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Generalizations of Sequential Lower Semicontinuity

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Dedicated to the memory of Guido Stampacchia

Abstract. – *In* [7] *W.A. Kirk and L.M. Saliga and in* [3] *Y. Chen, Y.J. Cho and L. Yang introduced lower semicontinuity from above, a generalization of sequential lower semicontinuity, and they showed that well-known results, such as some sufficient conditions for the existence of minima, Ekeland's variational principle and Caristi's fixed point theorem, remain still true under lower semicontinuity from above. In the second of the above papers the authors also conjectured that, for convex functions on normed spaces, lower semicontinuity from above is equivalent to weak lower semicontinuity from above. In the present paper we exhibit an example showing that such conjecture is false; moreover we introduce and study a new concept, that generalizes lower semicontinuity from above and consequently also sequential lower semicontinuity; moreover we show that:* (1) *such concept, for convex functions on normed spaces, is equivalent to its weak counterpart,* (2) *the above quoted results of* [3] *regarding sufficient conditions for minima remain still true for such a generalization,* (3) *the hypothesis of lower semicontinuity can be replaced by this generalization also in some results regarding well-posedness of minimum problems.*

1. – Introduction.

In [3] Y. Chen, Y.J. Cho and L. Yang noted that sequential lower semicontinuity, although important, is not essential for solving some minimization problems and proposed the following generalization ([3], Definition 1.2 and Definition 1.5).

DEFINITION 1.1. – *Let* (X,τ) *be a topological space. Let* $x \in X$. *A function* $f\colon X \to [-\infty,+\infty]$ *is said to be sequentially lower semicontinuous from above at* x *if* $f(x) \leq \lim_{n\to+\infty} f(x_n)$ *for every sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of* X *such that* $x_n \to x$ *and* $(f(x_n))_{n\in\mathbb{N}}$ *is a weakly decreasing sequence. Moreover* f *is said to be sequentially lower semicontinuous from above if it is sequentially lower semicontinuous from above at* x *for every* $x \in X$.

Actually the same definition was previously considered by W.A. Kirk and L.M. Saliga in [7] (Section 2, definition above Theorem 2.1).

Both in [7] and in [3] this concept is called *lower semicontinuity from above*, in [3] it is used only when X is metrizable or X is a normed space endowed with its weak topology, in [7] in a lightly different situation (see also [7], Section 2, definition above Remark 1); furthermore also J.M. Borwein and Q.J. Zhu in [1] (Exercise 2.1.4) used the same concept, naming it *partial lower semicontinuity*; here we are calling it *sequential lower semicontinuity from above*, as it is a generalization of *sequential lower semicontinuity*.

Moreover the authors of [3] conjectured that, for convex functions on normed spaces, sequential lower semicontinuity from above is equivalent to weak sequential lower semicontinuity from above (see [3], some rows below Definition 1.5).

Here we exhibit an example showing that such conjecture is false (Example 3.1 (see also Examples sketched in Remarks 3.1)). Then we define a new concept (Definitions 4.1), called by us *inf-sequential lower semicontinuity*, that generalizes sequential lower semicontinuity from above and we show that:

(a) for convex functions on normed spaces, such concept is equivalent to its weak counterpart (Theorem 4.1),

(b) some results listed in [3], such as sufficient conditions for the existence of minima remain still true under this generalization (Section 5).

Moreover, with Theorems 5.5, 5.7, 5.8 and 5.9, we wish to give some examples in which the hypothesis of lower semicontinuity can be weakened by means of the one of inf-sequential lower semicontinuity when we deal with well-posedness in the sense of Tykhonov or in the generalized sense.

We also give another example (Example 4.1) which shows that, for not convex functions on Hilbert spaces, the stronger condition of lower semicontinuity with respect to the topology induced by the norm is not sufficient to get inf-sequential lower semicontinuity with respect to the weak topology.

Moreover in Section 3, besides the above-cited Example 3.1 (and those included in Remarks 3.1), we supply further examples and results regarding the condition of sequential lower semicontinuity from above; furthermore in Section 4 we also give some examples and results in which we compare sequential lower semicontinuity from above with inf-sequential lower semicontinuity, both in the general case and in the case of convex functions. Further examples will be given in [2].

2. – Notations and Preliminaries

NOTATIONS. – In the sequel, unless otherwise specified, all linear spaces will be considered on the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. As convention, in $[-\infty, +\infty]$, $\inf \emptyset = +\infty$ and the product $0 \cdot (+\infty)$ is considered equal to 0. By $\mathbb{N}$ we denote the set of natural numbers (0 included), while $\mathbb{Z}_+ := \{n \in \mathbb{Z} : n > 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$; $\delta_{n,m}$ is the Kronecker symbol. If Z is a metric space, $A \subseteq Z$, let $\alpha(A) := \inf\{\delta > 0: A$ has a finite cover consisting of sets with diameter $< \delta\}$ be the *Kuratowski measure of noncompactness of* A. If Z is a linear space on $\mathbb{R}$ or on $\mathbb{C}$, let $\dim Z$ denote the dimension of Z, and, if $A \subseteq Z$, let $\operatorname{sp} A$ and $\operatorname{co} A$ denote respectively the linear subspace of Z that is generated by A and the convex hull of A; if $x, y \in Z$ let $[x, y] := \{\lambda x + (1-\lambda)y: \lambda \in [0,1]\}$ and, if $x \neq y$, let $]x, y] := \{\lambda x + (1-\lambda)y: \lambda \in [0,1[\}$. If Z is a topological linear space, let Z' denote the continuous dual of Z; if $A \subseteq Z$, let $\overline{\operatorname{co}} A$ be the closure of $\operatorname{co} A$. If Z is a normed space, then $|z|_Z$ indicates the norm in Z of an element $z \in Z$ and $S_Z(a, r) := \{z \in Z: |z - a|_Z < r\}$ $(a \in Z,\ r \in \mathbb{R}_+)$. Let ℓ^2, c_0 and $\mathcal{C}^0([0,1], \mathbb{F})$ respectively denote the real, or complex, Banach spaces of the sequences whose squares of moduli of coordinates are summable, of the infinitesimal sequences, and of the continuous functions on $[0,1]$. If Z is a Hilbert space, then $\langle z, w\rangle_Z$ indicates the scalar product in Z between two elements $z, w \in Z$. If A and B are sets, if $C \subseteq A$ and $f: A \to B$ is a function, then $f_{|C}$ means the restriction of f to C; if $g: A \to [-\infty, +\infty]$ is a function, then $\operatorname{epi} g := \{(x, \gamma) \in A \times \mathbb{R}: g(x) \leq \gamma\}$ and $\operatorname{dom} g := \{x \in A: g(x) < +\infty\}$ denote respectively the epigraph and the effective domain of g; moreover let $\arg\min(A, g) := \{x \in A: g(x) = \inf_A g\}$. If Z is a topological space and if $A \subseteq Z$, let ∂A be the boundary of A. Moreover let $\mathcal{B}([0,1])$ indicate the Borel σ-algebra on $[0,1]$ with respect to the euclidean topology and $|A|$ is the Lebesgue measure of a set A. If $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of elements of $[-\infty, +\infty]$ and if $\ell \in [-\infty, +\infty]$, then $\tau_n \searrow \ell$ (respectively $\tau_n \nearrow \ell$) means that $(\tau_n)_{n\in\mathbb{N}}$ is a weakly decreasing (respectively increasing) sequence with $\lim\limits_{n\to+\infty} \tau_n = \ell$. Henceforth we shall shorten both lower semicontinuous and lower semicontinuity in "*lsc*", both sequentially lower semicontinuous and sequentially lower semicontinuity in "*slsc*", both sequentially lower semicontinuous from above and sequential lower semicontinuity from above in "*d-slsc*".

DEFINITION 2.1. – *Let X be a linear space on $\mathbb{F}$, $A \subseteq X$, $y \in A$. Then (in accordance with* [6]):

(*a*) *the point y is said to be an internal point of A if for every $x \in X$ there exists a $\gamma_x \in \mathbb{R}_+$ such that $y + \lambda x \in A$ for all $\lambda \in [0, \gamma_x]$;*

(*b*) *the point y is said to be an extreme point of A if $x, z \in A$, $\lambda \in]0,1[$ for which $y = \lambda x + (1-\lambda)z$ implies $x = z = y$;*

(*c*) *a normed space Y is said to be strictly convex if every point $y \in \partial S_Y(0,1)$ is an extreme point of $\overline{S_Y(0,1)}$.*

THEOREM 2.1. – *Let Y be a normed space, C a convex subset of $\overline{S_Y(0,1)}$, $B \subseteq C \cap \partial S_Y(0,1)$ such that*

$$C \cap \partial S_Y(0,1) \cap \operatorname{co} B = B. \tag{2.1}$$

Then $(C \setminus \partial S_Y(0,1)) \cup B$ is a convex set.

PROOF. – Let $y,\, z \in (C \setminus \partial S_Y(0,1)) \cup B$ and let $\lambda \in]0,1[$. Then $\lambda y + (1-\lambda)\cdot z \in C$, as C is convex. Now, if $\lambda y + (1-\lambda)z \in S_Y(0,1)$, we have $\lambda y + (1-\lambda)z \in C \cap S_Y(0,1) = C \setminus \partial S_Y(0,1) \subseteq (C \setminus \partial S_Y(0,1)) \cup B$; besides, if at least one of the points y and z belongs to $C \setminus \partial S_Y(0,1)$, it holds $|\lambda y + (1-\lambda)z|_Y \le \lambda |y|_Y + (1-\lambda)|z|_Y < 1$; so it remains to examine the case in which $y,\, z \in B$ and $\lambda y + (1-\lambda)z \in \partial S_Y(0,1)$: in such case $\lambda y + (1-\lambda)z \in C \cap \partial S_Y(0,1) \cap \operatorname{co} B = B$ and hence we can conclude.

Also if we shall not use it, here we note a simple corollary of previous theorem.

COROLLARY 2.1. – *Let Y be a strictly convex normed space, C a convex subset of $\overline{S_Y(0,1)}$, $A \subseteq \partial S_Y(0,1)$. Then $C \setminus A$ is a convex set.*

PROOF. – Since Y is strictly convex, if $B \subseteq C \cap \partial S_Y(0,1)$, we have $\operatorname{co} B \subseteq B \cup S_Y(0,1)$, hence $C \cap \partial S_Y(0,1) \cap \operatorname{co} B = C \cap \partial S_Y(0,1) \cap B = B$ and hence (2.1) is verified; therefore, applying Theorem 2.1 to $B := (C \cap \partial S_Y(0,1)) \setminus A$ we obtain that $C \setminus A = (C \setminus \partial S_Y(0,1)) \cup B$ is convex.

REMARK 2.1. – Note that Theorem 2.1 can be seen also introducing the following definition of extreme sets and proving for them a simple property.

Let X be a linear space on $\mathbb{F}$, $A \subseteq X$, $S \subseteq A$. Then S is said to be an *extreme set of* A if $x,\ z \in A$, $\lambda \in]0,1[$ for which $\lambda x + (1-\lambda)z \in S$ implies that at least one point between x and z belongs to S.

