Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **1** (2008), n.1, p. 243–253.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_1_243_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2008.

Bollettino U. M. I.
(9) **I** (2008), 243-253

On the Regularity of *p*-Harmonic Functions in the Heisenberg Group

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Sunto. – *Descriviamo alcuni recenti risultati ottenuti in* [29], *dove si dimostrano teoremi di regolarità per soluzioni di equazioni sub-ellittiche in forma di divergenza orizzontale, nel gruppo di Heisenberg. I risultati coprono il caso di operatori a crescita p, come il p-Laplaciano nel gruppo di Heisenberg, e sono ottenuti sotto l'ipotesi adimensionale* $p \in [2,4)$.

Abstract. – *We describe some recent results obtained in* [29], *where we prove regularity theorems for sub-elliptic equations in (horizontal) divergence form defined in the Heisenberg group, and exhibiting polynomial growth of order p. The main result tells that when* $p \in [2,4)$ *solutions to possibly degenerate equations are locally Lipschitz continuous with respect to the intrinsic distance. In particular, such result applies to p-harmonic functions in the Heisenberg group. Explicit estimates are obtained, and eventually applied to obtain the suitable Calderón-Zygmund theory for the associated non-homogeneous problems.*

1. – Introduction.

In this note we shall describe the results obtained in [29], concerning the local regularity of solutions to non-linear sub-elliptic equations in the Heisenberg group of the type

$$\mathrm{div}_H\, a(\mathfrak{X}u) = \sum_{i=1}^{2n} X_i a_i(\mathfrak{X}u) = 0\,. \tag{1.1}$$

The assumptions we shall consider are the natural ones for elliptic equations in divergence form with p-growth, initially considered by Ladyzhenskaya & Ural'tseva in the Euclidean setting [24]. The equation (1.1) is therefore defined in a bounded, open sub-domain Ω of the Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$, $n \geq 1$, while the vector field $a = (a_i)\colon \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ is assumed to be of class C^1 and satisfying the following growth and ellipticity conditions:

$$|Da(z)|(\mu^2 + |z|^2)^{\frac{1}{2}} + |a(z)| \leq L(\mu^2 + |z|^2)^{\frac{p-1}{2}}, \tag{1.2}$$

and

$$\nu(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \sum_{i,j=1}^{2n} D_{z_j} a_i(z)\lambda_i\lambda_j, \tag{1.3}$$

for every $z, \lambda \in \mathbb{R}^{2n}$, where $0 < \nu \leq 1 \leq L$, $\mu \in [0,1]$, $p \geq 2$. The parameter μ clearly serves to describe the possible degeneration, with respect to the horizontal gradient $\mathfrak{X}u$, of the considered operator; therefore we shall refer to the case $\mu = 0$ as the degenerate one. In the last case the relevant model example covered here involves the familiar horizontal p-Laplacean operator on the left-hand side:

$$\operatorname{div}_H \left(|\mathfrak{X}u|^{p-2}\mathfrak{X}u \right) = 0 \,. \tag{1.4}$$

The last equation, whose solutions are indeed called p-harmonic functions (in the Heisenberg group), has been intensively studied in the last years, especially in its original Euclidean version, and plays an important role in the modern Geometric Function Theory.

The notation we are using here is the standard one: we are denoting points $x \in \mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ by mean of the usual exponential coordinates $x = (x_1, \ldots, x_{2n}, t)$; for $i \in \{1, \ldots, n\}$, the horizontal vector fields are defined by

$$X_i \equiv X_i(x) = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \qquad X_{n+i} \equiv X_{n+i}(x) = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t, \tag{1.5}$$

while the remaining vertical vector field is $T \equiv T(x) = \partial_t$, which can be realized as a commutator, that is $T = [X_i, X_{i+n}]$ for every $i \in \{1, \ldots, n\}$; we also denote the *horizontal gradient* by $\mathfrak{X}u = (X_1u, X_2u, \ldots, X_{2n}u)$. The natural functional ambient for the problem (1.1) under the assumptions (1.2)-(1.3) is of course the horizontal Sobolev space $HW^{1,p}(\Omega)$, which consists of those L^p-functions whose distributional horizontal gradient is L^p-integrable. Therefore, the solutions u considered here are initially assumed to belong to $L^p(\Omega)$ and to satisfy

$$\mathfrak{X}u \in L^p(\Omega, \mathbb{R}^{2n}) \,, \tag{1.6}$$

while no additional regularity is assumed about Tu; as we shall see in a few lines, this lack of initial vertical integrability is a major source of problems when dealing with regularity of solutions. We recall that if $F \equiv (F_i) : \Omega \to \mathbb{R}^{2n}$ is an L^1-vector field in the following we shall denote the horizontal divergence operator by $\operatorname{div}_H F \equiv \sum_{i=1}^{2n} X_i F_i$, which is obviously defined in the distributional sense. Such operator obviously serves to define weak solutions of equation (1.1), which are in fact defined in the usual distributional way. We refer to [29] for more on the notation used here and more references and results on the Heisenberg groups $\mathbb{H}^n$, $n \in \mathbb{N}$.

