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On the Qualitative Behavior of the Solutions to the 2-D Navier-Stokes Equation(*).

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Abstract. – *This talk, based on a research in collaboration with E. Caglioti and F. Rousset, deals with a modified version of the two-dimensional Navier-Stokes equation wich preserves energy and momentum of inertia. Such a new equation is motivated by the occurrence of different dissipation time scales. It is also related to the gradient flow structure of the 2-D Navier-Stokes equation. The hope is to understand intermediate asymptotics.*

1. – Introduction.

Let us consider the Navier-Stokes equation in the plane for the vorticity $\omega = \omega(x, t)$:

$$(\partial_t + u \cdot \nabla)\omega(x, t) = \nu \Delta \omega(x, t). \tag{1.1}$$

Here $x \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and $u = u(x, t) \in \mathbb{R}^2$ is the velocity field defined as:

$$u = \nabla^\perp \psi, \quad \psi = -\Delta^{-1}\omega. \tag{1.2}$$

Explicitely:

$$u = K * \omega, \qquad K(x) = \nabla^\perp g(x) = -\frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \tag{1.3}$$

where

$$g(x) = -\frac{1}{2\pi}\log|x| \tag{1.4}$$

is the fundamental solution for the Poisson equation in the plane. Finally $\nu > 0$ is the viscosity coefficient.

The time asymptotics for the Navier-Stokes evolution is trivial, namely $\omega \to 0$ when $t \to \infty$ in all L^p norms with $p > 1$. On the other hand the occurrence of

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coherent structures an a suitable time scales suggests to look for an intermediate asymptotics.

An attempt to understand those typical profiles has been given in terms of the Statistical Mechanics of the vortex system, which is an Hamiltonian finite dimensional version of the Euler flow (see e.g. [MP]).

Let us remind the Euler equation. It is as the Navier-Stokes equation for $\nu = 0$:

$$(\partial_t + u \cdot \nabla)\omega(x, t) = 0. \tag{1.5}$$

It has many invariants:

$$E(\omega) = \frac{1}{2}\int \psi\omega dx, \qquad \text{(energy)} \tag{1.6}$$

$$M(\omega) = \int x\omega dx, \qquad \text{(center of vorticity)} \tag{1.7}$$

$$I(\omega) = \frac{1}{2} \int (x - M)^2 \omega dx, \qquad \text{(momentum of inertia)} \tag{1.8}$$

$$F_\phi(\omega) = \int \phi(\omega) dx \qquad \text{(Casimirs).} \tag{1.9}$$

From now on we set $\nu = 1$ in eq.n (1.1) and $M = 0$ costantly in time for both equations (1.1) and (1.5). We also restrict our analysis to vorticity profiles which are also probability density i.e. $\omega \geq 0$ and $\int \omega = 1$.

Following the Statistical Mechanical prescriptions, we select a special class of stationary solutions to the Euler equation, characterized as the solutions of the following nonlinear elliptic problem:

$$\omega = -\Delta\psi = \frac{e^{b\psi + a\frac{x^2}{2}}}{Z} \tag{1.10}$$

where

$$Z = \int e^{b\psi + a\frac{x^2}{2}} \tag{1.11}$$

is a normalization. Here a and b are suitable parameters.

For fixed values of $a < 0$ and b in a suitable range, eq.n (1.10), called Mean Field Equation, has been obtained as the Mean-Field limit of the Gibbs measures of the point vortex system (see [O], [LP] and [MJ]). For a mathematical study see [CLMP1], [CLMP2], [K], [KL] and the Appendix of [CPR 1].

Solutions to eq.n (1.10) are good candidates to explain the coherent structures observed in the Navier-Stokes evolution and they should be explained in terms of that flow rather than in terms of statistical mechanical principles. A naive attempt is to look at the dissipation rates of the invariants (1.6), (1.8) and (1.9) for the Navier-Stokes evolution. They are:

$$\dot{E} = -\int \omega^2, \tag{1.12}$$

$$\dot{I}(\omega) = 2, \tag{1.13}$$

$$\dot{F}_\phi = -\int \phi''(\omega)|\nabla\omega|^2. \tag{1.14}$$

The idea is that $-\dot{F}_\phi$, for a convex ϕ, may be much larger than $-\dot{E}$ and $\dot{I}$ in many situations.

