{1-(a+s)^2}}-\frac{1}{\sqrt{1-(a-s)^2}}\right)ds.$$

But it is clear that the r.h.s. of (3.27) is strictly positive for $a>0$ and strictly negative for $a<0$, so in either case we are led to a contradiction.

STEP 4: Conclusion.

Going back to (3.23) we can now write

$$(\phi_x^0)^2=c^2-\frac{\mu}{2}\sin^2 2\phi^0=\frac{\mu}{2}(b-\sin 2\phi^0)(b+\sin 2\phi^0)\quad \text{on } [0,1],$$

with $b=c\sqrt{2/\mu}$. The equation (3.25) now reads

$$(3.28)\qquad \sqrt{2\mu}=\int_{-b}^{b}\frac{ds}{\sqrt{(b-s)(b+s)(1-s^2)}}=\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\sqrt{1-b^2\sin^2\theta}}.$$

Since we assume that $\mu>\frac{\pi^2}{2}$, it follows that there is a *unique* $b>0$ for which (3.28) holds.

Next, there is a unique point $x_0 \in (0,1)$ where $0 = \phi^0(x_0) = \sin 2\phi^0(x_0)$. At that point, $\phi^0_x(x_0) = b\sqrt{\mu/2}$. The function $\tilde{\phi}(x) = -\phi^0(2x_0 - x)$ solves the equation

$$-\tilde{\phi}_{xx} = \mu \sin 2\tilde{\phi} \cos 2\tilde{\phi} \quad \text{in } (0,1)\,, \tag{3.29}$$

with the initial conditions

$$\tilde{\phi}(x_0) = \phi^0(x_0) = 0 \quad \text{and} \quad \tilde{\phi}_x(x_0) = \phi^0_x(x_0) = \sqrt{\frac{\mu}{2}}\,b\,. \tag{3.30}$$

Since there is a unique solution to (3.29)–(3.30), it follows that $\phi^0 = \tilde{\phi}$. Since $\tilde{\phi}_x(2x_0) = 0$ we must have $x_0 = 1/2$ and the symmetry property (3.15) holds. The uniqueness assertion of the proposition follows from the uniqueness for the initial problem (3.29)–(3.30) for $x_0 = 1/2$. □

Next we present a convergence result that will be used in our main theorem.

PROPOSITION 3.3. – *For each $\mu > 0$, any sequence of minimizers $\{\boldsymbol{m}_{\varepsilon_n}\}$, with $\varepsilon_n \to 0$, has a subsequence which converges in $H^1(0,1)$ and in $C[0,1]$ to $\boldsymbol{m}^0 \in C^\infty([0,1];S^1)$ which is a minimizer for $I(\mu)$.*

PROOF. – Note that $F_{\mu,\varepsilon}(\boldsymbol{m}^\varepsilon) \le F_{\mu,\varepsilon}(\alpha) = E_\mu(\alpha) = 0$ for any constant $\alpha \in S^1$. Using (3.5) we conclude that for $\varepsilon < \dfrac{1}{2\mu}$ we have

$$\int_0^1 |\boldsymbol{m}^\varepsilon_x|^2\,dx \le C \quad \text{and} \quad \frac{1}{\varepsilon}\int_0^1 (1 - |\boldsymbol{m}^\varepsilon|^2)^2\,dx \le C\,,$$

for some constant C (which is independent of ε). Since $H^1(0,1)$ is compactly embedded in $C[0,1]$, we can extract a subsequence, still denoted by $\{\boldsymbol{m}_{\varepsilon_n}\}$, that converges weakly in $H^1(0,1)$ and strongly in $C[0,1]$ to a limit $\boldsymbol{m}^0 \in H^1((0,1);S^1)$. Since for each ε, and each $\boldsymbol{m} \in H^1((0,1);S^1)$, $F_{\mu,\varepsilon}(\boldsymbol{m}^\varepsilon) \le E_\mu(\boldsymbol{m})$, we get that

$$\limsup_{\varepsilon_n \to 0} F_{\mu,\varepsilon_n}(\boldsymbol{m}^{\varepsilon_n}) \le E_\mu(\boldsymbol{m})\,, \quad \forall \boldsymbol{m} \in H^1((0,1);S^1)\,. \tag{3.31}$$

On the other hand, the weak lower-semicontinuity of the L^2-norm of the gradient, combined with the uniform convergence of $\{\boldsymbol{m}^{\varepsilon_n}\}$ towards $\boldsymbol{m}^0$, yields

$$E_\mu(\boldsymbol{m}^0) \le \liminf_{\varepsilon_n \to 0} F_{\mu,\varepsilon_n}(\boldsymbol{m}^{\varepsilon_n})\,. \tag{3.32}$$

Combining (3.31) with (3.32) we deduce that $E_\mu(\boldsymbol{m}^0) \le E_\mu(\boldsymbol{m})$, $\forall \boldsymbol{m} \in H^1((0,1);S^1)$, i.e., $\boldsymbol{m}^0$ is a minimizer for $I(\mu)$. It also follows that the convergence $\boldsymbol{m}^{\varepsilon_n} \to \boldsymbol{m}^0$ is actually strong in $H^1(0,1)$. □

We are now in position to state our main result for the minimization problem (3.2).