Then, if A is convex and S is an extreme set of A, the set $A \setminus S$ is convex: indeed, if $x,\ z \in A \setminus S$ and $\lambda \in]0,1[$, then $\lambda x + (1-\lambda)z \in A$ for the convexity of A and $\lambda x + (1-\lambda)z \notin S$ because otherwise at least one point between x and z should be in S, that is absurd.

(The proof of the above statement is also requested in [6] (Exercise 2.4.1), but for a slightly different class of sets, therein called extreme sets too).

Therefore, for obtaining Theorem 2.1, it is enough to note that, in the hypothesis of such theorem, $C \cap \partial S_Y(0,1) \setminus B$ is an extreme set of C: indeed, if $x,\ z \in C$, $\lambda \in]0,1[$, $\lambda x + (1-\lambda)z \in C \cap \partial S_Y(0,1) \setminus B$, then $1 = |\lambda x + (1-\lambda)z|_Y \le \lambda |x|_Y + (1-\lambda)|z|_Y \le 1$; hence $|x|_Y = |z|_Y = 1$; now, if we had $x,\ z \in B$, we would get $\lambda x + (1-\lambda)z \in \operatorname{co} B \cap C \cap \partial S_Y(0,1)$; from this, using (2.1), we would obtain a contradiction.

THEOREM 2.2. – *Let Y be a real normed space, $f\colon Y \to [-\infty, +\infty]$ a convex function for which there exist $x_0 \in \operatorname{dom} f$ and $r_0 \in \mathbb{R}_+$ such that*

$$\inf_{x \in \underline{S_Y(x_0, r_0)}} f(x) > -\infty. \tag{2.2}$$

Then there exist $g \in Y'$ and $\gamma \in \mathbb{R}$ such that $f(x) > g(x) + \gamma$ for every $x \in Y$ (consequently $f(x) > -\infty$ for every $x \in Y$).

PROOF. – It is sufficient to use the proof of [3] (Theorem 1.7) that works also in the above hypotheses, observing previously that $f(x) > -\infty$ for every $x \in Y$ (really, if by absurd there exists $y_0 \in Y$ such that $f(y_0) = -\infty$, then, by the convexity of f, it holds $f(\lambda x_0 + (1-\lambda) y_0) = -\infty$ for every $\lambda \in [0,1[$, in opposition to (2.2)) and that it is enough to prove the inequality of the thesis for every $x \in \operatorname{dom} f$.

DEFINITION 2.2 ([4], at the beginning of Sections 1 and 6 of Chapter I). – *Let* (X,τ) *be a topological space and let* $f: X \to]-\infty, +\infty]$ *be a not identically* $+\infty$ *function. Then the problem* (X,f) *to minimize* f *on* X *will be said:*

(*a*) Tykhonov well-posed *if there exists one and only one point* $z \in \arg\min(X,f)$ *and if* $x_n \to z$ *for every sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of* X *such that* $f(x_n) \to \inf_X f$;

(*b*) well-posed in the generalized sense *if every sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of* X *such that* $f(x_n) \to \inf_X f$ *admits a convergent subsequence towards some element of* $\arg\min(X,f)$.

3. – Examples and some results about sequentially lower semicontinuous from above functions.

EXAMPLE 3.1. – For every infinite dimensional Hilbert space Y there exists a function $f: Y \to [0,+\infty]$ that is convex, *d-slsc* with respect to the topology induced on Y by its norm, but that is not *d-slsc* with respect to the weak topology on Y.

Let E be a complete orthonormal set for Y, let $e_n \in E$ ($n \in \mathbb{N}$) (with $e_n \neq e_m$ if $n, m \in \mathbb{N}$, $n \neq m$) and let $g: Y \to [0,+\infty[$ be defined by $g(y) = \sup\{|\langle y, w\rangle_Y| : w \in E\}$ for every $y \in Y$. For every $k \in \mathbb{Z}_+$ let $y_k = (1-\frac{1}{k})e_0 + e_k$, then

$$\begin{aligned} &\langle y_k, w\rangle_Y = 0 \text{ for every } w \in E \setminus \{e_0, e_k\},\\ &\langle y_k, e_0\rangle_Y = 1 - \frac{1}{k},\\ &\langle y_k, e_k\rangle_Y = 1, \end{aligned} \tag{3.1}$$

hence $g(y_k) = 1$. Let $C := \{y \in Y: \langle y, w\rangle_Y = 0 \text{ for every } w \in E \setminus \{e_n : n \in \mathbb{N}\}, 0 \le \langle y, e_n\rangle_Y \le 1 \text{ for every } n \in \mathbb{N}\}$ and $B := \{y_k : k \in \mathbb{Z}_+\}$; then C is a closed convex subset of Y and, if $y \in (\operatorname{co} B) \setminus B$, there exist $N \in \mathbb{Z}_+$, $\lambda_0, \dots, \lambda_N \in]0,1[$ and $k_0, \dots, k_N \in \mathbb{Z}_+$, with $k_h \neq k_j$ if $h \neq j$, such that $\sum_{h=0}^{N} \lambda_h = 1$ and $y = \sum_{h=0}^{N} \lambda_h y_{k_h}$; then from (3.1) follows that $\langle y, w\rangle_Y = 0$ for every $w \in E \setminus \{e_m : m \in \mathbb{N}\}$ and

$0 \leq \langle y, e_n \rangle_Y < 1$ for every $n \in \mathbb{N}$ and, being $\langle y_k, e_n \rangle_Y = 0$ for every $n > k$ ($k \in \mathbb{Z}_+$), it is $\langle y, e_n \rangle_Y = 0$ for every $n > \max\{k_0, \ldots, k_N\}$; hence $g(y) < 1$; consequently, if we define $B(E) = \{\varphi : E \to \mathbb{F} \text{ bounded functions}\}$ and $|\varphi|_{B(E)} = \sup\{|\varphi(w)| : w \in E\}$ (where $\mathbb{F}$ is the field on which Y is Hilbert space) and $\Phi : Y \to B(E)$ such that $\Phi(y) = (w \in E \mapsto \langle y, w \rangle_Y \in \mathbb{F})$ for every $y \in Y$, being $g(y) = |\Phi(y)|_{B(E)}$ for every $y \in Y$, we get that

$$\Phi(B) \subseteq \Phi(C) \cap \partial S_{B(E)}(0,1) \cap \operatorname{co} \Phi(B) =$$

$$= \Phi(C) \cap \partial S_{B(E)}(0,1) \cap \Phi(\operatorname{co} B) \subseteq \partial S_{B(E)}(0,1) \cap \Phi(\operatorname{co} B) \subseteq \Phi(B);$$

so $\Phi(C)$ and $\Phi(B)$ verify the condition (2.1). Applying Theorem 2.1, if $D := (C \setminus \{y \in Y : g(y) = 1\}) \cup B = \{y \in C : g(y) < 1\} \cup B$, we obtain therefore that $\Phi(D) = (\Phi(C) \setminus \partial S_{B(E)}(0,1)) \cup \Phi(B)$ is a convex set and hence, being Φ injective, also D is a convex set.

Moreover B is a closed set, because for every couple of distinct elements of B, (y_k, y_m) with $k,\ m \in \mathbb{Z}_+$, $k \neq m$, it is $|y_k - y_m|_Y > \sqrt{2}$.

Now let $f : Y \to [0, +\infty]$ be defined by $f(y) = \begin{cases} g(y) & \text{if } y \in D \\ +\infty & \text{if } y \in Y \setminus D \end{cases}$; then, by the convexity of D and the convexity of g, consequence of the linearity of Φ and of the convexity of $|\ |_{B(E)}$, f is a convex function.

First we shall verify that f is *d-slsc* with respect to the topology induced on Y by its norm. Let $z,\ z_k \in Y$ ($k \in \mathbb{N}$), $z_k \to z$ such that $(f(z_k))_{k \in \mathbb{N}}$ is a weakly decreasing sequence. It is not restrictive to suppose that $z_k \in D$ ($k \in \mathbb{N}$), hence $z \in C$; moreover it is $g(z) = \lim\limits_{k \to +\infty} g(z_k)$ because g is continuous, being Φ and $|\ |_{B(E)}$ continuous; now, if $z \in C \cap \{y \in Y : g(y) = 1\}$, then $z_k \in D \cap \{y \in Y : g(y) = 1\}$ ($k \in \mathbb{N}$) since $(g(z_k))_{k \in \mathbb{N}} = (f(z_k))_{k \in \mathbb{N}}$ is a weakly decreasing sequence; therefore $z_k \in B$ ($k \in \mathbb{N}$) and $z \in B$ because B is closed; so $z \in D$ and the thesis follows from the continuity of g.

Now we shall prove that f is not *d-slsc* with respect to the weak topology on Y. Since $(1 - \frac{1}{k})e_0 \to e_0$ and $e_k \rightharpoonup 0$, it is $y_k \rightharpoonup e_0$; furthermore $f(y_k) = 1$ for every $k \in \mathbb{Z}_+$, so $(f(y_k))_{k \in \mathbb{N}}$ is a weakly decreasing sequence, but $f(e_0) = +\infty > 1 = \lim\limits_{k \to +\infty} f(y_k)$; so f is not *d-slsc* in the point e_0 with respect to the weak topology on Y and we have concluded.

REMARK 3.1. – Making some little changes to the Example 3.1, it is possible to construct other examples of convex functions on Y that are *d-slsc* with respect to the topology induced on Y by its norm and that are not *d-slsc* with respect to the weak topology on Y. Here we sketch two of them.

(a) First of all, we define a function, enjoying the above properties, on $Y = c_0$ endowed with the "sup" norm. For every $k \in \mathbb{Z}_+$ let $y_k = (1 - \frac{1}{k})e_0 + e_k$, then

$$0 \leq (y_k)_n < (y_k)_k = 1 \text{ for every } n \in \mathbb{N} \setminus \{k\},$$

hence $|y_k|_Y = 1$. Let $C := \{x = (x_n)_{n\in\mathbb{N}} \in Y: 0 \leq x_n \leq 1 \text{ for every } n \in \mathbb{N}\}$ and $B := \{y_k: k \in \mathbb{Z}_+\}$; then C is a closed convex subset of $\overline{S_Y(0,1)}$ and, analogously to what we proved in Example 3.1, if $z \in (\operatorname{co} B) \setminus B$ then $|z|_Y < 1$ and consequently $C \cap \partial S_Y(0,1) \cap \operatorname{co} B = B$; so C and B verify the condition (2.1). Applying Theorem 2.1, we obtain therefore that $D := (C \setminus \partial S_Y(0,1)) \cup B = \{x \in C: |x|_Y < 1\} \cup B$ is a convex set. Moreover B is a closed set (it can be proved likewise in Example 3.1). Now let $f: Y \to [0,+\infty]$ be defined by $f(z) = \begin{cases} |z|_Y & \text{if } z \in D \\ +\infty & \text{if } z \in Y \setminus D \end{cases}$; then, by the convexity of D and the convexity of $|\ |_Y$, f is a convex function. The verifications that f is *d-slsc* with respect to the topology induced on Y by its norm and that it is not *d-slsc* with respect to the weak topology on Y are completely analogous to those of the corresponding conditions proved in Example 3.1, using the $|\ |_{c_0}$ instead of g.

