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# Varieties of Algebras of Polynomial Growth (*)

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**Abstract.** – *Let $\mathcal{V}$ be a proper variety of associative algebras over a field $F$ of characteristic zero. It is well-known that $\mathcal{V}$ can have polynomial or exponential growth and here we present some classification results of varieties of polynomial growth. In particular we classify all subvarieties of the varieties of almost polynomial growth, i.e., the subvarieties of* $\mathrm{var}(G)$ *and* $\mathrm{var}(UT_2)$*, where $G$ is the Grassmann algebra and $UT_2$ is the algebra of* $2 \times 2$ *upper triangular matrices.*

## 1. – Introduction.

Let $F$ be a field of characteristic zero and let $F\langle X\rangle$ be the free associative algebra on a countable set $X$ over $F$.

If $\mathcal{V}$ is a variety of associative algebras over $F$, we denote by $\mathrm{Id}\,(\mathcal{V})$ the T-ideal of $F\langle X\rangle$ associated to $\mathcal{V}$. Recall that $\mathrm{Id}\,(\mathcal{V})$ is a two-sided ideal invariant under all endomorphisms of $F\langle X\rangle$ and consists of the polynomial identities satisfied by the algebras of $\mathcal{V}$. If $A$ is a generating algebra for the variety, we write $\mathcal{V} = \mathrm{var}\,(A)$ and $\mathrm{Id}\,(\mathcal{V}) = \mathrm{Id}\,(A)$.

An important invariant of a variety is given by its growth which is defined as follows. If $B$ is the relatively free algebra of countable rank of the variety $\mathcal{V}$, then its $n$-th codimension $c_n(\mathcal{V})$ is defined as the dimension of its multilinear part in $n$ standard generators. Then the growth of the variety $\mathcal{V}$ is the growth of the sequence $c_n(\mathcal{V}), n = 1, 2, \ldots$. We write also $c_n(\mathcal{V}) = c_n(A)$ if $A$ generates $\mathcal{V}$.

It is well known (see [18]) that if $A$ satisfies some non-trivial polynomial identity and, so, $\mathcal{V} = \mathrm{var}\,(A)$ is a proper variety, then the sequence of codimensions of $\mathcal{V}$ is exponentially bounded, i.e., there exist constants $a, a > 0$ such that $c_n(\mathcal{V}) \leq aa^n$ for all $n$. Kemer in [13, 14] characterized those varieties with a polynomially bounded codimension sequence. From his description it follows that there exists no variety with intermediate growth of the codimensions between polynomial and exponential, i.e, either $c_n(\mathcal{V})$ is polynomially bounded or $c_n(\mathcal{V})$ grows exponentially.

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Moreover, if $c_n(\mathcal{V})$ is polynomially bounded, i.e., there exist $a, t$ such that $c_n(\mathcal{V}) \leq an^t$, then it was proved in [2] that $c_n(\mathcal{V}) = qn^k + O(n^{k-1}) \approx qn^k$, $n \to \infty$, $q \in \mathbb{Q}$.

For general PI-algebras the exponential rate of growth was computed in [7] and [8] and it turns out to be a non-negative integer.

In case the codimensions are polynomially bounded, Kemer in [14] gave the following characterization. Let $G$ be the infinite dimensional Grassmann algebra over $F$ and let $UT_2$ be the algebra of $2 \times 2$ upper triangular matrices. Then $c_n(\mathcal{V})$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $G, UT_2 \notin \mathcal{V}$.

Hence $\mathrm{var}\,(G)$ and $\mathrm{var}\,(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

Recently in [16] the author determined a complete list of finite dimensional algebras generating the subvarieties of $\mathrm{var}\,(G)$ and $\mathrm{var}\,(UT_2)$.

A classification of varieties of polynomial growth was started in [5] and in [6]. More precisely the authors gave a complete list of finite dimensional algebras generating varieties of at most linear growth and, in the unitary case, of at most cubic growth.

The purpose of this paper is to present in a complete fashion these results regarding the classification of varieties of polynomially codimension growth.

## 2. – Codimensions and cocharacters.

Throughout we shall denote by $F$ a field of characteristic zero and by $\mathcal{V} = \mathrm{var}\,(A)$ a variety of associative algebras over $F$ generated by $A$. Let $F\langle X\rangle$ denote the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over $F$.