THEOREM 3.2. –

(i) *For each* $\mu < \lambda_2/2$ *there exists* $\varepsilon_0(\mu) > 0$ *such that for* $\varepsilon \le \varepsilon_0(\mu)$ *we have* $\mathcal{F}_{\mu,\varepsilon} = 0$ *and the only minimizers for* (3.2) *are constant functions* $\boldsymbol{m}^\varepsilon \equiv \alpha \in S^1$.

(ii) *For* $\mu > \lambda_2/2$ *we have* $\mathcal{F}_{\mu,\varepsilon} < 0$ *for every* $\varepsilon > 0$. *For each* $\varepsilon > 0$ *we may choose a representative for the minimizer* $\boldsymbol{m}^\varepsilon$ *(by replacing* $\boldsymbol{m}^\varepsilon$ *with* $\mathcal{S}_i(\boldsymbol{m}^\varepsilon)$*, see* (3.4)*) such that* $\lim_{\varepsilon\to 0} \boldsymbol{m}^\varepsilon = \boldsymbol{m}^0$ *in* $H^1(0,1)$ *and in* $C[0,1]$, *where* $\boldsymbol{m}^0 \in C^\infty([0,1];S^1)$ *is a non-trivial minimizer for* $I(\mu)$.

(iii) *In the limiting case* $\mu = \lambda_2/2$, *we have for a subsequence,* $\lim_{\varepsilon_n\to 0} \boldsymbol{m}^{\varepsilon_n} = \alpha$ *in* $H^1(0,1)$ *and in* $C[0,1]$, *for some constant* $\alpha \in S^1$.

PROOF. – (i) By Proposition 3.3 we have, in particular, that $\lim_{\varepsilon\to 0} |\boldsymbol{m}^\varepsilon| = 1$, uniformly on $[0,1]$. Hence, for any $\delta > 0$ we have, for $\varepsilon \le \varepsilon_1(\delta)$,

$$1-\delta \le |\boldsymbol{m}^\varepsilon(x)| \le 1+\delta\,, \quad x \in [0,1]\,. \tag{3.33}$$

In particular, if $\delta \le 1/2$, say, then we may write $\boldsymbol{m}^\varepsilon = \rho e^{i\phi}$, with $\rho = |\boldsymbol{m}^\varepsilon|$. A simple computation gives

$$F_{\mu,\varepsilon}(\boldsymbol{m}^\varepsilon) = \frac{1}{2}\int_0^1 (\rho^2|\phi_x|^2 + |\rho_x|^2)dx + \frac{1}{4\varepsilon}\int_0^1 (1-\rho^2)^2\,dx \\ -\frac{\mu}{4}\int_0^1 \left(\rho^2\sin 2\phi - \int_0^1 \rho^2 \sin 2\phi\, dt\right)^2 dx. \tag{3.34}$$

By the Cauchy-Schwarz inequality we get,

$$\int_0^1 \left(\rho^2 \sin 2\phi - \int_0^1 \rho^2 \sin 2\phi\, dt\right)^2 dx \tag{3.35}$$
$$= \int_0^1 \left(\left(\sin 2\phi - \int_0^1 \sin 2\phi\, dt\right) + (\rho^2-1)\sin 2\phi - \int_0^1 (\rho^2-1)\sin 2\phi\, dt\right)^2 dx$$
$$\le (1+\delta)\int_0^1 \left(\sin 2\phi - \int_0^1 \sin 2\phi\, dt\right)^2 dx$$
$$+\left(1+\frac{1}{\delta}\right)\left(\int_0^1 (\rho^2-1)^2 \sin^2 2\phi\, dx - \left(\int_0^1 (\rho^2-1)\sin 2\phi\, dx\right)^2\right)$$
$$\le (1+\delta)\int_0^1 \left(\sin 2\phi - \int_0^1 \sin 2\phi\, dt\right)^2 dx + \left(1+\frac{1}{\delta}\right)\int_0^1 (1-\rho^2)^2\,dx\,.$$