Finally, let us mention that applications of our results to the Calculus of Variations in the Heisenberg group are immediate. In fact, via the use of the related associated Euler-Lagrange equations we can assert regularity results for local minima of variational integrals of the type:

$$v \mapsto \int_\Omega f(\mathfrak{X}v)\, dx \,,$$

where of course $v \in HW^{1,p}(\Omega)$, under suitable convexity and regularity assumptions of the energy density $f\colon \mathbb{R}^{2n} \to \mathbb{R}$. Essentially, we require that the vector field ∂f satisfies the conditions (1.2)-(1.3) when considering $a(\,\cdot\,) \equiv \partial f(\,\cdot\,)$. In particular, the results presented in the following section apply to local minima of the functional

$$v \mapsto \int_\Omega \left(\mu^2 + |\mathfrak{X}v|^2\right)^{p/2} dx, \qquad\qquad \mu \in [0,1]\,.$$

2. – Gradient regularity.

The study of regularity properties of weak solutions to equations as in (1.1) has a rather long and distinguished history. The first results are in the classical paper of Hörmander [19], which is concerned with the linear case, i.e. $a(z) \equiv z$; further linear results are for instance in [18, 17, 22]. The first non-linear paper dealing with gradient regularity is due to Capogna [5], and concerns the case $p = 2$, so that equations as those in (1.4) are not covered by his theory for $p \neq 2$. Indeed, the case $p > 2$ is another story, and presents remarkable, additional difficulties; Hölder continuity of u has been obtained in [6, 25], while when considering the gradient of solutions only partial regularity results are available, that is, the regularity of the gradient outside a closed, negligible subset of the domain Ω; this fact has first been established by Capogna & Garofalo in [7], see also [16] for another proof. Up to now, the everywhere continuity of Du has been established only assuming that p is not "too far from 2", that is under an assumption of the type

$$2 \leq p < 2 + o_n \tag{2.1}$$

where $o_n > 0$ denotes a rather awkward, and only in principle explicitly computable quantity, such that $o_n \searrow 0$ when $n \nearrow \infty$. Clearly, an unpleasant feature of an assumption such as (2.1) is that for a fixed p in the range $[2,4)$ only low dimensional Heisenberg groups can be dealt with.

Let us summarize the situation; Domokos [11], extending earlier, pioneering results of Marchi [28], showed that $Tu \in L^p_{\mathrm{loc}}(\Omega)$ if $p < 4$, which proved to be an up-to-now unavoidable upper bound on p, coming in a particularly natural way from the analysis of (1.1). Proving that $Tu \in L^p_{\mathrm{loc}}(\Omega)$ is of course the first fundamental step towards the regularization of solutions u to (1.1), since for them the initial regularity information is just (1.6). As for the higher regularity of Du or $\mathfrak{X}u$, a few Hölder regularity results are available in [8, 12, 13, 26]; all such papers feature a condition like (2.1). In particular, Hölder continuity of the gradient is proved in the degenerate case in [12], for a not explicitly computed quantity o_n, while in [26], for the non-degenerate case $\mu > 0$, the authors find a bound of the type $o_n \approx 1/n$, and only the lower dimensional cases $\mathbb{H}^1$ and $\mathbb{H}^2$ can be treated when considering the full range $p \in [2, 4)$.

The first novelty of our paper [29] is that, for the first time, it is possible to obtain *dimension-free pointwise regularity results for gradients of solutions*, therefore completely avoiding the use of any dimensional assumptions of the type (2.1). Second, up to a certain extent, we are treating the degenerate case $\mu = 0$, thereby *covering the sub-elliptic p-Laplacean equation* (1.4).