(*b*) Now, on $Y = \mathcal{C}^0([0,1],\mathbb{R})$ endowed with the "sup" norm, we define a function, satisfying the above properties. Let $n \in \mathbb{N}$ and let $y_n \in Y$ be defined by

$$y_n(t) = \begin{cases} -4nt + 1 - \frac{1}{n+1} & \text{if } t \in [0, \frac{1}{4(n+1)}] \\ 4(n+1)t - 1 & \text{if } t \in]\frac{1}{4(n+1)}, \frac{1}{2(n+1)}] \\ -2t + 1 + \frac{1}{n+1} & \text{if } t \in]\frac{1}{2(n+1)}, \frac{1}{n+1}[\\ 1 - \frac{1}{n+1} & \text{if } t \in [\frac{1}{n+1}, 1] \end{cases}; \text{ then}$$

(3.2) $0 = y_n\left(\frac{1}{4(n+1)}\right) < y_n(t) < y_n\left(\frac{1}{2(n+1)}\right) = 1$ for every $t \in [0,1] \setminus \left\{\frac{1}{4(n+1)}, \frac{1}{2(n+1)}\right\}$,

hence $|y_n|_Y = 1$. Let $C := \overline{S_Y(0,1)}$ and $B := \{y_n: n \in \mathbb{N}\}$; then, if $y \in (\operatorname{co} B) \setminus B$, from (3.2) follows that $0 \leq y(t) < 1$ for every $t \in [0,1]$ and hence $|y|_Y < 1$ and consequently $C \cap \partial S_Y(0,1) \cap \operatorname{co} B = \partial S_Y(0,1) \cap \operatorname{co} B = B$, so C and B verify the condition (2.1). Applying Theorem 2.1, we obtain therefore that $D := S_Y(0,1) \cup B$ is a convex set. Moreover B is a closed set, as any sequence of elements of B which is uniformly convergent to $z \in \overline{B} \setminus B$ is also pointwise converging to z and therefore the function z is the constant 1, but this is not possible, because for every $n \in \mathbb{N}$ there exists a point in $[0,1]$ in which y_n assumes the value 0. Now let $f: Y \to [0,+\infty]$ be defined by $f(y) = \begin{cases} |y|_Y & \text{if } y \in D \\ +\infty & \text{if } y \in Y \setminus D \end{cases}$; then, by the convexity of D and the convexity of $|\ |_Y$, f is a convex function. The verification that f is *d-slsc* with respect to the topology induced on Y by its norm is completely similar to the analogue one of Example 3.1, using $|\ |_{\mathcal{C}^0([0,1],\mathbb{R})}$ instead of g.

Finally we shall prove that f is not *d-slsc* with respect to the weak topology on Y. Let $z \in Y$ be defined by $z(t) = 1$ for every $t \in [0,1]$. Then, if $\mu: \mathcal{B}([0,1]) \to \mathbb{R}$ is a measure, being $(y_n)_{n\in\mathbb{N}}$ pointwise converging to z and $|y_n(t)| \leq 1$ for every $t \in [0,1]$, by Lebesgue dominated convergence's theorem, we obtain that $\int y_n d\mu \to \int z d\mu$ and hence $y_n \rightharpoonup z$; furthermore $f(y_n) = 1$ for every $n \in \mathbb{N}$, so $(f(y_n))_{n\in\mathbb{N}}$ is a weakly decreasing sequence, but $f(z) = +\infty > 1 = \lim_{n\to+\infty} f(y_n)$; so

f is not *d-slsc* in the point z with respect to the weak topology on Y and we have concluded.

EXAMPLE 3.2. – In [3] (Example 1.3) an example is given of a function from $\mathbb{R}$ to $\mathbb{R}$ that is *d-slsc*, but not *lsc*. Here we shall exhibit another example of *d-slsc* function from $\mathbb{R}$ to $\mathbb{R}$ for which the set of points of *lsc* is $\mathbb{Q}$ and that does not have points of continuity; this example shows, in a certain way, how much these functions can be irregular unlike the case of *lsc* functions (see [6], Theorem 3.1.7, where it is proved that every *lsc* function from an open subset A of a complete metric space to $\mathbb{R}$ is continuous on a dense G_δ subset of A).

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(y) = \begin{cases} -1 - \frac{1}{q} & \text{if } y \in \mathbb{Q},\ y = \frac{p}{q} \text{ with } p \in \mathbb{Z},\ q \in \mathbb{Z}_+ \text{ each other prime} \\ 0 & \text{if } y \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Then if $y \in \mathbb{R}$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence of real numbers different from y converging to y such that $(f(y_n))_{n\in\mathbb{N}}$ is a converging sequence too, it follows that there are two possibilities:

(i) $y_n \in \mathbb{R} \setminus \mathbb{Q}$ definitely

(ii) $y_n \in \mathbb{Q}$ definitely;

moreover in the case (ii), if $p_n \in \mathbb{Z}$, $q_n \in \mathbb{Z}_+$ are each other prime and such that $y_n = \frac{p_n}{q_n}$ for every $n \in N := \{m \in \mathbb{N}\colon y_m \in \mathbb{Q}\}$ it must be $q_n \to +\infty$, otherwise it should exist a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers belonging to N such that $(q_{n_k})_{k\in\mathbb{N}}$ is bounded and hence constituted by elements of a finite set; from this and being $(y_{n_k})_{k\in\mathbb{N}}$ a converging sequence, analogously also $(p_{n_k})_{k\in\mathbb{N}}$ could assume only a finite number of values; but then $(\frac{p_{n_k}}{q_{n_k}})_{k\in\mathbb{N}}$ should have values in a finite set and could not converge to y.

Consequently and using densities in $\mathbb{R}$ of $\mathbb{Q}$ and of $\mathbb{R} \setminus \mathbb{Q}$, it results that f is *lsc* in y if and only if $y \in \mathbb{Q}$ and does not admit points of continuity. However f is *d-slsc*, because if $y \in \mathbb{R} \setminus \mathbb{Q}$, if $y_n \to y$ and if $(y_n)_{n\in\mathbb{N}}$ verifies (ii) then, with the above notations and as seen above, $q_n \to +\infty$ and therefore $(f(y_n))_{n\in\mathbb{N}}$ cannot be weakly decreasing.

EXAMPLE 3.3. – Here we shall show that there exist functions, also from $\mathbb{R}$ to $[0, +\infty]$, that are convex, *d-slsc*, but that are not *lsc* (clearly, on the contrary, it is not possible to obtain a similar example by means of a function from $\mathbb{R}$ to $\mathbb{R}$, because such a function is always continuous, being convex).

Let $f\colon \mathbb{R} \to [0, +\infty], f(x) = \begin{cases} x^2 & \text{if } x \in]-1, 1[\\ 2 & \text{if } x \in \{-1, 1\} \\ +\infty & \text{if } |x| > 1 \end{cases}$. Then f is convex, *d-slsc* (if $x \in \{-1, 1\}$, if $x_n \in \mathbb{R}$ ($n \in \mathbb{N}$), $x_n \to x$ and $(f(x_n))_{n\in\mathbb{N}}$ is a weakly decreasing sequence, then $|x_n| \geq 1$ for every $n \in \mathbb{N}$ and hence $2 = f(x) \leq \lim_{n\to+\infty} f(x_n)\big)$, but it is not *lsc* in the points -1 and 1.

For certain convex functions the result becomes true:

THEOREM 3.1. – *Let X be a real normed space, $x_0 \in X$, Y a vector subspace of X, $f: X \to]-\infty, +\infty]$ a d-slsc function such that $f_{|x_0+Y}: x_0 + Y \to \mathbb{R}$ is the restriction of an affine function and $f_{|X\setminus(x_0+Y)} = +\infty$. Then $f_{|x_0+Y}$ is continuous, Y is closed and hence f is lsc.*

PROOF. – Let $g: Y \to \mathbb{R}$ be a linear function and $\gamma \in \mathbb{R}$ such that $f(x) = g(x - x_0) + \gamma$ for every $x \in x_0 + Y$. First we shall prove that g is continuous (so $f_{|x_0+Y}$ is continuous too). From the *d-slsc* of f, we can deduce that $\liminf_{x\to x_0} f(x) \in]-\infty, +\infty]$, otherwise there should be $y_n \in X$ $(n \in \mathbb{N})$ such that $y_n \to x_0$ and $f(y_n) \searrow -\infty$, that is in contradiction with the *d-slsc* of f at x_0; so $\liminf_{x\to 0} g(x) \in]-\infty, +\infty]$ and therefore g has a lower bound in a suitable neighbourhood of 0 and, by its linearity, it is bounded in a suitable neighbourhood of 0 and hence it is continuous.

Now we shall prove that Y is closed. Let $\tilde{g}: \overline{Y} \to \mathbb{R}$ be the continuous extension of g to $\overline{Y}$. Let $y_n \in Y$, $y \in X$ such that $y_n \to y$. Then $y \in \overline{Y}$ and $g(y_n) = \tilde{g}(y_n) \to \tilde{g}(y)$; therefore $(g(y_n))_{n\in\mathbb{N}}$ admits a subsequence $(g(y_{n_k}))_{k\in\mathbb{N}}$ that is weakly monotone.

(*i*) If $(g(y_{n_k}))_{k\in\mathbb{N}}$ is a weakly decreasing sequence, then $(f(y_{n_k} + x_0))_{k\in\mathbb{N}}$ is a weakly decreasing sequence too; hence, since f is *d-slsc*, $f(y + x_0) \leq \liminf_{k\to+\infty} f(y_{n_k} + x_0) = \lim_{k\to+\infty} (g(y_{n_k}) + \gamma) = \tilde{g}(y) + \gamma \in \mathbb{R}$ and so we obtain $f(y + x_0) < +\infty$, that implies $y \in Y$.

(*ii*) If $(g(y_{n_k}))_{k\in\mathbb{N}}$ is a weakly increasing sequence, being $g(-y_{n_k}) = -g(y_{n_k})$ for every $k \in \mathbb{N}$, we have that $(f(-y_{n_k} + x_0))_{k\in\mathbb{N}}$ is a weakly decreasing sequence; hence, being $\tilde{g}(-y) = -\tilde{g}(y)$, $-y_{n_k} \to -y$ and $-g(y_{n_k}) \to -\tilde{g}(y)$, from the *d-slsc* of f, we have $f(-y + x_0) \leq \liminf_{k\to+\infty} f(-y_{n_k} + x_0) = \lim_{k\to+\infty} (-g(y_{n_k}) + \gamma) = -\tilde{g}(y) + \gamma \in \mathbb{R}$ and so we obtain $f(-y + x_0) < +\infty$, that implies $-y \in Y$ and therefore $y \in Y$.