Combining (3.35) with (3.34) and (3.33) yields

$$F_{\mu,\varepsilon}(\boldsymbol{m}^{\varepsilon}) \geq \frac{(1-\delta)^2}{2}\int_0^1 |\phi_x|^2 - \frac{\mu(1+\delta)}{4}\int_0^1\left(\sin 2\phi - \int_0^1 \sin 2\phi\, dt\right)^2 + \left(\frac{1}{4\varepsilon} - \frac{\mu}{4}\left(1+\frac{1}{\delta}\right)\right)\int_0^1 (1-\rho^2)^2\, dx. \tag{3.36}$$

Since $\mu < \lambda_2/2$ we can fix δ small enough so that

$$\tilde{\mu} := \frac{1+\delta}{(1-\delta)^2}\mu < \frac{\lambda_2}{2}.$$

For ε small enough such that $\frac{1}{8\varepsilon} \geq \frac{\mu}{4}(1+1/\delta)$ we obtain from (3.36)

$$0 \geq F(\boldsymbol{m}^{\varepsilon}) \geq (1-\delta)^2\left\{\frac{1}{2}\int_0^1 |\phi_x|^2\, dx - \frac{\tilde{\mu}}{4}\int_0^1\left(\sin 2\phi - \int_0^1 \sin 2\phi\, dt\right)^2 dx\right\} + \frac{1}{8\varepsilon}\int_0^1 (1-\rho^2)^2\, dx \geq 0. \tag{3.37}$$

By Proposition 3.1 strict inequality holds for the last inequality on the r.h.s. of (3.37), unless $\boldsymbol{m}^{\varepsilon}$ equals identically a constant of modulus one, hence the result.

(ii) By Proposition 3.1 we have in this case,

$$\mathcal{F}_{\mu,\varepsilon} \leq I(\mu) < 0.$$

The convergence assertion follows from Proposition 3.3 and the uniqueness follows from Proposition 3.2.

(iii) This part is a direct consequence of Proposition 3.3 and Proposition 3.1. □

REMARK 3.2. – We do not know whether in the the limiting case $\mu = \lambda_2/2$ (case (iii)) the minimizer $\boldsymbol{m}^{\varepsilon}$ is necessarily a constant for ε small enough, as in case (i).

4. – The analysis of the gradient flow equation.

Let T be a positive number, we define $Q_T = \Omega \times (0,T)$ and $((\cdot,\cdot))$ the scalar product in $L^2(\Omega)$ and in $\boldsymbol{L}^2(\Omega)$. Consider the initial boundary value problem

$$\boldsymbol{u}_t = \boldsymbol{u}_{xx} - \varepsilon^{-1}(|\boldsymbol{u}|^2 - 1)\boldsymbol{u} + \mu\Lambda(\boldsymbol{u})\left[\Lambda(\boldsymbol{u})\cdot\boldsymbol{u} - \int_0^1 \Lambda(\boldsymbol{u})\cdot\boldsymbol{u}\, dx\right], \tag{4.1}$$

with the boundary conditions

$$\boldsymbol{u}_x(0,t) = \boldsymbol{u}_x(1,t) = 0, \qquad t \in (0,T)\,, \tag{4.2}$$

and the initial condition

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \equiv (0,1). \tag{4.3}$$

Provided the solution $\boldsymbol{u}(t)$ of (4.1), (4.2), (4.3) exists for all t, we show that $\lim_{t\to\infty} \boldsymbol{u}(t) = \boldsymbol{u}_\infty$ exists and, for suitable choice of the initial datum $\boldsymbol{u}_0$, the function $\boldsymbol{u}_\infty$ is a negative energy solution to (2.21), (2.22).

The following existence and uniqueness theorem holds.

THEOREM 4.1. – *Let $\boldsymbol{u}_0(\boldsymbol{x}) \in \boldsymbol{H}^1(\Omega)$ and $\varepsilon^{-1} > 2\mu$ and set*

$$\boldsymbol{N}(\boldsymbol{u}) = -\varepsilon^{-1}(|\boldsymbol{u}|^2 - 1)\boldsymbol{u} + \mu \Lambda(\boldsymbol{u})[\Lambda(\boldsymbol{u}) \cdot \boldsymbol{u} - \int_0^1 \Lambda(\boldsymbol{u}) \cdot \boldsymbol{u}\, dx]. \tag{4.4}$$

Then, there exists a unique solution $\boldsymbol{u} \in \boldsymbol{L}^\infty(Q_T)$ such that