Recall that a polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity for $A$ and we write $f \equiv 0$ if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. Then

$$\mathrm{Id}\,(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$$

is a T-ideal of $F\langle X\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$. It is well known that in characteristic zero $\mathrm{Id}\,(A)$ is completely determined by its multilinear polynomials and we denote by $V_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ the vector space of multilinear polynomials in the variables $x_1, \ldots, x_n$. The non-negative integer

$$c_n(A) = \dim_F \frac{V_n}{V_n \cap Id(A)}$$

is called the $n$-th codimension of $A$.

In case $A$ is an algebra with 1, $\mathrm{Id}\,(A)$ is completely determined by its multilinear proper polynomials (see for instance [3]).

Recall that $f(x_1,\dots,x_n)\in V_n$ is a proper polynomial if it is a linear combination of products of (long) Lie commutators $[x_{i_1},\dots,x_{i_k}]$.

We denote by $\Gamma_n$ the subspace of $V_n$ of proper polynomials in $x_1,\dots,x_n$, we put also $\Gamma_0=\mathrm{span}\{1\}$. Then, the sequence of proper codimensions is defined as $c_n^p(A)=\dim\frac{\Gamma_n}{\Gamma_n\cap Id(A)}, n=0,1,2,\dots$.

For a unitary algebra $A$, the relation between ordinary codimensions and proper codimensions (see for instance [4]), is given by the formula

$$c_n(A)=\sum_{i=0}^{n}\binom{n}{i}c_i^p(A)\ n=1,2\dots. \tag{1}$$

In particular, if $A$ is a unitary algebra whose sequence of codimensions is polynomially bounded, then $c_n(A)=qn^k+\cdots$ is a polynomial with rational coefficients ([2], [6]).

One of the main tool in the study of the T-ideals is provided by the representation theory of the symmetric group. Recall that the symmetric group $S_n$ acts on the space $V_n$ by permuting the variables; if $\sigma\in S_n$ and $f(x_1,\dots,x_n)\in V_n$, $\sigma f(x_1,\dots,x_n)=f(x_{\sigma(1)},\dots,x_{\sigma(n)})$. This action is very useful since T-ideals are invariant under renaming of the variables. Hence $\frac{V_n}{V_n\cap Id(A)}$ becomes an $S_n$-module. Similarly $\frac{\Gamma_n}{\Gamma_n\cap Id(A)}$ is an $S_n$-module under the induced action. We denote by $\chi_n(A)$ and $\chi_n^p(A)$ the characters of the $S_n$-modules $\frac{V_n}{V_n\cap Id(A)}$ and $\frac{\Gamma_n}{\Gamma_n\cap Id(A)}$, respectively. They are called the $n$-th cocharacter and the $n$-th proper cocharacter of $A$.

By complete reducibility $\chi_n(A)$ and $\chi_n^p(A)$ decompose into irreducibles and let

$$\chi_n(A)=\sum_{\lambda\vdash n}m_\lambda\chi_\lambda,\ \ \chi_n^p(A)=\sum_{\lambda\vdash n}m'_\lambda\chi_\lambda,$$

where $\chi_\lambda$ is the irreducible $S_n$-character associated to the partition $\lambda$ and $m_\lambda, m'_\lambda$ are the corresponding multiplicities. We refer the reader to [1] for an account of the relations between $\chi_n(A)$ and $\chi_n^p(A)$.

Another tool used in the study of the T-ideals is the representation theory of the general linear group.

Let $F_m\langle X\rangle=F\langle x_1,\dots,x_m\rangle$ denote the free associative algebra in $m$ variables and let $U=\mathrm{span}_F\{x_1,\dots,x_m\}$. The group $GL(U)\cong GL_m$ acts naturally on the left on the space $U$ and we can extend this action diagonally to get an action on $F_m\langle X\rangle$.