2.1 – *The degenerate case $\mu = 0$.*

The main result in the degenerate case is the following:

THEOREM 2.1. – *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation* (1.1) *under the assumptions* (1.2)-(1.3) *with $\mu = 0$, where $2 \le p < 4$. Then*

$$\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}), \qquad \textit{and} \qquad Tu \in L^q_{\mathrm{loc}}(\Omega) \qquad \textit{for every} \quad q < \infty \, . \tag{2.2}$$

Moreover there exists a constant c, depending on $n, p, L/\nu$, but otherwise independent of the solution u, and of the vector field $a(\cdot)$, such that the following inequality holds for any CC-ball $B_R \subset \Omega$:

$$\sup_{B_{R/2}} |\mathfrak{X}u| \le c \left(\fint_{B_R} |\mathfrak{X}u|^p \, dx \right)^{1/p} . \tag{2.3}$$

Finally, for every $q < \infty$ there exists a constant $\tilde{c}$ depending only on $n, p, L/\nu, q$ such that

$$\left(\fint_{B_{R/2}} |Tu|^q \, dx \right)^{1/q} \le \frac{\tilde{c}}{R} \left(\fint_{B_R} |\mathfrak{X}u|^p \, dx \right)^{1/p} . \tag{2.4}$$

See also Corollaries 2.1 and 2.2 below. We remark that, for a fixed p in the considered range, the previous theorem applies to equations as (1.1), considered in Heisenberg groups of arbitrary dimension n; this cannot be obtained in previous papers such as [12, 26]. We also observe that, here as in the following, the balls considered are those generated by the intrinsic distance $d_{cc}(\cdot,\cdot)$, that is the intrinsic Carnot-Carathèodory metric generated by the horizontal vector fields $\mathfrak{X}u$.

2.2 – *Non-degenerate equations.*

When considering the non-degenerate case $\mu > 0$, better results can be obtained. In this situation the basic model example is given by the non-degenerate sub-elliptic p-Laplacean equation, that is

$$\mathrm{div}_H\left((\mu^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}}\mathfrak{X}u\right) = 0 \qquad \mu > 0\,. \tag{2.5}$$

In fact, the classical Euclidean Hölder continuity of the gradient is retrieved.

THEOREM 2.2. – *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation* (1.1) *under the assumptions* (1.2)-(1.3) *with $2 \le p < 4$ and $\mu > 0$. Then the Euclidean gradient Du is locally Hölder continuous in Ω.*

Related local estimates are of course available in this case too. We have also an a-priori L^∞ local estimates for Tu, but this degenerates when μ approaches zero; this fact prevents us to prove higher regularity results in the degenerate case.

THEOREM 2.3. – *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation* (1.1) *under the assumptions* (1.2)-(1.3) *with $2 \le p < 4$ and $\mu > 0$. Then there exists a constant c, depending on n, p and L/υ, but otherwise independent of μ, of the solution u, and of the vector field $a(\cdot)$, such that the following inequalities hold for any CC-ball $B_R \subset \Omega$:*

$$\sup_{B_{R/2}} |\mathfrak{X}u| \le c\left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p\, dx\right)^{1/p}, \tag{2.6}$$

and

$$\sup_{B_{R/2}} |Tu| \le c\frac{\mu^{\frac{Q(2-p)}{4}}}{R}\left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p\, dx\right)^{\frac{1}{p}+\frac{Q(p-2)}{4p}}. \tag{2.7}$$

Finally, for every $1 < q < \infty$ there exists a constant $\tilde{c}$ depending only on $n, p, L/\upsilon, q$ such that

$$\left(\fint_{B_{R/2}} |Tu|^q\, dx\right)^{1/q} \le \frac{\tilde{c}}{R}\left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p\, dx\right)^{1/p}. \tag{2.8}$$

In the specific situation of the equation (2.5), where the considered vector field $a(z) \equiv (\mu^2 + |z|^2)^{\frac{p-2}{2}} z$ is smooth, the previous theorem allows to prove the arbitrary smoothness of solutions via standard boot-strap methods; see for instance [5, 26].

2.3 – *Horizontal Lipschitz continuity.*

In both the degenerate and the non-degenerate case the boundedness of the horizontal gradient naturally yields a-priori Lipschitz bounds

COROLLARY 2.1. – *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation* (1.1) *under the assumptions* (1.2)-(1.3) *with $2 \leq p < 4$ and $\mu \in [0,1]$. Then u is locally Lipschitz continuous in Ω with respect to the CC-metric in $\mathbb{H}^n$. Moreover there exists a constant c, depending only on $n, p, L/\nu$, but otherwise independent of μ, of the solution u, and of the vector field $a(\cdot)$, such that*

$$|u(x) - u(y)| \leq c \left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p \, dx \right)^{1/p} d_{cc}(x,y) \,, \tag{2.9}$$

holds whenever $B_R \subset \Omega$, and $x, y \in B_{R/2}$.