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{L}^2(0,T;H^1(\Omega)), \quad \boldsymbol{u}_t \in \boldsymbol{L}^2(0,T;H^1(\Omega)'), \\ \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(Q_T)} \le B, \quad (B \textit{ independent of } T), \\ \dfrac{d}{dt}((\boldsymbol{u},\boldsymbol{v})) + \displaystyle\int_\Omega \boldsymbol{u}_x \cdot \boldsymbol{v}_x dx = ((\boldsymbol{N}(\boldsymbol{u}),\boldsymbol{v})), \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \text{ in } \mathcal{D}'(0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{4.5}$$

Then $\boldsymbol{u}$ is a weak solution of (4.1)-(4.3) *and since $\boldsymbol{N}(\boldsymbol{u})$ is bounded this is also a strong solution.*

PROOF. – We use the Galerkin method. We consider $w_1, ..., w_n$ an orthogonal basis in $L^2(\Omega)$ of eigenvectors for the Neumann problem

$$\begin{cases} -w_{xx} = \lambda w \quad \text{in } \Omega \\ w_x(0) = w_x(1) = 0. \end{cases} \tag{4.6}$$

We consider then

$$\boldsymbol{u}_n = (u_{n,1}, u_{n,2}), \quad u_{n,j} = \sum_{i=1}^n y_{i,j}(t) w_i \quad j = 1,2, \tag{4.7}$$

solution to the Cauchy problem

$$\begin{cases} \boldsymbol{u}_n' = (\boldsymbol{u}_n)_{xx} + \boldsymbol{N}(\boldsymbol{u}_n) \quad t \in (0,T), \\ u_{n,j}(0) = \sum_{i=1}^n ((w_i, u_{0,j})) w_i, \quad j = 1,2. \end{cases} \tag{4.8}$$

It is clear that (4.8) is a nonlinear system of ode's with $2n$ unknowns. It has a unique solution locally.

CLAIM 1: $\boldsymbol{u}_n(0)$ is bounded in $\boldsymbol{H}^1(\Omega)$. Indeed for $j = 1, 2$ one has

$$(4.9)\qquad \begin{cases} \displaystyle\int_0^1 |(u_{n,j}(0))_x|^2 = \sum_{i=1}^n ((w_i, u_{0,j}))^2 \int_0^1 |w_{ix}|^2 = \sum_{i=1}^n ((w_i, u_{0,j}))^2 \lambda_i \\ \displaystyle\qquad\qquad \le \sum_{i=1}^\infty ((w_i, u_{0,j}))^2 \lambda_i = \int_0^1 |(u_{0,j})_x|^2 \\ \displaystyle\int_0^1 |u_{n,j}(0)|^2 = \sum_{i=1}^n ((w_i, u_{0,j}))^2 \int_0^1 |w_i|^2 = \sum_{i=1}^n ((w_i, u_{0,j}))^2 \le \int_0^1 |u_{0,j}|^2. \end{cases}$$

To simplify our notation we do not write the measures of integration.

CLAIM 2: $\boldsymbol{u}_n$ is bounded in $\boldsymbol{L}^\infty(\Omega \times (0,t))$ by a constant independent of n and t.

We multiply the first equation of (4.8) by $\boldsymbol{u}_n'$ and integrate on $Q_t = \Omega \times (0,t)$ to get

$$\int_{Q_t} |\boldsymbol{u}_n'|^2 = \int_{Q_t} (\boldsymbol{u}_n)_{xx} \cdot \boldsymbol{u}_n' - \varepsilon^{-1} \int_{Q_t} (|\boldsymbol{u}_n|^2 - 1) \boldsymbol{u}_n \cdot \boldsymbol{u}_n'$$
$$+ \mu \int_{Q_t} \Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n' \left[\Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n - \int_0^1 \Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n \right].$$

We remark then that

$$\boldsymbol{u}_n \cdot \boldsymbol{u}_n' = \left(\frac{1}{2} |\boldsymbol{u}_n|^2 \right)', \qquad \Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n' = (u_{n,1} u_{n,2})' = \frac{1}{2} (\Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n)'.$$

Then we obtain

$$\int_{Q_t} |\boldsymbol{u}_n'|^2 = -\int_0^t \frac{1}{2} \left(\int_\Omega |\boldsymbol{u}_{nx}|^2 \right)' - \frac{\varepsilon^{-1}}{4} \int_{Q_t} ((|\boldsymbol{u}_n|^2 - 1)^2)'$$
$$+ \frac{\mu}{4} \int_0^t \left(\int_\Omega (\Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n)^2 \right)' - \frac{\mu}{4} \int_0^t \left[\left(\int_\Omega \Lambda(\boldsymbol{u}_n) \cdot \boldsymbol{u}_n \right)^2 \right]'.$$