This would suggest to consider, in the first approximation, E and I as constants, by looking at a master equation which modifies the Navier-Stokes equation leaving constant both energy and moment of inertia, but retaining all the other features of the Navier-Stokes dynamics. Unfortunately such a procedure is ambiguous as we shall see in a moment. For instance let us consider the heat equation:

$$\partial_t\omega = \Delta\omega, \tag{1.15}$$

with the aim of leaving invariant the moment of inertia I. One way is to consider simply

$$\partial_t\omega = \partial^2_{\theta,\theta}\omega. \tag{1.16}$$

However eq.n (1.16) is not so good because it has too many invariants and does not select a unique invariant state. On the other hand the theory of gradient flows produces other more interesting equations. Indeed the heat equation can be written in the form

$$\partial_t\omega = -\nabla^g F \equiv \operatorname{div}\left[g(\omega)\nabla\frac{\delta F}{\delta\omega}\right] \tag{1.17}$$

where F is of the form (1.9) and $g = g(\omega) = \dfrac{1}{\phi''(\omega)}$ is a suitable weight function. Such a formulation is connected with the metric on the space of the absolutely continuous probability measures with respect to which the gradient ∇^g is computed. See [V] and [AGS] for details.

If we want to constrain eq.n (1.17) to have a constant I, we just project it on the manifold I=const. Since

$$\nabla^g I = \operatorname{div}\left(\frac{x}{\phi''(\omega)}\right)$$

we readly arrive to

$$\partial_t \omega = \operatorname{div}\left[\nabla\omega + a\left(\frac{x}{\phi''(\omega)}\right)\right], \tag{1.18}$$

where a is a multiplicator choosen in such a way that I is mantained constant. This implies

$$a = \frac{2}{\int \frac{x^2}{\phi''(\omega)}}. \tag{1.19}$$

The asymptotic state for the nonlinear evolution (1.18) is expected to be the minimizer of F at I constant. Of course such a state depends crucially of F which is arbitrary at this level.

A special choice is when $\phi(r) = r \log r$, namely F is the Entropy functional. In this case $a = \frac{1}{I}$ is constant (I is the moment of inertia of the initial condition) and the resulting equation is

$$\partial_t \omega = \operatorname{div}\left[\nabla\omega + \frac{1}{I}(x\omega)\right]. \tag{1.20}$$

We also note that eq.n (1.20) can be recovered by the classical change of variables:

$$x \to \frac{x}{\sqrt{1+t}}; \qquad t \to \log(1+t),$$

rescalig suitably ω. The above change of variables is suggested by the fact that the exact solution

$$\omega(x,t) = \frac{1}{4\pi(1+t)} e^{-\frac{x^2}{4(1+t)}}$$

is made constant.

We finally remark that the above choice of the Entropy functional, for which $g(\omega) = \omega$, induces as metric function the Wasserstein distance among the probability measures. The corresponding gradient is denoted as $\nabla^\omega = \nabla^W$.

Coming back to the Navier-Stokes equation, following [V] we write eq.n (1.1) (for $\nu = 1$) as

$$\begin{aligned} \partial_t \omega &= J\nabla^W E - \nabla^W S \\ &= -\operatorname{div}\left[\omega\nabla^\perp \frac{\delta E}{\delta\omega}\right] + \operatorname{div}\left[\omega\nabla \frac{\delta S}{\delta\omega}\right], \end{aligned} \tag{1.21}$$

where $S(\omega) = \int \omega \log \omega$ is the Entropy functional. Moreover the antigradient is defined as

$$J\nabla^g = -\operatorname{div}\left[g(\omega)\nabla^\perp\frac{\delta}{\delta\omega}\right].$$

We note that, if we want that the (conservative) Euler part is an antigradient of the Energy functional, necessarily $g(\omega) = \omega$ and hence the Entropy functional is authomatically selected if we want to express the conservative and dissipative part of the equation in terms of the same metric.