Another consequence of Theorem 2.1 and of the standard, Euclidean Sobolev-Morrey embedding theorem, is now the following:

COROLLARY 2.2. – *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation* (1.1) *under the assumptions* (1.2)-(1.3) *with $2 \leq p < 4$ and $\mu \in [0,1]$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha < 1$.*

We finally mention that the previous theorems are stated for $2 \leq p < 4$ for completeness, since in the automatically non-degenerate case $p = 2$ they are essentially due to Capogna [5].

3. – The Calderón-Zygmund theory.

As we shall see in a few lines, estimates of the type (2.3) open the way to a non-linear version of estimates of Calderón-Zygmund type in the Heisenberg group, up to now developed only in the case of linear sub-elliptic equations [3]. Here we shall deal with non-linear sub-elliptic equations.

Let us recall that in the Euclidean setting the validity of a non-linear Calderón-Zygmund theory is a classical achievement dating back to T. Iwaniec

[20], who dealt with the scalar case; later on extensions to systems of p-Laplacean type have been obtained by DiBenedetto & Manfredi in [10], see also [1]. The equations considered in [20, 10] are modeled by the non-homogeneous p-Laplacean equation, that is

$$\text{div}\,(|Du|^{p-2}Du) = \text{div}\,(|F|^{p-2}F)\,, \tag{3.1}$$

considered in open subsets of $\mathbb{R}^n$. The results obtained basically assert that $F \in L^q_{\text{loc}}$ implies $Du \in L^q_{\text{loc}}$ for every $q \geq p$ - see also [21] for some results when $q < p$. Observe that when $p = 2$ this is essentially the outcome of the classical Calderón-Zygmund theory developed in the fifties.

More recently, very general Calderón-Zygmund type estimates valid for solutions to general non-linear elliptic systems have been proposed in [1, 23], following and extending the techniques from [4]. The main point in the last two papers is the possibility of getting integrability results for solutions to non-homogeneous equations once certain reverse Hölder type inequalities have been established for solutions to homogeneous ones. Now, observe that the estimates found in (2.3)-(2.6) imply the validity of every type of reverse Hölder type inequalities for solutions to homogenous sub-elliptic equations. This fact allows us to extend the techniques used in [23]. We shall in fact take [23] as a starting point, and via the combined use of suitable maximal operators, of estimates (2.3)-(2.6), and of a-priori higher integrability results in the style of Gehring's lemma, we shall obtain a far-reaching generalization of both the non-linear results valid in the Euclidean setting, and of the linear results known in the sub-elliptic one. Due to the techniques adopted is not a surprise that at the end all the integrability results obtained will come along with explicit reverse type inequalities asserted in the corresponding integrability spaces.

Specifically, in [29] the equations we are considering are the natural horizontal version of (3.1), involving possibly discontinuous coefficients of VMO type, that is

$$\text{div}_H\,[b(x)a(\mathfrak{X}u)] = \text{div}_H\,(|F|^{p-2}F)\,, \tag{3.2}$$

with

$$b(\,\cdot\,) \in \text{VMO}_{\text{loc}}(\Omega) \qquad \text{and} \qquad \upsilon \leq b(x) \leq L\,. \tag{3.3}$$

The space $\text{VMO}_{\text{loc}}(\Omega)$ is the natural sub-elliptic version of the usual Euclidean VMO-space of John and Nirenberg: CC-balls are used in the definition instead of the usual Euclidean ones; see [29] for more. The prototype of (3.2) is clearly the non-homogeneous p-Laplacean equation with VMO-coefficients, that is

$$\text{div}_H\left(b(x)|\mathfrak{X}u|^{p-2}\mathfrak{X}u\right) = \text{div}_H\left(|F|^{p-2}F\right)\,, \tag{3.4}$$

where the coefficient function $b(\,\cdot\,)$ satisfies (3.3), and $F \in L^p(\Omega, \mathbb{R}^{2n})$. The main result is the following:

THEOREM 3.1. – *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation* (3.2) *under the assumptions* (1.2)-(1.3) *with $2 \le p < 4$, and* (3.3). *Then*

$$F \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}) \qquad \textit{implies that} \qquad \mathfrak{X}u \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}),$$

whenever $p < q < \infty$. Moreover there exists a constant c, depending only on $n, p, L/\nu, q$, and the function $b(\cdot)$, such that the following reverse-Hölder type inequality holds for any CC-ball $B_R \Subset \Omega$:

$$\left(\fint_{B_{R/2}} |\mathfrak{X}u|^q \, dx \right)^{1/q} \le c \left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p \, dx \right)^{1/p} + c \left(\fint_{B_R} |F|^q \, dx \right)^{1/q}. \tag{3.5}$$