By integration we obtain

$$(4.10)\qquad \int_{Q_t} |\boldsymbol{u}_n'|^2 = F(\boldsymbol{u}_n)(0) - F(\boldsymbol{u}_n)(t)$$

where we have set

$$F(\boldsymbol{u}_n) = \frac{1}{2}\int_\Omega |(\boldsymbol{u}_n)_x|^2 + \frac{\varepsilon^{-1}}{4}\int_\Omega (|\boldsymbol{u}_n|^2 - 1)^2 - \frac{\mu}{4}\int_\Omega (\Lambda(\boldsymbol{u}_n)\cdot\boldsymbol{u}_n)^2 + \frac{\mu}{4}\left(\int_\Omega \Lambda(\boldsymbol{u}_n)\cdot\boldsymbol{u}_n\right)^2.$$

By the Claim 1, $\boldsymbol{u}_n(0)$ is bounded in $\boldsymbol{H}^1(\Omega)$, and then also in $\boldsymbol{L}^\infty(\Omega)$, by a constant independent of n. It follows that $F(\boldsymbol{u}_n)(0)$ is bounded by a constant A independent of n, so from (4.10) we derive

$$F(\boldsymbol{u}_n)(t) \le A\,. \tag{4.11}$$

Now we have

$$\int_\Omega (\Lambda(\boldsymbol{u}_n)\cdot\boldsymbol{u}_n)^2 \le \int_\Omega |\boldsymbol{u}_n|^4 = \int_\Omega (|\boldsymbol{u}_n|^2 - 1 + 1)^2 \le 2\int_\Omega (|\boldsymbol{u}_n|^2 - 1)^2 + 2\,.$$

Then, from (4.11) and the definition of $F(\boldsymbol{u}_n)$ we get

$$\frac{1}{2}\int_\Omega |(\boldsymbol{u}_n)_x|^2 + \frac{\varepsilon^{-1} - 2\mu}{4}\int_\Omega (|\boldsymbol{u}_n|^2 - 1)^2 - \frac{\mu}{2} + \frac{\mu}{4}\left(\int_\Omega \Lambda(\boldsymbol{u}_n)\cdot\boldsymbol{u}_n\right)^2 \le A.$$

Since $\varepsilon^{-1} - 2\mu > 0$ it follows that

$$\int_\Omega |(\boldsymbol{u}_n)_x|^2 \le 2A + \mu, \qquad \int_\Omega (|\boldsymbol{u}_n|^2 - 1)^2 \le \frac{4A + 2\mu}{\varepsilon^{-1} - 2\mu}. \tag{4.12}$$

Due to the inequality $\int_\Omega (|\boldsymbol{u}_n|^2 - 1) \le \left\{\int_\Omega (|\boldsymbol{u}_n|^2 - 1)^2\right\}^{1/2}$ we have that $\boldsymbol{u}_n$ is bounded in $\boldsymbol{H}^1(\Omega)$ by a constant independent of n and t and the Claim 2 follows from the imbedding of $H^1(\Omega)$ into $L^\infty(\Omega)$.

As a consequence of the Claim 2 the solution to (4.8) is global on $(0, T)$. It is also unique due to the fact that for $\boldsymbol{u}$ bounded, $\boldsymbol{N}(\boldsymbol{u})$ is Lipschitz continuous. Moreover $\boldsymbol{u}_n$ is also smooth in x and t.

Let us denote by B the constant which bounds, uniformly in n and t, the function $\boldsymbol{u}_n$ and set

$$K = \{\boldsymbol{v} \in \boldsymbol{L}^2(Q_T) \quad | \quad |\boldsymbol{v}| \le B \ \text{ a.e. in } Q_T\}.$$

It is clear that K is a closed convex set of $\boldsymbol{L}^2(Q_T)$. Due to the preceding analysis and the equation (4.8) it follows that for some constant C independent of n and T we have

$$\|\boldsymbol{u}_n\|_{\boldsymbol{L}^\infty(0,T;H^1(\Omega))} \le C, \quad \|(\boldsymbol{u}_n)_t\|_{\boldsymbol{L}^\infty(0,T;H^1(\Omega)')} \le C, \quad \|\boldsymbol{u}_n\|_{\boldsymbol{L}^\infty(0,T;L^2(\Omega))} \le C.$$