Finally we can deduce that f is *lsc*: let y, $y_n \in X$ $(n \in \mathbb{N})$, $y_n \to y$; since $f_{|X\setminus(x_0+Y)} = +\infty$, without restrictions we can consider $y_n \in x_0 + Y$ for every $n \in \mathbb{N}$; then, from the closure of Y, we deduce that $y \in x_0 + Y$ and so, from the continuity of f on $x_0 + Y$, consequence of the continuity of g on Y, we obtain $f(y) = \lim_{n\to+\infty} f(y_n) = \liminf_{n\to+\infty} f(y_n)$, from which the thesis follows.

REMARK 3.2. – Another result, in which under suitable hypothesis it is true the implication "*d-slsc* $\Rightarrow$ *slsc*", is the following one.

Let X be a topological linear space and let $f: X \to [0, +\infty]$ be a function such that $f(\gamma x) = \gamma f(x)$ for every $\gamma \in [0, +\infty[$ and for every $x \in X$. Suppose that f is *d-slsc*. Then f is *slsc* too.

We shall show it in a successive paper, as a particular case of a more general result (see [2], Theorem 3.5).

LEMMA 3.1. – *Let X be a topological linear space and let $f: X \to [-\infty, +\infty]$ be a convex function. Suppose that x_n ($n \in \mathbb{N}$), $x \in X$, $\gamma \in [-\infty, +\infty[$ are such that x is an internal point of* $\operatorname{dom} f$ *(and hence $f(x) < +\infty$), $x_n \to x$, $f(x_n) \nearrow \gamma < f(x)$. Then there exists another sequence $(y_n)_{n \in \mathbb{N}}$ of elements of X such that $y_n \to x$, $f(y_n) = \gamma$ for every $n \in \mathbb{N}$.*

PROOF. – From the hypotheses it follows that $f(x_n) < +\infty$ for every $n \in \mathbb{N}$ and $f(x) \in \mathbb{R}$. Moreover, if $\gamma = -\infty$, then $f(x_n) = -\infty$ for every $n \in \mathbb{N}$ and we can conclude choosing $y_n = x_n$ for every $n \in \mathbb{N}$. On the contrary, if $\gamma \in \mathbb{R}$, then we can suppose without restrictions that also $f(x_n) \in \mathbb{R}$ for every $n \in \mathbb{N}$. So, in this case, being f a convex function, being $f(x) \in \mathbb{R}$ and being x an internal point of $\operatorname{dom} f$, for every $n \in \mathbb{N}$ it must exist a $\mu < 0$ such that $f(\mu x_n + (1 - \mu)x) \in \mathbb{R}$ and $f(z) \in \mathbb{R}$ for every $z \in [x_n, \mu x_n + (1 - \mu)x]$; then the restriction $f_{|]x_n, x]}$ is continuous and $f(x) > \gamma \geq f(x_n) \geq \lim_{\lambda \to 1^-} f(\lambda x_n + (1 - \lambda)x)$ for the convexity of f, therefore $f_{|[x_n, x]}$ assumes every value in $[\gamma, f(x)]$; hence it is enough, for every $n \in \mathbb{N}$, to choose $y_n \in [x_n, x]$ such that $f(y_n) = \gamma$: in fact, with such a choice, to show that $y_n \to x$, it suffices to note that for every $n \in \mathbb{N}$ there exists $\lambda_n \in [0, 1]$ such that $y_n = \lambda_n x + (1 - \lambda_n)x_n$, whence $y_n - x = (1 - \lambda_n)(x_n - x)$, and to take into account that $x_n \to x$ and that 0 has a neighbourhood basis constituted by balanced sets.

THEOREM 3.2. – *Let X be a topological linear space and let $f: X \to [-\infty, +\infty]$ be a convex, d-slsc function. Then f is slsc in the points of* $(\operatorname{dom} f)^\circ$.

PROOF. – Let $x_n, x \in (\operatorname{dom} f)^\circ$, $x_n \to x$ and let $\gamma = \liminf_{n \to +\infty} f(x_n)$. Then there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $f(x_{n_k}) \to \gamma$ and there exists a further subsequence $(x_{n_{k_h}})_{h \in \mathbb{N}}$ of $(x_{n_k})_{k \in \mathbb{N}}$ such that $(f(x_{n_{k_h}}))_{h \in \mathbb{N}}$ is a weakly monotone sequence. Now, if by absurd $\gamma < f(x)$, exploiting Lemma 3.1 there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that $y_n \to x$ and $f(y_n) \searrow \gamma$; so it is enough to use the *d-slsc* of f to get a contradiction.

REMARK 3.3. – Concerning Example 3.1 (and examples sketched in Remarks 3.1) it can be observed that the construction of an example satisfying the statement of those numbers is possible only by means of a function whose points of not *lsc* are all belonging to $\partial(\operatorname{dom} f)$ (where ∂ is considered with respect to the topology induced by the norm) and the same thing happens for the points in which it is not *d-slsc* with respect to the weak topology.

In fact, if (X, τ) is a topological linear space, if $f: X \to [-\infty, +\infty]$ and if $g: X \to [-\infty, +\infty]$ is defined by

$$g(x) = \min\{f(x), \liminf_{y \to x} f(y)\} \text{ for every } x \in X, \tag{3.3}$$

then, for Corollary I.2.1 of [5] (whose demonstration works also if X is a topological linear space, even on the complex field), we get that $\operatorname{epi} g = \overline{\operatorname{epi} f}$ and g is a *lsc* function; furthermore it is convex because its epigraph is a convex set, being the closure of a convex set; moreover, if (X, τ) is a locally convex topological linear space, for [5] (Corollary I.2.2, that works as well when the scalar field is the complex one), g is also *lsc* with respect to the weak topology of X.

Moreover, if f is convex and *d-slsc* (with respect to the topology τ), it happens that $\overline{\operatorname{dom} f}$ is a closed convex and therefore weakly closed set (see [11], Theorem III.6.3), whence $g(x) = f(x) (= +\infty)$ for every $x \in X \setminus \overline{\operatorname{dom} f}$ by definition of g. If in addition the topology of X satisfies the first axiom of countability, then from Theorem 3.2 it follows that f is *lsc* in the points of $(\operatorname{dom} f)^\circ$; therefore from (3.3) we get $g(x) = f(x)$ also for every $x \in (\operatorname{dom} f)^\circ$. Hence, being $g \leq f$ for (3.3), we get $f(x) = g(x) \leq \liminf_{y \to x} g(y) \leq \liminf_{y \to x} f(y)$ for every $x \in (\operatorname{dom} f)^\circ$ and where the "lim inf" can be considered indifferently with respect to topology τ or weak topology. So $f(x) = g(x)$ for every $x \in X \setminus \partial(\operatorname{dom} f)$ and f is both strongly τ-*lsc* and weakly *lsc* in every points of $X \setminus \partial(\operatorname{dom} f)$ and therefore it is also *d-slsc* with respect to the weak topology in the same points.

4. – A new concept, weaker than sequential lower semicontinuity from above.

DEFINITION 4.1. – *Let (X, τ) be a topological space. Let f be a function, $f: X \to [-\infty, +\infty]$. Then f is said to be:*

(i) inf-sequentially lower semicontinuous at $x \in X$ *("i-slsc at x") if one of the following equivalent conditions is verified (see Remarks* 4.1*):*

(a) *$(x_n)_{n \in \mathbb{N}}$ sequence of elements of X for which $x_n \to x$ and $\lim_{n \to +\infty} f(x_n) = \inf f$, implies $f(x) = \inf f$,*

(b) *$(x_n)_{n \in \mathbb{N}}$ sequence of elements of X for which $x_n \to x$ and $f(x_n) \searrow \inf f$, implies $f(x) = \inf f$;*

(ii) inf-sequentially lower semicontinuous *("i-slsc") if one of the following equivalent conditions is verified (see Remarks* 4.1*):*

(A) *condition* (a) *holds at every point of X,*

(B) *condition* (b) *holds at every point of X.*

REMARK 4.1. – With the hypotheses and notations of Definitions 4.1, here we shall prove the cited equivalences.

We have

$(a) \Rightarrow (b)$ is obvious;

$(b) \Rightarrow (a)$ because from $\lim_{n \to +\infty} f(x_n) = \inf f$ we deduce that there exists a subsequence $(f(x_{n_k}))_{k \in \mathbb{N}}$ of $(f(x_n))_{n \in \mathbb{N}}$ constant or strictly decreasing: in any case such subsequence is weakly decreasing.

Moreover

$(A) \Rightarrow (B)$ is obvious;

$(B) \Rightarrow (A)$ because $(b) \Rightarrow (a)$.

REMARK 4.2. – Let (X, τ) be a topological space. Let $x \in X$. Let f be a function, $f: X \to [-\infty, +\infty]$. We note that:

(a) if f is *d-slsc* at x, then f is *i-slsc* at x, because, if $x_n \to x$ and $f(x_n) \searrow \inf f$, then $f(x) \le \inf f$;

(b) if f is *d-slsc*, then f is *i-slsc*, for (a).

REMARK 4.3. – Note that *i-slsc* can be expressed equivalently in the following way:

f is *i-slsc* if z (sequentially) cluster point of a minimizing sequence implies f *slsc* in z.

Moreover the concept of *i-slsc* means indeed that the set of the (sequentially) cluster points of the minimizing sequences coincides with the set of minimum points, that is *i-slsc* of f can be characterized in the following way, through a property of arg min(X,f).

Let (X, τ) be a topological space. Let $f: X \to [-\infty, +\infty]$. Then f is *i-slsc* if and only if $\{\lim_{n \to +\infty} x_n : (x_n)_{n \in \mathbb{N}}$ sequence of elements of X for which $\lim_{n \to +\infty} f(x_n) = \inf f$ and there exists $\lim_{n \to +\infty} x_n\} = \arg\min(X,f)$.

In fact the inclusion $\subseteq$ is a direct consequence of (A) of Definitions 4.1, while the inclusion $\supseteq$ can be obtained considering constant sequences.

Perhaps *i-slsc* may be more easily compared with some type of well-posedness, as we shall do in Remark 5.3, but here we are denominating this concept *i-slsc* also because it can be considered also an extreme case of some definitions we shall give in [2], that are extensions of “sequential lower semicontinuity from above” and that are more in the direction of “sequential lower semicontinuity”.

EXAMPLE 4.1. – There exists a Hilbert space Y on which there is a function that is *lsc* with respect to the topology induced on Y by its norm, but that is not *i-slsc* with respect to the weak topology on Y.

The present example shows hence that, without hypothesis of convexity on the function, lower semicontinuity with respect to the topology induced by the norm does not imply with respect to the weak topology (besides, as it is well known, lower semicontinuity) the weaker (see (*b*) of Remarks 4.2) *i-slsc*.