Let us just recall that in the Euclidean case there is a wide literature on Calderón-Zygmund type estimates for linear problems with VMO-coefficients starting from the Euclidean work of Chiarenza & Frasca & Longo [9]. In the subelliptic setting, the theory is confined to the linear case [3], where equations involving general Hörmander vector fields are considered. We also remark that the integrability results obtained in [29] are new already in the case $b(x) \equiv 1$ - that is, when no coefficients are actually involved. Moreover, the result of Theorem 3.1 extends to a family of more general equations with continuous coefficients, and to equations with discontinuous BMO coefficients satisfying a small BMO-seminorm condition; the corresponding statements are again presented in [29].

4. – The technical approach - novelties.

The approach developed in [29] strongly differs from those proposed in earlier papers on the subject. Hörmander's original strategy [19], and subsequent linear works [17, 18], prescribe to first obtain a certain maximal regularity for the vertical gradient Tu, and then, using such an additional information, obtaining regularity results for the horizontal part $\mathfrak{X}u$. Such an approach also works for non-linear problems when $p = 2$, as shown in [5], but does not seem to yield results when $p \neq 2$ and the equation becomes in a certain sense heavily non-linear. In [29] we instead introduce a double-bootstrap method: we shall obtain regularity for Tu using the one obtained for $\mathfrak{X}u$, and vice-versa. More precisely we shall prove that

$$Tu \in L^{q_k}_{\mathrm{loc}} \Longrightarrow \mathfrak{X}u \in L^{p_k}_{\mathrm{loc}} \quad \text{and} \quad Xu \in L^{p_k}_{\mathrm{loc}} \Longrightarrow Tu \in L^{q_{k+1}}_{\mathrm{loc}} \tag{4.1}$$

where $\{p_k\}$ and $\{q_k\}$ are two sequences diverging to infinity; in some sense we repeat Hörmander's original strategy breaking it in a countable number of steps. As a first consequence we obtain that

$$\mathfrak{X}u, Tu \in L^q_{\mathrm{loc}} \qquad \text{for every} \quad q < \infty . \tag{4.2}$$

The use of such a mixed iteration is a direct consequence of the non-linearity of the problem (1.1), since, after a preliminary differentiation of the equation, Tu cannot be realized as a solution of a linear equation with bounded coefficients, and a deeper interaction between the horizontal and the vertical parts of the gradient must be exploited. The implementation of (4.1) requires a rather delicate interaction between: suitable Caccioppoli type estimates - also called energy estimates - for the horizontal and vertical gradients; interpolation inequalities of Gagliardo-Nirenberg type in the Heisenberg group; integration-by-parts methods; a certain kind of non-standard energy estimates of mixed type. More precisely, we shall replace the usual Moser's iteration by a different kind of iteration where, at each step, the gain in the integrability exponent is not achieved by Sobolev inequality, but, rather, by an interpolation estimate of Gagliardo-Nirenberg type via an integration-by-parts procedure.

Once the integrability information in (4.2) is gained, a suitable variant of the standard Moser's iteration technique will lead to $\mathfrak{X}u \in L^{\infty}_{\mathrm{loc}}$. Finally, in the non-degenerate case $\mu > 0$ this will lead to $Tu \in L^{\infty}_{\mathrm{loc}}$ via the results in [26], and eventually to the local Hölder continuity of the Euclidean gradient, which is a standard implication after the work in [5, 26].

An important background of our technique is the observation of the natural analogy between sub-elliptic equations of the type (1.1), and the more classical Euclidean non-uniformly elliptic equations, or “equations with non-standard growth conditions”, or with “(p, q)-growth conditions”, as very often called in the setting of the Calculus of Variations [2, 14, 15, 27]. Problems with non-standard growth indeed involve equations featuring ellipticity properties which appear to be weaker in certain special spatial directions: this immediately reminds of the situation of horizontal quasi-linear equations in the Heisenberg group as (1.1), where the vertical derivative Tu does not appear directly in the operator. It rather appears only in an intrinsic way, via the horizontal vector fields $\mathfrak{X}u$ and after commutation, and therefore the vertical direction is clearly playing a very special role. Such a lack of “vertical ellipticity” is in fact the basic source of problems in the theory of elliptic equations in the Heisenberg group.

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Received October 29, 2007 and in revised form November 23, 2007