Since the imbedding

$$\{v \quad | \quad v \in L^2(0,T;H^1(\Omega)),\ v_t \in L^2(0,T;H^1(\Omega)')\} \ \subset \ L^2(0,T;L^2(\Omega))$$

is compact – up to a subsequence – there exists $\boldsymbol{u}$ in $\boldsymbol{L}^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned} \boldsymbol{u}_n &\rightharpoonup \boldsymbol{u} \quad \text{in } \boldsymbol{L}^2(0,T;H^1(\Omega)), \\ \boldsymbol{u}_n &\to \boldsymbol{u} \quad \text{in } \boldsymbol{L}^2(0,T;L^2(\Omega)), \\ (\boldsymbol{u}_n)_t &\rightharpoonup \boldsymbol{u}_t \quad \text{in } \boldsymbol{L}^2(0,T;H^1(\Omega)'). \end{aligned}$$

Of course $\boldsymbol{u} \in K$. Going back to (4.4) we have

$$\boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{N}_1(\boldsymbol{u}) + \boldsymbol{N}_2(\boldsymbol{u})\,,$$

where we have set

$$\begin{aligned} \boldsymbol{N}_1(\boldsymbol{u}) &= -\varepsilon^{-1}(|\boldsymbol{u}|^2-1)\boldsymbol{u} + \mu(\Lambda(\boldsymbol{u})\cdot\boldsymbol{u})\Lambda(\boldsymbol{u}) \\ \boldsymbol{N}_2(\boldsymbol{u}) &= -\mu\Lambda(\boldsymbol{u})\int_\Omega (\Lambda(\boldsymbol{u})\cdot\boldsymbol{u})\,dx. \end{aligned}$$

Since $\boldsymbol{N}_1(\boldsymbol{u})$ is a smooth function we have for some constant L_1

$$|\boldsymbol{N}_1(\boldsymbol{u}) - \boldsymbol{N}_1(\boldsymbol{v})| \le L_1|\boldsymbol{u}-\boldsymbol{v}|, \quad \forall\,\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^2, \text{ bounded.} \tag{4.13}$$

Moreover for $\boldsymbol{u},\boldsymbol{v} \in K$

$$\begin{aligned} |\boldsymbol{N}_2(\boldsymbol{u}) - \boldsymbol{N}_2(\boldsymbol{v})| &= \Big| -\mu\Lambda(\boldsymbol{u})\int_\Omega (\Lambda(\boldsymbol{u})\cdot\boldsymbol{u}) + \mu\Lambda(\boldsymbol{v})\int_\Omega (\Lambda(\boldsymbol{v})\cdot\boldsymbol{v})\Big| \\ &= \Big| -\mu(\Lambda(\boldsymbol{u})-\Lambda(\boldsymbol{v}))\int_\Omega (\Lambda(\boldsymbol{u})\cdot\boldsymbol{u}) + \mu\Lambda(\boldsymbol{v})\int_\Omega (\Lambda(\boldsymbol{v})\cdot\boldsymbol{v} - \Lambda(\boldsymbol{u})\cdot\boldsymbol{u})\Big| \\ &\le C_1|\boldsymbol{u}-\boldsymbol{v}| + C_2\Big\{\int_\Omega |\boldsymbol{u}-\boldsymbol{v}|^2dx\Big\}^{1/2}. \end{aligned} \tag{4.14}$$

From these estimates it follows that

$$\boldsymbol{N}(\boldsymbol{u}_n) \to \boldsymbol{N}(\boldsymbol{u}) \quad \text{in } \boldsymbol{L}^2(0,T;L^2(\Omega)).$$

We take now $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ to get from (4.8)

$$\frac{d}{dt}(\!(\boldsymbol{u}_n,\boldsymbol{v})\!) = -\int_\Omega \boldsymbol{u}_{nx}\cdot\boldsymbol{v}_x + \int_\Omega \boldsymbol{N}(\boldsymbol{u}_n)\cdot\boldsymbol{v}, \qquad \forall\, t \in (0,T).$$

Passing to the limit in n we get easily the third equation of (4.5).

Let now $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and let φ be a smooth function such that

$$\varphi(0) = 1, \quad \varphi(T) = 0.$$

From (4.5) we have

$$\int_0^T \frac{d}{dt}(\!(\boldsymbol{u},\boldsymbol{v})\!)\varphi = -\int_{Q_T} \boldsymbol{u}_x\cdot\boldsymbol{v}_x\,\varphi + \int_{Q_T} N(\boldsymbol{u})\cdot\boldsymbol{v}\,\varphi$$

$$= \lim_n \; -\int_{Q_T} \boldsymbol{u}_{nx}\cdot\boldsymbol{v}_x\,\varphi + \int_{Q_T} N(\boldsymbol{u}_n)\cdot\boldsymbol{v}\,\varphi = \lim_n \int_0^T \frac{d}{dt}\,(\!(\boldsymbol{u}_n,\boldsymbol{v})\!)\,\varphi$$