Starting from eq.n (1.21), in order that the functional E and I are separately invariant, we project the dissipation part on the manifold $E =$ const and $I =$ const. This is equivalent to write

$$\partial_t\omega = -\operatorname{div}\left[\omega\nabla^\perp\frac{\delta E}{\delta\omega}\right] + \operatorname{div}\left[\omega\nabla\left(\frac{\delta S}{\delta\omega} - a\frac{\delta I}{\delta\omega} - b\frac{\delta E}{\delta\omega}\right)\right], \tag{1.22}$$

determining the multipliers a and b to guarantee the constance of E and I.

We arrive to:

$$\begin{aligned}\partial_t\omega + u\cdot\nabla\omega &= \operatorname{div}(\nabla\omega - b\omega\nabla\psi - a\omega x)\\ &= \operatorname{div}\left[\omega\nabla\left(\log\omega - b\psi - a\frac{x^2}{2}\right)\right],\end{aligned} \tag{1.23}$$

where

$$b = \frac{2I\int\omega^2 + 2V}{2I\int\omega|\nabla\psi|^2 - V^2}; \quad a = -\frac{2\int\omega|\nabla\psi|^2 + V\int\omega^2}{2I\int\omega|\nabla\psi|^2 - V^2}, \tag{1.24}$$

and

$$V = \int\omega x\cdot\nabla\psi = \int dx\int dy\,\omega(x)\omega(y)x\cdot\nabla g(x-y) = -\frac{1}{4\pi}. \tag{1.25}$$

Eq.n (1.23) has been introduced in [CPR 1].

Let us now argue on the asymptotic behavior of eq.n (1.23). The invariant states satisfy

$$\operatorname{div}\left[\omega\nabla\left(\log\omega - b\psi - a\frac{x^2}{2}\right)\right] = 0.$$

Therefore

$$\omega = \frac{e^{b\psi + a\frac{x^2}{2}}}{Z} \tag{1.26}$$

where

$$Z = \int e^{b\psi + a\frac{x^2}{2}} \tag{1.27}$$

is a normalization.

Recalling that $\omega = -\Delta\psi$, we realize that eq.n (1.26) is a nonlinear elliptic equation which is, however, intractable at this stage, if a and b are functionals of the solution itself (see eq.n (1.24). Fortunately the parameters a and b are uniquely determined by the values of the energy and momentum of inertia as we shall see in a moment.

We first note that, for fixed values of a and b, eq.n (1.26) is the same as eq.n (1.10) introduced above. Here we summarize the main mathematical results concerning this equation. Before doing this we first introduce a useful functional, the free-energy, defined as

$$F_{(b,a)}(\omega) = S(\omega) - bE(\omega) - aI(\omega) \tag{1.28}$$

for a given pair $a < 0$ and $b > 0$. $F_{(b,a)}$ is defined on the space Γ of all the probability densities on $\mathbb{R}^2$ with finite entropy, energy and moment of inertia. We define

$$F(b,a) = \inf_{\omega \in \Gamma} F_{(b,a)}(\omega). \tag{1.29}$$

Moreover, for $E \in \mathbb{R}$ and $I > 0$ let us introduce

$$\Gamma_{(E,I)} = \{\omega \in \Gamma | E(\omega) = E, I(\omega) = I\} \tag{1.30}$$

and set

$$S(E,I) = \inf_{\omega \in \Gamma_{(E,I)}} S(\omega). \tag{1.31}$$

The above variational problems are related to the solutions of the Mean-Field equation (1.26).