Let $Y = \ell^2$ with the usual inner product and let C be the convex subset of Y defined by $C := \{y = (y_m)_{m\in\mathbb{N}} \in Y\colon y_m \geq 0$ for every $m \in \mathbb{N}$, there exists $M_y \in \mathbb{N}$ such that $y_m = 0$ for every $m > M_y$, $\sum_{m=0}^{M_y} y_m = 1\}$; for every $y = (y_m)_{m\in\mathbb{N}} \in C$ let $K_y := \min\{m \in \mathbb{N}\colon y_m > 0\}$ and let $N_y := \max\{m \in \mathbb{N}\colon y_m > 0\}$. Now let $f\colon Y \to [0, +\infty]$ be defined by $f(y) = \begin{cases} N_y - K_y & \text{if } y \in C \\ +\infty & \text{if } y \in Y\setminus C \end{cases}$.

First we shall verify that f is *lsc* with respect to the topology induced on Y by its norm. Let $y = (y_m)_{m\in\mathbb{N}}$, $w_n = (w_{n,m})_{m\in\mathbb{N}} \in Y$ $(n \in \mathbb{N})$, $w_n \to y$. Since $f_{|Y\setminus C} = +\infty$, without restrictions we can consider $w_n \in C$ for every $n \in \mathbb{N}$. Since $|w_{n,m} - y_m| \leq |w_n - y|_Y \xrightarrow[n\to+\infty]{} 0$ $(m \in \mathbb{N})$, it holds:

$$\lim_{n\to+\infty} w_{n,m} = y_m \text{ for every } m \in \mathbb{N},$$

therefore $y_m \geq 0$ for every $m \in \mathbb{N}$ and

(4.1) $\quad y_m > 0 \Rightarrow$ there exists $H_m \in \mathbb{N}$ such that $w_{n,m} > 0$ for every $n > H_m$;

moreover, since $w_n \to y$ also weakly, considering the inner product of $w_n - y$ with the points $z_k = (z_{k,m})_{m\in\mathbb{N}} \in Y$ such that $z_{k,m} = 1$ if $m \leq k$, $z_{k,m} = 0$ if $m > k$ $(k \in \mathbb{N})$ we obtain

$$\lim_{n\to+\infty} \sum_{m=0}^{k} w_{n,m} = \sum_{m=0}^{k} y_m \text{ for every } k \in \mathbb{N}, \tag{4.2}$$

then $\sum_{m=0}^{k} y_m \leq 1$ for every $k \in \mathbb{N}$ and hence

$$0 \leq \sum_{m\in\mathbb{N}} y_m \leq 1. \tag{4.3}$$

Now let $e_n := (\delta_{n,m})_{m\in\mathbb{N}}$ for every $n \in \mathbb{N}$; we shall distinguish four cases:

(*i*) $y \in C$,

(*ii*) $y \in \mathrm{sp}\{e_m\colon m \in \mathbb{N}\} \setminus (C \cup \{0\})$,

(*iii*) $y \in Y \setminus \mathrm{sp}\{e_m\colon m \in \mathbb{N}\}$,

(*iv*) $y = 0$.

(*i*) If $y \in C$, then from (4.1) we obtain that there exists a $\nu_y \in \mathbb{N}$ such that $N_{w_n} \geq N_y$ and $K_{w_n} \leq K_y$ for every $n > \nu_y$, so $f(y) = N_y - K_y \leq \liminf_{n\to+\infty} (N_{w_n} - K_{w_n}) = \liminf_{n\to+\infty} f(w_n)$.

(*ii*) If $y \in \mathrm{sp}\{e_m\colon m \in \mathbb{N}\} \setminus (C \cup \{0\})$, then we can define K_y and N_y as when

$y \in C$; from (4.3) it follows that $\sum_{m \in \mathbb{N}} y_m < 1$; moreover, as in the previous case, there exists a $\nu_y \in \mathbb{N}$ such that $K_{w_n} \leq K_y$ for every $n > \nu_y$; now, if $\liminf_{n \to +\infty} N_{w_n} < +\infty$, it should exist a subsequence $(w_{n_h})_{h \in \mathbb{N}}$ of $(w_n)_{n \in \mathbb{N}}$ such that $\{N_{w_{n_h}} : h \in \mathbb{N}\}$ is bounded, consequently there should be a $H \in \mathbb{N}$ such that N_y, $N_{w_{n_h}} \leq H$ for every $h \in \mathbb{N}$ and hence from (4.2) it should hold $1 = \lim_{h \to +\infty} \sum_{m=0}^{H} w_{n_h,m} = \sum_{m=0}^{H} y_m < 1$, that is absurd; so $\liminf_{n \to +\infty} N_{w_n} = +\infty$ and $f(y) = +\infty = \liminf_{n \to +\infty} (N_{w_n} - K_{w_n}) = \liminf_{n \to +\infty} f(w_n)$.

(*iii*) If $y \in Y \setminus \mathrm{sp}\{e_m : m \in \mathbb{N}\}$, as when $y \in C$ we can define K_y and we obtain that there exists a $\nu_y \in \mathbb{N}$ such that $K_{w_n} \leq K_y$ for every $n > \nu_y$; moreover, since in this case there exist arbitrarily great values of $m \in \mathbb{N}$ such that $y_m > 0$, for (4.1) we have $\liminf_{n \to +\infty} N_{w_n} = +\infty$ and hence $f(y) = +\infty = \liminf_{n \to +\infty} (N_{w_n} - K_{w_n}) = \liminf_{n \to +\infty} f(w_n)$.

(*iv*) If $y = 0$ and if $\liminf_{n \to +\infty} (N_{w_n} - K_{w_n}) < +\infty$, it should exist a subsequence $(w_{n_h})_{h \in \mathbb{N}}$ of $(w_n)_{n \in \mathbb{N}}$ such that there is $L \in \mathbb{N}$ for which $N_{w_{n_h}} - K_{w_{n_h}} \leq L$ for each $h \in \mathbb{N}$; consequently, since $\sum_{m \in \mathbb{N}} w_{n_h,m} = 1$ $(h \in \mathbb{N})$, there should be for every $h \in \mathbb{N}$ a $m_h \in \mathbb{N} \cap [K_{w_{n_h}}, N_{w_{n_h}}]$ such that $w_{n_h,m_h} \geq \frac{1}{L+1}$ and hence it should hold $|w_{n_h} - y|_Y = |w_{n_h}|_Y \geq |w_{n_h,m_h}| \geq \frac{1}{L+1}$, contradictorily to $|w_{n_h} - y|_Y \xrightarrow[h \to +\infty]{} 0$; therefore $f(y) = f(0) = +\infty = \liminf_{n \to +\infty} (N_{w_n} - K_{w_n}) = \liminf_{n \to +\infty} f(w_n)$.

Finally we shall prove that f is not *i-slsc* with respect to the weak topology on Y. For this, it suffices to note that $f(e_n) = N_{e_n} - K_{e_n} = 0 = \min f$ for every $n \in \mathbb{N}$, $\lim_{n \to +\infty} e_n = 0$ with respect to the weak topology on Y, but $+\infty = f(0) > 0 = \liminf_{n \to +\infty} f(e_n)$.

THEOREM 4.1. – *Let X be a normed space, let $f : X \to [-\infty, +\infty]$ be a convex, i-slsc function with respect to the topology induced on X by its norm. Then f is i-slsc also with respect to the weak topology on X.*

PROOF. – Let $x \in X$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of X for which $x_n \rightharpoonup x$ and $f(x_n) \searrow \inf f$. Now we shall distinguish two cases:

(*i*) there exists a sequence $(t_n)_{n \in \mathbb{N}}$ of elements of $\mathrm{co}\{x_m : m \in \mathbb{N}\}$ such that $t_n \to x$ and $f(t_n) = \inf f$ for every $n \in \mathbb{N}$,

(*ii*) there does not exist a sequence as in (*i*).

In the case (*i*) we can conclude, simply using the *i-slsc* of f at x with respect to the topology induced on X by its norm, relatively to the sequence $(t_n)_{n \in \mathbb{N}}$.

If we are in the case (*ii*), then there exists $\gamma > 0$ such that

$$S_X(x, \gamma) \cap \mathrm{co}\{x_m : m \in \mathbb{N}\} \subseteq \{y \in X : f(y) > \inf f\}; \tag{4.4}$$

besides, being $x \in \overline{\mathrm{co}\{x_k : k > n\}}$ for every $n \in \mathbb{N}$ (as a convex closed subset of X

is weakly closed too), there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of $S_X(x,\gamma)$, strongly converging to x, such that $y_n \in \mathrm{co}\,\{x_k : k > n\}$ for every $n \in \mathbb{N}$; therefore by (4.4) it is

$$f(y_n) > \inf f \text{ for every } n \in \mathbb{N}; \tag{4.5}$$

now for every $n \in \mathbb{N}$ let $N_n \in \mathbb{N}$, $N_n \geq n+1$ such that there exist $\lambda_{j,n} \in [0,1]$ for $j \in \{n+1,\dots,N_n\}$ with $\sum_{j=n+1}^{N_n} \lambda_{j,n} = 1$ for which $y_n = \sum_{j=n+1}^{N_n} \lambda_{j,n} x_j$; since $\lim_{n\to+\infty} f(x_n) = \inf f$, for every $a > \inf f$ there is $n_a \in \mathbb{N}$ such that $f(x_n) < a$ for every $n > n_a$ and hence for $n > n_a$, by the convexity of f, it is

$$f(y_n) \leq \sum_{j=n+1}^{N_n} \lambda_{j,n} f(x_j) < \sum_{j=n+1}^{N_n} \lambda_{j,n} a = a$$

and therefore $\lim_{n\to+\infty} f(y_n) = \inf f$; since (4.5) is verified, it is possible to define for induction a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that:

$$n_0 = 0,\ n_{k+1} = \min\{n > n_k : f(y_n) \leq f(y_{n_k})\} \text{ for every } k \in \mathbb{N};$$

in this way we obtain a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to x$ and $f(y_{n_k}) \searrow \inf f$; so we can conclude using the *i-slsc* of f at x with respect to the topology induced on X by its norm, relatively to the sequence $(y_{n_k})_{k\in\mathbb{N}}$.

EXAMPLE 4.2. – Here we shall exhibit in (a), in (b), in (c) and in (d) some examples of convex functions which are not *i-slsc*.

(a) For every non trivial normed space X there exist a convex function $k: X \to [0,+\infty]$ and a point $w \in X$ such that k is not *i-slsc* at w.

(b) For every infinite dimensional real normed space X, there exists a linear function $j: X \to \mathbb{R}$ such that j is not *i-slsc* at u whatever $u \in X$ is considered. Such a function j can be chosen as whatever a linear not continuous functional on X.

(c) If X is an infinite dimensional normed space, then there exists a function j as in (a) but taking values in $\mathbb{R}$; moreover such function j can be chosen in such a way as to have that j is a linear function with respect to the structure of linear space on $\mathbb{R}$ of X (see also (b)) and that, for every $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that $x_n \to x$ and $j(x_n) \to -\infty = \inf_{y\in X} j(y)$.

(d) If X is an infinite dimensional Banach space, then there exists a function j as in (a) but with values in $[0,+\infty[$.