$$= \lim_n \int_0^T \frac{d}{dt}[(\!(\boldsymbol{u}_n,\boldsymbol{v})\!)\,\varphi] - \int_0^T (\!(\boldsymbol{u}_n,\boldsymbol{v}\varphi)\!)' = -\lim_n (\!(\boldsymbol{u}_n(0),\boldsymbol{v})\!) - \int_0^T (\!(\boldsymbol{u},\boldsymbol{v})\!)\varphi' =$$

$$= -(\!(\boldsymbol{u}_0,\boldsymbol{v})\!) - \int_0^T (\!(\boldsymbol{u},\boldsymbol{v})\!)\,\varphi'.$$

Integrating the left hand side of this equality we arrive to

$$(\!(\boldsymbol{u}(0),\boldsymbol{v})\!) = (\!(\boldsymbol{u}_0,\boldsymbol{v})\!), \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}^1(\Omega)\,,$$

which completes the existence result.

For uniqueness, starting from two solutions $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ we have

$$\frac{d}{dt}(\boldsymbol{u}_1-\boldsymbol{u}_2) = (\boldsymbol{u}_1-\boldsymbol{u}_2)_{xx} + \boldsymbol{N}(\boldsymbol{u}_1) - \boldsymbol{N}(\boldsymbol{u}_2).$$

Multiplying by $(\boldsymbol{u}_1-\boldsymbol{u}_2)$ and integrating in Ω we get by (4.13), (4.14)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_1-\boldsymbol{u}_2|^2 \le C\int_\Omega |\boldsymbol{u}_1-\boldsymbol{u}_2|^2\,,$$

and the result follows. □

COROLLARY 4.1. – *Let* $\boldsymbol{u}$ *be the solution of the problem* (4.1), (4.2), (4.3). *Then,*

$$\int_{Q_t} |\boldsymbol{u}_t|^2 = F(\boldsymbol{u})(0) - F(\boldsymbol{u})(t)\,. \tag{4.15}$$

Moreover, there exists a positive constant $\bar{A}$ *independent of t, such that*

$$\int_{Q_t} |\boldsymbol{u}_t|^2 + \int_\Omega |\boldsymbol{u}_x|^2 + \int_\Omega (|\boldsymbol{u}|^2-1)^2 \le \bar{A}\,. \tag{4.16}$$

PROOF. – The equality (4.15) easily follows from (4.10). Moreover, we have

$$F(\boldsymbol{u}) \ge \frac{1}{2}\int_\Omega |\boldsymbol{u}_x|^2 + \frac{\varepsilon^{-1}-2\mu}{4}\int_\Omega (|\boldsymbol{u}|^2-1)^2 - \frac{\mu}{2} + \frac{\mu}{4}\left(\int_\Omega \Lambda(\boldsymbol{u})\cdot\boldsymbol{u}\right)^2$$

for $\varepsilon^{-1} - 2\mu \ge \bar{a} > 0$. We get then the estimate (4.16). □

LEMMA 4.1. – *Let* $\boldsymbol{u}$ *the solution of the problem* (4.1), (4.2), (4.3). *Then, there exists a positive constant* $\bar{K}$ *such that the following estimate holds*

$$\int_0^T \left| \frac{d}{dt} \|\boldsymbol{u}_t\|^2_{L^2(0,1)} \right| dt \le \bar{K} . \tag{4.17}$$

PROOF. – We look at the equation (4.1) in the form

$$\boldsymbol{u}_t = \boldsymbol{u}_{xx} + \boldsymbol{N}(\boldsymbol{u}) .$$

Differentiating with respect to t and multiplying by $\boldsymbol{u}_t$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 |\boldsymbol{u}_t|^2 \, dx + \int_0^1 |\boldsymbol{u}_{xt}|^2 dx = \int_0^1 \frac{d}{dt} \boldsymbol{N}(\boldsymbol{u}) \cdot \boldsymbol{u}_t \, dx. \tag{4.18}$$

Recall that $\boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{N}_1(\boldsymbol{u}) + \boldsymbol{N}_2(\boldsymbol{u})$ where $\boldsymbol{N}_1$ is a C^∞-function and

$$\boldsymbol{N}_2(\boldsymbol{u}) = -\mu \Lambda(\boldsymbol{u}) \int_\Omega (\Lambda(\boldsymbol{u}) \cdot \boldsymbol{u}) \, dx.$$