THEOREM 1. – *For $a < 0$ and $0 < b < 8\pi$:*

i) *There exists a unique, radially symmetric minimizer $\omega \in \Gamma$ of the problem* (1.29). *$\omega = \omega_{(b,a)}$ is the unique radially symmetric solution to eq.n* (1.26).

ii) *When $b \to 8\pi$, ω converges (weakly) to a δ at the origin.*

iii) *$F(b,a)$ is a concave smooth function and*

$$\frac{\partial F}{\partial a} = I(\omega_{(b,a)}), \qquad \frac{\partial F}{\partial b} = -E(\omega_{(b,a)})$$

iv) *For $E \in \mathbb{R}$ and $I > 0$ define*

$$S^*(I,E) = \sup_{a,b} (F(b,a) + bE + aI)$$

and denote by $b(I,E)$ and $a(I,E)$ the unique maximizers. Then $S(I,E) = S^*(I,E)$ and hence S is a smooth convex function.

v) *The variational problem* (1.31) *has a unique minimizer $\omega(I,E)$ and*

$$\omega(I,E) = \omega_{(b(I,E),a(I,E))}.$$

Note that when $b \leq 0$ the theory is easier. Indeed the functional $F_{(b,a)}(\omega)$ is convex so the minimazion problem is standard and eq.n (1.16) has a unique (radial) solution [GL] .

Note also that eq.n (1.26) has a natural statistical mechanical interpretation, being its solutions, Gibbs states with a self-consistent interaction. Therefore $-b$ is an inverse temperature. Hence $b > 0$ implies negative temperature states, as predicted by Onsager [O] in terms of point vortex theory.

Now we want to study the solutions to the initial value problem associated to eq.n (1.23). The first difficulty one faces in dealing with this equation is the fact that the denominator entering in the definition of a and b

$$D = 2I\int \omega|\nabla\psi|^2 - V^2, \tag{1.32}$$

can vanish. We first remark that, by the Chauchy-Schwarz inequality

$$V^2 = \left(\int \omega x \cdot \nabla\psi\right)^2 \leq 2I\int \omega|\nabla\psi|^2 \tag{1.33}$$

and hence D is positive if inequality (1.33) holds strictly. Moreover $b > 0$ if $\int \omega^2 > \dfrac{1}{4\pi I}$ and $a < 0$ if $2\int \omega|\nabla\psi|^2 > 1/4\pi \int \omega^2$.

Note that D vanishes when x and $\nabla\psi$ are collinear in $L^2(\omega dx)$ and this happens for the one-dimensional family of circular vortex patches:

$$\omega = \frac{1}{\pi R^2}\chi_{B(0,R)} \tag{1.34}$$

where $\chi_{B(0,R)}$ is the characteristic function of $B(0,R)$, the disk of center 0 and radius R. Then:

$$\omega\nabla\psi = -\frac{\omega}{2\pi R^2} \tag{1.35}$$

and the right hand side of eq.n (1.23) does not make sense.

To avoid this difficulty we studied in [CPR 1] the simpler problem of the Navier-Stokes equation constrained to the single manifold $E \pm I =$const. In this framework we proved the existence and uniqueness of smooth solutions for small initial data or data close enough to the equilibrium. However the techniques developed in [CPR1] allows us to prove the existence of solutions of the original problem (1.23) in the vicinity of the equilibria, characterized by the solutions of the Mean-Field equation (1.26). In this way we avoid the solution to be close to the singular set (1.31).

The main feature of eq.n (1.23) is the decreasing of the entropy functional. Infact:

$$
\begin{aligned}
(1.36)\qquad \frac{dS(\omega)}{dt} &= \frac{dS(\omega)}{dt} - b\frac{dE(\omega)}{dt} - a\frac{dI(\omega)}{dt}\\
&= \int \left(\frac{\delta S}{\delta\omega} - b\frac{\delta E}{\delta\omega} - a\frac{\delta I}{\delta\omega}\right) \operatorname{div}\left[\omega\nabla\left(\frac{\delta S}{\delta\omega} - a\frac{\delta I}{\delta\omega} - b\frac{\delta E}{\delta\omega}\right)\right]\\
&= -\int \omega\left|\nabla\left(\frac{\delta S}{\delta\omega} - a\frac{\delta I}{\delta\omega} - b\frac{\delta E}{\delta\omega}\right)\right|^2 = -\int \omega\left|\nabla\left(\log\omega - b\psi - a\frac{x^2}{2}\right)\right|^2.
\end{aligned}
$$

The decreasing of the entropy will play a crucial role in the proof of the existence of the solutions and in the control of the asymptotic behavior.