(a) Let $k(x) = \begin{cases} 0 & \text{if } x \in S_X(0,1) \\ +\infty & \text{if } x \in X \setminus S_X(0,1) \end{cases}$ (or, with a slight change, it can be obtained a function k such that $\{x \in X : k(x) < +\infty\}$ is also a closed set: it suffices to define $k(x) = \begin{cases} 0 & \text{if } x \in S_X(0,1) \\ 1 & \text{if } x \in \partial S_X(0,1) \\ +\infty & \text{if } x \in X \setminus \overline{S_X(0,1)} \end{cases}$) and let w be whatever a point of $\partial S_X(0,1)$; then it suffices to consider $w_n = (1 - \frac{1}{n+1})w$ for every $n \in \mathbb{N}$ to get that $w_n \to w$, $(k(w_n))_{n\in\mathbb{N}}$ (that is constantly 0) is weakly decreasing and $k(w) > \lim_{n\to+\infty} k(w_n) = \inf k$.

If $X = \mathcal{C}^0([0,1], \mathbb{F})$ with the "sup" norm, we can construct also the following, perhaps more expressive, example.

Let $C := \{x \in X : x(t) \geq 0 \text{ for every } t \in [0,1]\}$ and let

$$k(x) = \begin{cases} |\{t \in [0,1] : x(t) = 0\}| & \text{if } x \in C \\ +\infty & \text{if } x \in X \setminus C \end{cases}.$$

Then C is a convex set and k is a convex function, because, if $x, y \in C$, $\lambda \in]0,1[$, then

$$\{t \in [0,1] : (\lambda x + (1-\lambda)y)(t) = 0\} = \{t \in [0,1] : x(t) = 0\} \cap \{t \in [0,1] : y(t) = 0\}$$

and hence

$$\begin{aligned} &|\{t \in [0,1] : (\lambda x + (1-\lambda)y)(t) = 0\}| \\ =&|\{t \in [0,1] : x(t) = 0\} \cap \{t \in [0,1] : y(t) = 0\}| \\ \leq&\lambda|\{t \in [0,1] : x(t) = 0\}| + (1-\lambda)|\{t \in [0,1] : y(t) = 0\}|. \end{aligned}$$

Now it results $\inf k = 0$ and, choosing as w whatever a point of C such that $|\{t \in [0,1] : w(t) = 0\}| > 0$, it is enough to consider $w_n(t) := \max\{w(t), \frac{1}{n+1}\}$ for every $t \in [0,1]$ $(n \in \mathbb{N})$, because with such choice it holds that $w_n \to w$ and $k(w_n) = 0$ for every $n \in \mathbb{N}$ (see (a) or (b) of Definition 4.1 (i)).

(b) It is enough to observe that first conclusion of Theorem 3.1 remains true if the hypothesis of *d-slsc* is replaced by the one of *i-slsc* (moreover it suffices to assume such condition at the point x_0): really with the same proof it is possible to show the following result.

Let X be a real normed space, $x_0 \in X$, Y a vector subspace of X, $f : X \to]-\infty, +\infty]$ a *i-slsc* at x_0 function, such that $f_{|x_0+Y} : x_0 + Y \to \mathbb{R}$ is the restriction of an affine function and $f_{|X\setminus(x_0+Y)} = +\infty$. Then $f_{|x_0+Y}$ is continuous.

(c) We shall work with the normed space X considered on the real field, also if X is a normed space on the complex field.

Let $e_n \in X$ be such that $|e_n|_X = 1$ $(n \in \mathbb{N})$, $e_n \neq e_m$ if $n, m \in \mathbb{N}$, $n \neq m$, $\{e_n : n \in \mathbb{N}\}$ linearly independent set of vectors, B a Hamel basis of X such that $\{e_n : n \in \mathbb{N}\} \subseteq B$. For every $x \in X$ let x_n, $x(b) \in \mathbb{R}$ $(n \in \mathbb{N},\ b \in B \setminus \{e_n : n \in \mathbb{N}\})$ be such that $x = \sum_{n\in\mathbb{N}} x_n e_n + \sum_{b \in B\setminus\{e_n : n\in\mathbb{N}\}} x(b)b$, where in these sums there is only a finite number of not null addenda, and $j(x) := \sum_{n\in\mathbb{N}} n x_n$.

Then j is a linear function with respect to the structure of linear space on $\mathbb{R}$ of X. Now let $x \in X$ and $z_n := x - \frac{1}{n+1} e_{(n+1)^2}$ for every $n \in \mathbb{N}$; consequently $z_n \to x$, because $|z_n - x|_X = \frac{1}{n+1} \to 0$, while $j(z_n) = j(x) - \frac{1}{n+1} j(e_{(n+1)^2}) = j(x) - \frac{(n+1)^2}{n+1} = j(x) - n - 1 \to -\infty$ and so we can conclude.

(d) Let Y be a Banach space, on the same field of X's one, and let $T: X \to Y$ be a linear surjective not continuous operator (in order that such an operator exists, it suffices to suppose that $\dim X \geq \dim Y \geq 1$ (for example let $e_n \in X$ be such that $|e_n|_X = 1$ ($n \in \mathbb{N}$), $e_n \neq e_m$ if n, $m \in \mathbb{N}$, $n \neq m$, $\{e_n : n \in \mathbb{N}\}$ linearly independent set of vectors, B a Hamel basis of X such that $\{e_n : n \in \mathbb{N}\} \subseteq B$; since $\dim X \geq \dim Y \geq 1$, if C is a Hamel basis of Y constituted by elements having norm 1, there exists $\varphi: B \to C$ surjective; then it is enough to define $T(e_n) = n\varphi(e_n)$ for every $n \in \mathbb{N}$, $T(b) = \varphi(b)$ for every $b \in B \setminus \{e_n : n \in \mathbb{N}\}$, T extended for linearity to all X)).

Let $j(x) := |T(x)|_X$ for each $x \in X$. Then j is convex, by the convexity of the norm and the linearity of T. Now by absurd we suppose that j is *i-slsc*; then, on account of (a) of Definitions 4.1, we obtain

(4.6) $(x_n)_{n \in \mathbb{N}}$ sequence of elements of X for which
$$x_n \to x \text{ and } \lim_{n \to +\infty} j(x_n) = 0, \text{ implies } j(x) = 0;$$

but from this we get that T is continuous: indeed, owing to the closed graph theorem, it is enough to prove that T is closed; besides, if $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements of X for which there exist x, $y \in X$ with $x_n \to x$ and $T(x_n) \to y$ and if $z \in X$ is such that $T(z) = y$, we have $T(x_n - z) \to 0$, hence $j(x_n - z) \to 0$ and because of (4.6) we infer that $j(x - z) = 0$ and so $T(x - z) = 0$, that is $T(x) = T(z) = y$ and then we have a contradiction with the assumption that T was not continuous.

5. – Sufficient conditions for the existence of minima; well-posedness.

THEOREM 5.1. – *Let X be a sequentially compact topological space and $f: X \to [-\infty, +\infty]$ a i-slsc function. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_X f$.*

PROOF. – Since X is sequentially compact, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ of elements of X as well as $x_0 \in X$ such that $y_n \to x_0, f(y_n) \searrow \inf_X f$. Now it suffices to use the *i-slsc* of f at x_0 for concluding.

REMARK 5.1. – In [3] Y. Chen, Y.J. Cho and L. Yang stated the following:

Theorem ([3], Theorem 1.6) Let X be a real reflexive Banach space and let $f: D(f) \to]-\infty, +\infty]$ be a *d-slsc* and convex function, not identically $+\infty$.

Suppose that $\lim_{|x|_X \to +\infty} f(x) = +\infty$. Then there exists $x_0 \in D(f)$ such that $f(x_0) = \inf_{x \in D(f)} f(x)$.

Actually such a statement can not be true, as we can see by means of the following example:

Let X be a not null real reflexive Banach space, $y \in \partial S_X(0,1)$, $D(f) = S_X(y,1)$ and $f(x) = |x|_X$ for every $x \in D(f)$: then the function f satisfies all the hypotheses of the above Theorem, but it does not verify its thesis, being $\inf_{x \in D(f)} f(x) = 0$ and $f(x) > 0$ for each $x \in D(f)$.

However the above result becomes true if the hypotheses that $D(f)$ is convex and closed are added, hypotheses, among other things, usually assumed in literature (see for example [5], Proposition II.1.2); both such hypotheses are exploited in the proof of [3]; anyhow, also in this context, the proof seems a little lacunary to us; we will not specify the details here, because we shall give a generalization of such result in the part (a) of Remarks 5.2.

THEOREM 5.2. – *Let X be a reflexive Banach space, let $f: X \to [-\infty, +\infty]$ be a i-slsc function with respect to the weak topology on X. Suppose that $\lim_{|x|_X \to +\infty} f(x) = +\infty$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.*

PROOF. – If f is $+\infty$ identically then the thesis is obvious; otherwise, if $(y_n)_{n \in \mathbb{N}}$ is a minimizing sequence for f, then the hypothesis of coercitivity on f assures that such a sequence must be bounded and therefore, being X reflexive, there exists a subsequence weakly converging to a point x_0 and it is enough to use *i-slsc* of f at x_0 with respect to the weak topology to conclude.

COROLLARY 5.1. – *Let X be a reflexive Banach space, let $f: X \to [-\infty, +\infty]$ be a convex, i-slsc function with respect to the topology induced on X by its norm. Suppose that $\lim_{|x|_X \to +\infty} f(x) = +\infty$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.*

PROOF. – It is enough to apply Theorem 4.1 and Theorem 5.2.

REMARK 5.2.

(a) Using Corollary 5.1, the following result can be proved:

Let X be a reflexive Banach space, D a closed, convex, not empty subset of X and $f: D \to [-\infty, +\infty]$ a convex, *i-slsc* function with respect to the relative topology on D of the one induced on X by its norm. If D is not bounded, suppose furthermore that $\lim_{|x|_X \to +\infty} f(x) = +\infty$. Then there exists $x_0 \in D$ such that $f(x_0) = \inf_{x \in X} f(x)$.

In fact it suffices to note that in the above hypotheses the function

$g: X \to [-\infty, +\infty]$ defined by $g(x) = \begin{cases} f(x) & \text{if } x \in D \\ +\infty & \text{if } x \in X \setminus D \end{cases}$ is *i-slsc* and convex.

Therefore, since *d-slsc* implies *i-slsc* (see part (*b*) of Remarks 4.2), if we add the hypothesis that $D(f)$ is non empty, convex and closed, then the result stated in Remark 5.1 becomes true.

Moreover note that the obtained proof for that result, with the added hypotheses above cited, is also shorter than the proof of [3], because it represents an example of proof in which, by means of Theorem 4.1 (and part (*b*) of Remarks 4.2), in spite of

convex, *d-slsc* with respect to the topology induced on X by its norm $\not\Rightarrow$
d-slsc with respect to the weak topology

(for a function $f: X \to [-\infty, +\infty]$, where X is a normed space) (see Example 3.1 and those sketched in Remarks 3.1), can be used the fact that such a function f convex, *d-slsc* with respect to the topology induced on X by its norm, is always *i-slsc* with respect to the weak topology on X.