From this we deduce

$$\frac{d}{dt} \boldsymbol{N}_2(\boldsymbol{u}) = -\mu \Lambda(\boldsymbol{u}_t) \int_\Omega (\Lambda(\boldsymbol{u}) \cdot \boldsymbol{u}) \, dx - 2\mu \Lambda(\boldsymbol{u}) \int_\Omega (\Lambda(\boldsymbol{u}) \cdot \boldsymbol{u}_t) \, dx ,$$

and thus

$$\left\| \frac{d}{dt} \boldsymbol{N}_2(\boldsymbol{u}) \right\| \le C \|\boldsymbol{u}_t\|.$$

Hence from (4.18), Theorem 4.1 and Corollary 4.1

$$\int_0^T \left| \frac{d}{dt} \|\boldsymbol{u}_t\|^2_{L^2(0,1)} \right| dt \le C \left| \int_{Q_T} |\boldsymbol{u}_t|^2 dx \right| \le \bar{K} ,$$

and the proof of the lemma easily follows. □

Now we can prove the following theorem

THEOREM 4.2. – *–Let* $\boldsymbol{u}$ *the solution of the problem* (4.1), (4.2), (4.3) *for* $T = \infty$. *Then, there exists a sequence* $t_k \to \infty$ *such that*

$$\boldsymbol{u}(\boldsymbol{x}, t_k) \rightharpoonup \boldsymbol{u}_\infty(\boldsymbol{x}) \quad \text{in } H^1(0,1) , \tag{4.19}$$

where $\boldsymbol{u}_\infty(\boldsymbol{x})$ *is a stationary point of* (4.1). *Moreover, all the weakly convergent sequences converge to stationary points.*

PROOF. – Let $\boldsymbol{u}^k = \boldsymbol{u}(\cdot, t_k)$ be the given solution of (4.1), (4.2), (4.3) at time t_k. From the estimate (4.16) it follows that, passing to a subsequence if necessary,

$$\boldsymbol{u}^k \rightharpoonup \boldsymbol{u}_\infty \quad \text{weakly} \quad \text{in } H^1(0,1)\,, \tag{4.20}$$

$$\boldsymbol{u}^k \to \boldsymbol{u}_\infty \quad \text{strongly} \quad \text{in } L^2(0,1)\,, \tag{4.21}$$

$$\boldsymbol{u}^k \cdot \Lambda(\boldsymbol{u}^k) \to \boldsymbol{u}_\infty \cdot \Lambda(\boldsymbol{u}_\infty) \quad \text{strongly} \quad \text{in } L^2(0,1)\,, \tag{4.22}$$

$$|\boldsymbol{u}^k|^2 \to |\boldsymbol{u}_\infty|^2 \quad \text{strongly} \quad \text{in } L^2(0,1)\,. \tag{4.23}$$

Now we have to prove that $\boldsymbol{u}_\infty$ is a solution of the stationary problem. For this we multiply the equation (4.1) by $\boldsymbol{v} \in \boldsymbol{H}^1(0,1)$ and integrate to get

$$\int_0^1 \boldsymbol{u}_t^k \cdot \boldsymbol{v}\, dx = -\int_0^1 \boldsymbol{u}_x^k \cdot \boldsymbol{v}_x\, dx - \varepsilon^{-1} \int_0^1 (|\boldsymbol{u}^k|^2 - 1)\boldsymbol{u}^k \cdot \boldsymbol{v} dx \tag{4.24}$$

$$+\, \mu \int_0^1 \Lambda(\boldsymbol{u}^k) \cdot \boldsymbol{v} \left[\Lambda(\boldsymbol{u}^k) \cdot \boldsymbol{u}^k - \int_0^1 \Lambda(\boldsymbol{u}^k) \cdot \boldsymbol{u}^k dx \right] \tag{4.25}$$

From Lemma 4.1 we have that $\|\boldsymbol{u}_t^k\|^2$ is a Cauchy sequence (see (4.17)) and the limit can only be 0 since $\int_0^\infty \|\boldsymbol{u}_t\|^2$ is bounded. From the convergence established above it follows that $\boldsymbol{u}_\infty$ is a weak solution of the stationary problem. □

COROLLARY 4.2. – *Let* $\boldsymbol{u}_0$ *be a function verifying the hypotheses of Theorem* 4.1. *If* $F(\boldsymbol{u}_0) < 0$ *then the limit function* $\boldsymbol{u}_\infty(\boldsymbol{x})$ *defined in Theorem* 4.2 *is a negative energy stationary point of* (3.1).

PROOF. – The proof easily follows from the energy estimate (4.15). Indeed since the system is dissipative we have

$$F(\boldsymbol{u}_\infty) \leq F(\boldsymbol{u}_0)$$

□

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