The mathematical study of eq.n (1.23) starts with the study of the existence problem briefly discussed in the next section.

We discuss only the more interesting (both from a physical and technical point of view) negative temperature case. The positive temperature case follows easily from the same arguments.

2. – Construction of the solutions and asymptotics.

To approach the initial value problem to eq.n (1.23), we limit ourselves to initial data close enough to a radial solution of (1.26).

Let us introduce the manifold of probability densities

$$
(2.1)\qquad \mathcal{M}(E, I) = \{\omega | E(\omega) = E, I(\omega) = I\}
$$

and fix the (radial) solution to eq.n (1.26) in $\mathcal{M}(E, I)$, denoted, in the sequel by ω_{MF}. Then we have:

THEOREM 2 ([CPR 1], [CPR2]). – *Let $\omega_0 \in L^p$, for some $p > 2$, be a probability distribution in $\mathcal{M}(E, I)$. Then there exists $\delta = \delta(\omega_{MF})$ depending on ω_{MF}, such that, if*

$$
(2.2)\qquad S(\omega_0) - S(\omega_{MF}) \le \delta,
$$

there exists a unique classical solution to eq.n (1.13), *with initial datum ω_0, for which*

$$
(2.3)\qquad \sup_{t>0} \|\omega(t)\|_{L^p} \le C(\omega_0).
$$

Moreover

$$\lim_{t\to\infty} \omega(t) = \omega_{MF} \tag{2.4}$$

in L^1.

The proof of Th.m 2 is based on the following steps.

Step 1. For all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\omega \in \mathcal{M}(E, I)$ and

$$S(\omega) - S(\omega_{MF}) \leq \delta, \tag{2.5}$$

then

$$\|\omega - \omega_{MF}\|_{L^1} \leq \varepsilon. \tag{2.6}$$

Step 2. If $\omega \in L^p$ with p sufficiently large, then

$$|b(\omega) - b(\omega_{MF})| + |a(\omega) - a(\omega_{MF})| \leq C\|\omega - \omega_{MF}\|_{L^1}^{a}, \tag{2.7}$$

for some $a < 1$.

Step 3. For any smooth solution $[0, T) \to \omega(t)$ to (1.23), as far as $b(\omega(t)) + |a(\omega(t))| \leq C$ then

$$\sup_{t>0} \|\omega(t)\|_{L^p} \leq C(\omega_0), \tag{2.8}$$

where $C(\omega_0)$ depends on ω_0 , but not on T.

Step 4. We conclude the existence part in the following way. Let $[0, T) \to \omega(t)$ be a local solution up to the time T for which, for $t < T$:

$$b(\omega(t)) < 2b(\omega_{MF}); \qquad |a(\omega(t))| < 2|a(\omega_{MF})|. \tag{2.9}$$

Note that (2.9) holds initially because we assume ω_0 close to ω_{MF} in the sense of the entropies and hence in the L^1 sense (Step 1). Suppose that at time T (2.9) is violated. By the decreasing of the entropy we have that (Step 1) $\|\omega_0 - \omega_{MF}\|_{L^1}$ remains small. Step 3 guarantees that $|b(\omega) - b(\omega_{MF})| + |a(\omega) - a(\omega_{MF})|$ is also small. Hence (2.9) cannot be violated and $T = +\infty$.

Step 1 is based on the Csiszar-Kullback inequality. Step 2 follows from explicit computations. Step 3 is obtained by a direct bound on the time derivative of the L^p norm of the solution and the GNS inequality, in the same spirit of [JL] and [BDP] for the Keller-Segel model. (See [CPR1] for details).

The asymptotic behavior follows by some regularity estimates, yielding L^1 compacteness, the entropy identity (1.36) and the uniqueness of the free-energy minimizer.

For details and other connected results [CPR 1] and [CPR 2].

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