(*b*) Note that, in consequence of Theorem 5.2, in the hypotheses of Theorem 5.2 and if f does not assume the value $-\infty$, we deduce that f is bounded from below; but the same result can be found also if X is simply a normed space, provided that f is supposed a convex and *i-slsc* function with respect to the topology induced on X by its norm, and maintaining the hypothesis of coercitivity on f: in fact it is enough to use the subsequent Theorem 5.3, applied to the normed space X considered with respect to its linear structure on the real field, exploiting that each affine function is bounded on every bounded set and that, owing to the hypothesis of coercitivity, f is bounded from below in $X \setminus B$ where B is a suitable bounded subset of X.

LEMMA 5.1. – *Let (X, τ) be a topological space satisfying the first axiom of countability. Let $f: X \to]-\infty, +\infty]$ be a i-slsc function. Then, if $x_0 \in X$ and if $f(x_0) \in \mathbb{R}$, there exists a neighbourhood U of x_0 such that $\inf_{x \in U} f(x) > -\infty$.*

PROOF. – Let U_n $(n \in \mathbb{N})$ be such that $U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$ and $\{U_n : n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods for x_0. By absurd let $\inf_{x \in U_n} f(x) = -\infty$ for every $n \in \mathbb{N}$; then for every $n \in \mathbb{N}$ there exists a point $y_n \in U_n$ such that $f(y_n) < -n$ and hence $y_n \to x_0$, but $\lim_{n \to +\infty} f(y_n) = -\infty = \inf_{x \in X} f(x) < f(x_0)$, that is contrary to (*a*) of Definitions 4.1 relatively to x_0.

THEOREM 5.3. – *Let Y be a real normed space, $f: Y \to]-\infty, +\infty]$ a convex, i-slsc function. Then there exist $g \in Y'$ and $\gamma \in \mathbb{R}$ such that $f(x) > g(x) + \gamma$ for every $x \in Y$.*

PROOF. – If f is $+\infty$ identically, then the thesis is obvious; otherwise it is enough to use Lemma 5.1 and Theorem 2.2.

REMARK 5.3. – As anticipated in Remark 4.3, now we shall compare our definition of *i-slsc* with some definitions of well-posedness considered in [4] (see Definitions 2.2): in particular we shall show below that *i-slsc* appears to be a slight generalization of well-posedness in the generalized sense; inspired thereof, we shall bring out that in some results of [4] it is possible to substitute the hypothesis of *lsc* with the one of *i-slsc* (see Theorems 5.5, 5.7, 5.8 and 5.9).

Let (X,τ) be a topological space and let $f\colon X \to \,]-\infty,+\infty]$ be a not identically $+\infty$ function. Then, besides the following obvious implication:

$$(X,f) \text{ Tykhonov well-posed} \Rightarrow (X,f) \text{ well-posed in the generalized sense,}$$

if X is a Hausdorff space, then:

$$(X,f) \text{ well-posed in the generalized sense} \Rightarrow f \text{ } i\text{-}slsc \text{ function.}$$

In fact, if $(x_n)_{n\in\mathbb{N}}$ sequence of elements of X and $x \in X$ are such that $x_n \to x$ and $f(x_n) \to \inf_X f$, then, using well-posedness in the generalized sense of (X,f), there exist $z \in \arg\min(X,f)$ and a subsequence of $(x_n)_{n\in\mathbb{N}}$ converging to z; hence, being X Hausdorff space, $x = z \in \arg\min(X,f)$.
Moreover, we can note also that:

$$(X,f) \text{ well-posed in the generalized sense} \Rightarrow f \text{ bounded from below,}$$

because of nonemptyness of $\arg\min(X,f)$, together with the fact that $f(x) > -\infty$ for every $x \in X$.

THEOREM 5.4. – *Let (X,d) be a compact metric space and $f\colon X \to \,]-\infty,+\infty]$ a i-slsc, not identically $+\infty$ function. If there exists $z \in X$ such that $\arg\min(X,f) = \{z\}$, then (X,f) is Tykhonov well-posed.*

PROOF. – Let $x_n \in X$ $(n \in \mathbb{N})$ be such that $f(x_n) \to \inf_X f$ and by absurd we suppose that $(f(x_n))_{n\in\mathbb{N}}$ does not converge to z; then there exist $y \in X \setminus \{z\}$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converging to y; therefore, by *i-slsc* of f, we obtain that $y \in \arg\min(X,f)$, that is in contradiction with the hypothesis.

THEOREM 5.5. – *Let (X,d) be a complete metric space and let $f\colon X \to \,]-\infty,+\infty]$ be a i-slsc, bounded from below, not identically $+\infty$ function such that*

$$\lim_{\varepsilon\to 0^+} \operatorname{diam}\{y \in X\colon f(y) \le \inf_X f + \varepsilon\} = 0. \tag{5.1}$$

Then (X,f) is Tykhonov well-posed.

(Note that (5.1) is a necessary condition for the Tykhonov well-posedness of (X,f), also if X is not complete and $f\colon X \to \,]-\infty,+\infty]$ is only supposed not identically $+\infty$: see [4], Theorem I.2.11).

PROOF. – The proof of [4] (Theorem I.2.11, see at page 9) works well also in this case.

REMARK 5.4. – Regarding some problems about well-posedness, as for example the one considered in Theorem 5.5, note that in [4] the authors suppose, in the "standing assumptions" (see at page 5), that the function by us called f and that they call I is real valued, affirming in a Remark that such hypothesis can be done "without loss of generality" (really the proof of [4] (Theorem I.2.11) works well, as already written above, also if such an assumption is omitted).

However, in sight of the hypothesis of completeness on the space, we think that, the fact that generality is not lost assuming the functional to be real valued, may be emphasized by the following result, in which we are introducing a function g, suitably tied to our f, that offers the advantage to be defined in a closed set; because of role played in these problems by the hypotheses of *i-slsc* and boundedness from below, in the following Theorem we are also comparing such properties for f and g.

THEOREM 5.6. – *Let X be a topological space that satisfies the first axiom of countability and let $f\colon X \to]-\infty,+\infty]$ be a i-slsc function, not identically $+\infty$. Let $Y := \overline{\mathrm{dom} f}$, let $\gamma \in \mathbb{R}$, $\gamma > \inf_X f$ and let $g\colon Y \to]-\infty,+\infty]$ defined by*

$$g(y) = \begin{cases} f(y) & \text{if } f(y) < +\infty \\ \liminf\limits_{\substack{x \to y \\ x \in \mathrm{dom} f}} f(x) & \text{if } y \in Y \setminus \mathrm{dom} f \text{ and } \liminf\limits_{\substack{x \to y \\ x \in \mathrm{dom} f}} f(x) < +\infty \\ \gamma & \text{if } y \in Y \setminus \mathrm{dom} f \text{ and } \liminf\limits_{\substack{x \to y \\ x \in \mathrm{dom} f}} f(x) = +\infty \end{cases}.$$

Then:

(*a*) *$g(y) < +\infty$ for every $y \in Y$, $g(x) = f(x)$ for every $x \in \mathrm{dom} f$, $\inf_Y g = \inf_X f$ and g is a i-slsc function;*

(*b*) *if $z \in X$, it holds*

$$f(z) = \inf_X f \Leftrightarrow z \in Y \text{ and } g(z) = \inf_Y g,$$

that is $\arg\min(X,f) = \arg\min(Y,g)$;

(*c*) *Tykhonov well-posedness of (Y,g) implies Tykhonov well-posedness of (X,f); well-posedness in the generalized sense of (Y,g) implies well-posedness in the generalized sense of (X,f).*

Moreover, if X is also a metrizable space, then:

(*d*) *(X,f) is Tykhonov well-posed if and only if (Y,g) is Tykhonov well-posed; (X,f) is well-posed in the generalized sense if and only if (Y,g) is well-posed in the generalized sense.*

PROOF. – (a) The only not obvious part is the proof that g is *i-slsc*. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of elements of Y such that $y_n \to y$ and $\lim\limits_{n\to+\infty} g(y_n) = \inf\limits_Y g$.

Now we shall prove that

$$\text{(5.2)} \quad \text{there exists a sequence } (z_k)_{k\in\mathbb{N}} \text{ of elements of } \operatorname{dom} f \text{ such that } z_k \to y \text{ and } \lim_{k\to+\infty} g(z_k) = \inf_Y g\,.$$

We shall distinguish two cases:
(i) $(y_n)_{n\in\mathbb{N}}$ admits a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of elements of $\operatorname{dom} f$;
(ii) $(y_n)_{n\in\mathbb{N}}$ does not admit a subsequence of elements of $\operatorname{dom} f$.
If we are in the case (i), it is enough to define $z_k = y_{n_k}$ for every $k \in \mathbb{N}$.
If we are in the case (ii), given $\{U_k : k \in \mathbb{N}\}$ base for the neighbourhood system of y, with U_k open $(k \in \mathbb{N})$ (existence of such a base being secured because X verifies the first axiom of countability) and owing to (ii), to the equality $\inf\limits_Y g = \inf\limits_X f$ and to the inequality $\gamma > \inf\limits_X f$, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \in U_k \cap (Y \setminus \operatorname{dom} f)$ and $g(y_{n_k}) < \gamma$ for every $k \in \mathbb{N}$; so $g(y_{n_k}) = \liminf\limits_{\substack{x\to y_{n_k}\\ x\in \operatorname{dom} f}} f(x)$ for every $k \in \mathbb{N}$; then, to obtain (5.2), it is sufficient to consider $z_k \in U_k \cap \operatorname{dom} f$ such that $f(z_k) < g(y_{n_k}) + \frac{1}{k+1}$ for every $k \in \mathbb{N}$.
So (5.2) is proved.

Now, if $y \in \operatorname{dom} f$, we can conclude using *i-slsc* of f. Instead, if $y \in Y \setminus \operatorname{dom} f$, owing to the equality $\inf\limits_Y g = \inf\limits_X f$ and to the inequality $\liminf\limits_{\substack{x\to y\\ x\in \operatorname{dom} f}} f(x) \le \liminf\limits_{k\to+\infty} f(z_k)$, we obtain that $g(y) = \liminf\limits_{\substack{x\to y\\ x\in \operatorname{dom} f}} f(x) = \inf\limits_Y g$.

(b) If $f(z) = \inf\limits_X f$, then $z \in \operatorname{dom} f$ and therefore, using (a), we get $g(z) = f(z) = \inf\limits_X f = \inf\limits_Y g$.
Conversely, if $z \in Y$, $g(z) = \inf\limits_Y g$, then, being $g(z) < \gamma$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of $\operatorname{dom} f$ such that $y_n \to z$ and $\lim\limits_{n\to+\infty} f(y_n) = g(z) = \inf\limits_Y g = \inf\limits_X f$ and so, using *i-slsc* of f, we deduce that $f(z) = \inf\limits_X f$.

(c) As (Y, g) is Tykhonov well-posed (resp. well-posed in the generalized sense), then $\arg\min(Y, g) = \{z\}$ (resp. $\arg\min(Y, g) \ne \emptyset$) and so, exploiting (b), the same is true for $\arg\min(X, f)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X such that $f(x_n) \to \inf\limits_X f$; then definitely $x_n \in \operatorname{dom} f$ and hence $g(x_n) = f(x_n) \to \inf\limits_X f = \inf\limits_Y g$, from whence, using Tykhonov well-posedness (resp. well-posedness in the generalized sense) of (Y, g), $x_n \to z$ (resp. $(x_n)_{n\in\mathbb{N}}$ admits a convergent subsequence towards whatever an element of $\arg\min(Y, g) = \arg\min(X, f)$, where the equality follows from (b)) and we conclude.

(d) Owing to (c), it is enough to prove that Tykhonov well-posedness (resp. well-posedness in the generalized sense) of (X, f) implies Tykhonov well-posedness (resp. well-posedness in the generalized sense) of (Y, g). Then we suppose that (X,f) is Tykhonov well-posed (resp. well-posed in the generalized sense).

Wherefore $\arg\min(X,f) = \{z\}$ (resp. $\arg\min(X,f) \neq \emptyset$) and so, exploiting ($b$), the same is true for $\arg\min(Y,g)$.

Let d be now a metric that induces X's topology. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of elements of Y such that $g(y_n) \to \inf_Y g$; then there exists $n_0 \in \mathbb{N}$ such that $g(y_n) < \gamma$ for every $n > n_0$; consequently, by definition of g, for every $n > n_0$ there is a sequence of elements of $\operatorname{dom} f$ converging to y_n and such that f, calculated on the elements of such a sequence, converges to $g(y_n)$ (being able to choose such a sequence identically equal to y_n if $y_n \in \operatorname{dom} f$); hence for every $n > n_0$ there is a point $x_n \in \operatorname{dom} f$ such that $d(y_n, x_n) < \frac{1}{n}$ and $f(x_n) < g(y_n) + \frac{1}{n}$; therefore, being $\inf_X f = \inf_Y g$, we get $f(x_n) \to \inf_X f$, from whence, using Tykhonov well-posedness (resp. well-posedness in the generalized sense) of (X,f), we deduce that $x_n \to z$ (resp. there exist $w \in \arg\min(X,f)$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $x_{n_k} \to w$ and moreover $w \in \arg\min(Y,g)$ owing to (b)) and hence $y_n \to z$ (resp. $y_{n_k} \to w$) too and we conclude.

THEOREM 5.7. – *Let (X,d) be a metric space and let $f: X \to]-\infty, +\infty]$ be a i-slsc function, not identically $+\infty$ and bounded from below. Then (X,f) is well-posed in the generalized sense if and only if every minimizing sequence admits at least a convergent subsequence. In such a case,* $\arg\min(X,f)$ *is compact nonempty. Moreover well-posedness in the generalized sense is equivalent to compactness of* $\arg\min(X,f)$ *and upper semicontinuity at* 0 *of the multifunction* $\varepsilon \in \mathbb{R}_+ \mapsto \{y \in X : f(y) \leq \inf_X f + \varepsilon\}$.

PROOF. – The proof of [4] (Proposition I.6.36, see at page 27) works well also in this case.

THEOREM 5.8. – *Let (X,d) be a metric space and let $f: X \to]-\infty, +\infty]$ be a i-slsc, bounded from below and not identically $+\infty$ function. Then:*

(a) *if (X,f) is well-posed in the generalized sense, it is*

$$\lim_{\varepsilon\to 0^+} \alpha\big(\{y \in X : f(y) \leq \inf_X f + \varepsilon\}\big) = 0; \tag{5.3}$$

(b) *if X is complete and* (5.3) *is verified, it follows that (X,f) is well-posed in the generalized sense.*

PROOF. – (a) The proof of [4] (Theorem I.6.38, see at page 28, from "conversely" forward) works also in this case, using that the set $M := \arg\min(X,f)$ is compact on account of Theorem 5.7.

(b) Let $(x_n)_{n\in\mathbb{N}}$ be a minimizing sequence; then from (5.3) we deduce that

$\{x_n : n \in \mathbb{N}\}$ is totally bounded and therefore $(x_n)_{n \in \mathbb{N}}$ admits a Cauchy subsequence; being X complete, such a subsequence is convergent; so conclusion follows from the first thesis of Theorem 5.7.

DEFINITION 5.1. – ([4], Section 2 of Chapter I, after Theorem 11) Let $D \subseteq [0, +\infty[$ such that $0 \in D$. Then $c : D \to [0, +\infty[$ is called a *forcing function* if $c(0) = 0$ and

$$a_n \in D \ (n \in \mathbb{N}), \ c(a_n) \to 0 \Rightarrow a_n \to 0.$$

THEOREM 5.9. – *Let (X, d) be a bounded metric space and let $f : X \to]-\infty, +\infty]$ be a i-slsc, bounded from below and not identically $+\infty$ function. We have:*

(*a*) *if (X, f) is well-posed in the generalized sense, then*

$$\text{(5.4)} \quad \textit{there exists a forcing function } c \textit{ such that } \inf_X f \le \sup_A f - c(\alpha(A))$$
$$\textit{for every } A \subseteq X \text{ such that } A \neq \emptyset, \ \sup_A f < +\infty$$

(*b*) *if X is complete and* (5.4) *is verified, it follows that (X, f) is well-posed in the generalized sense.*

PROOF. – It is sufficient to use Theorem 5.8 and the equivalence between conditions (5.3) and (5.4), that is proved in the demonstration of [4] (Theorem I.6.39, see at page 28) without the use of completeness of X and of any lower semicontinuity hypotheses.

REMARK 5.5. – Here we wish to compare our definition of *i-slsc* (see Definitions 4.1) and some of results of this Section with some definitions and results introduced (also in sequential spaces) and studied in [8], [9] and [10].

Let (X, τ) be a topological space. Let $f : X \to [-\infty, +\infty]$ be a function. Then f is said to be:

(i) ([8], Definition 2.4 (i)) *sequentially lower pseudocontinuous at* $x \in X$ ("*slp* at x") if $z \in X, f(z) < f(x)$, implies $f(z) < \liminf_{n \to +\infty} f(x_n)$ for every sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X for which $x_n \to x$; *sequentially lower pseudocontinuous* ("*slp*") if f is *slp* at x for every $x \in X$;

(ii) ([10], Definition 2.2) *sequentially lower weakly pseudocontinuous at* $x \in X$ ("*slwp* at x") (sequentially lower quasicontinuous in [9], Definition 3.1) if $z \in X, f(z) < f(x)$, implies $f(z) \le \liminf_{n \to +\infty} f(x_n)$ for every sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X for which $x_n \to x$; *sequentially lower weakly pseudocontinuous* ("*slwp*") if f is *slwp* at x for every $x \in X$;

(*iii*) ([9], Definition 2.1) *sequentially transfer weakly lower continuous at* $x \in X$ ("*stwlc* at x") if $z \in X, f(z) < f(x)$, implies that there exists $z' \in X$ such that $f(z') \leq \liminf_{n\to+\infty} f(x_n)$ for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X for which $x_n \to x$; *sequentially transfer weakly lower continuous* ("*stwlc*") if f is *stwlc* at x for every $x \in X$.

The following facts are easily verifiable:

(*a*) if $x \in X$ and f is *slp* at x, then f is *i-slsc* at x;

(*b*) if $x \in X$ and f is *i-slsc* at x and if there exists $(x_n)_{n\in\mathbb{N}}$ minimizing sequence of elements of X for which $x_n \to x$, then f is *slp* at x;

(*c*) f can be *slwp* and not *i-slsc* (it is enough to define $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } y > 0 \\ 1 & \text{if } y \leq 0 \end{cases}$, that is not *i-slsc* at 0), f can be *i-slsc* and not *slwp* (it is sufficient to consider $f\colon \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 0 & \text{if } y \geq 0 \\ 1 & \text{if } y \in]-\infty, -2[\cup]-2,-1[\\ 2 & \text{if } y = -2 \\ 3 & \text{if } y \in [-1, 0[\end{cases}$, that is not *slwp* at -1, as can be seen showing that, with $z = -2$, the condition considered in (*ii*) is not verified).

Moreover:

(*d*) if the topology of X satisfies the first axiom of countability and f is *i-slsc*, then f is *stwlc* too.

In fact, if $x \in X$ and if $z \in X, f(z) < f(x)$, then

$$\inf\{\liminf_{n\to+\infty} f(x_n) : (x_n)_{n\in\mathbb{N}} \text{ sequence of elements of } X \text{ such that } x_n \to x\} > \inf_X f,$$

because, otherwise, satisfying the topology of X the first axiom of countability, there should exist a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of X such that $y_n \to x$ and $\lim_{n\to+\infty} f(y_n) = \inf_X f$ from whence, using the *i-slsc* of f at x, x should be a minimum point for f, that is impossible, being $f(z) < f(x)$; hence for concluding it suffices to choose a point $z' \in X$ such that $f(z') < \inf\{\liminf_{n\to+\infty} f(x_n) : x_n \in X \ (n \in \mathbb{N}), x_n \to x\}$.

(*e*) There exist a topological space X that does not satisfy the first axiom of countability and a function $f\colon X \to \mathbb{R}$ that is *i-slsc*, but is not *stwlc* at 0.

Let X be an infinite dimensional Hilbert space endowed with its weak topology and $e_n \in X$ $(n \in \mathbb{N})$ (with $e_n \neq e_m$ if $n, m \in \mathbb{N}, n \neq m$) elements of an orthonormal set for X. Let $f\colon X \to \mathbb{R}_+$ be a function defined by $f(x) = \begin{cases} \frac{1}{k} & \text{if } k \in \mathbb{Z}_+ \text{and } x \in \{ke_n : n \in \mathbb{N}\} \\ 1 & \text{if } x \in X \setminus \{ke_n : k \in \mathbb{Z}_+,\ n \in \mathbb{N}\} \end{cases}$. Then $\inf_X f = 0$; moreover $k\,e_n \rightharpoonup 0$ as $n \to +\infty$ for every $k \in \mathbb{Z}_+$, therefore $\inf\{\liminf_{n\to+\infty} f(x_n) : x_n \in X \ (n \in \mathbb{N}),$

$x_n \rightharpoonup 0\} = 0$, whence f is not *stwlc* at 0, because $f(x) > 0$ for every $x \in X$; moreover, being $|k\, e_n|_X = k$ for $k \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that $\lim_{n\to+\infty} f(x_n) = 0$ is not bounded and hence not weakly convergent; so f is *i-slsc*.

So Theorem 2.1, Corollary 3.1 of [9] (and Lemma 4.1 of [8]) are similar to our Theorem 5.1, but not directly comparable to it; the same thing happens for a result about well-posedness: Theorem 3.1 of [10] and our Theorem 5.4. Furthermore, as regards other results about well-posedness, we give with Theorem 5.7 (using that a *i-slsc* function $f: X \to]-\infty, +\infty]$, in the hypothesis of compactness on X, is bounded from below because it admits minimum, for Theorem 5.1) and Theorem 5.5 respectively generalizations of Theorems 3.2 and 3.3 of [10], because in both cases the hypothesis of *slp* is by us replaced by the weaker *i-slsc*.

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