The space $F_m\langle X\rangle\cap \mathrm{Id}(A)$ is invariant under this action, hence

$$F_m(A)=\frac{F_m\langle X\rangle}{F_m\langle X\rangle\cap Id(A)}$$

inherits a structure of left $GL_m$-module. If $F_m^n$ denotes the space of homogeneous polynomials of degree $n$ in the variables $x_1,\dots,x_m$

$$F_m^n(A)=\frac{F_m^n}{F_m^n\cap Id(A)}$$

is a $GL_m$-submodule of $F_m(A)$ whose character is denoted by $\psi_n(A)$. Write

$$\psi_n(A) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda$$

where $\psi_\lambda$ is the irreducible $GL_m$-character associated to the partition $\lambda$ and $\bar{m}_\lambda$ is the corresponding multiplicity.

The $S_n$-module structure of $V_n/(V_n \cap \mathrm{Id}\,(A))$ and the $GL_m$-module structure of $F_m^n(A)$ are related by the following: if $\chi_n(A) = \sum m_\lambda \chi_\lambda$ is the decomposition of the $n$-th cocharacter of $A$ then $m_\lambda = \bar{m}_\lambda$, for all $\lambda \vdash n$ whose corresponding diagram has height at most $m$ (see for instance [3]).

It is also well known that any irreducible submodule of $F_m^n(A)$ corresponding to $\lambda$ is generated by a non-zero polynomial $f_\lambda$, called highest weight vector, of the form

$$f_\lambda = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(x_1, \ldots, x_{h_i(\lambda)}) \sum_{\sigma \in S_n} a_\sigma \sigma,$$

where $a_\sigma \in F$, the right action of $S_n$ on $F_m^n(A)$ is defined by place permutation, $h_i(\lambda)$ is the height of the $i$-th column of the diagram of $\lambda$ and $St_r(x_1, \ldots, x_r) = \sum_{\tau \in S_r} (\mathrm{sgn}\,\tau) x_{\tau(1)} \cdots x_{\tau(r)}$ is the standard polynomial of degree $r$.

## 3. – Algebras with 1 of polynomial codimension growth.

Throughout this section we shall denote by $A$ a unitary algebra whose sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded. Hence,

$$c_n(A) = \sum_{i=0}^{k} \binom{n}{i} c_i^p(A) \approx qn^k \tag{2}$$

is a polynomial of degree $k$, for some $k \geq 0$, with rational coefficients.

In [4] it was proved that in case $k > 1$ the leading coefficient $q$ is a rational number satisfying the inequality

$$\frac{1}{k!} \leq q \leq \sum_{j=2}^{k} \frac{(-1)^j}{j!} \to \frac{1}{e}, \quad k \to \infty, \tag{3}$$

where $e = 2.71\ldots$. In the non-unitary case, for any $q \in \mathbb{Q}$ there exists an algebra $A$ such that $c_n(A) \approx qn^k$ for a suitable $k$.

For $k$ odd the lowest bound was improved in [6]. The authors proved that if $c_n(A) \approx qn^k$, for some odd integer $k > 1$ and rational number $q$, then $q \geq \frac{k-1}{k!}$.

Moreover, they proved that for any $k$ the highest and the lowest bound of $q$ are actually reached.

In this section we exhibit PI-algebras realizing the smallest and the largest value of $q$ (see for instance [6]).

We start by constructing an algebra of triangular matrices realizing the largest value of $q$. Here the $e_{ij}$'s denote the usual matrix units.

DEFINITION 1. – *Let*

$$U_k = U_k(F) = \{aE + \sum_{1 \le i < j \le k} a_{ij} e_{ij} \mid a, a_{ij} \in F\},$$

*where $E = E_{k \times k}$ denotes the identity $k \times k$ matrix.*

In what follows Lie commutators are left-normed, i.e., $[[\cdots[[x_1, x_2], x_3], \ldots], x_k] = [x_1, \ldots, x_k]$. The next theorem shows that the algebra $U_k$ has the largest possible polynomial growth of degree $k-1$, namely $c_n(U_k) \approx qn^{k-1}$ as $n \to \infty$, where $q = \sum_{j=2}^{k-1} \frac{(-1)^j}{j!}$.

THEOREM 2 [6, Theorem 3.1]. – *Let $F$ be an infinite field. Then*

1) *a basis of the identities of $U_k$ is given by all products of commutators of total degree $k$*

$$[x_1, \ldots, x_{a_1}][x_{a_1+1}, \ldots, x_{a_2}] \cdots [x_{a_{r-1}+1}, \ldots, x_{a_r}] \tag{4}$$

*with $a_r = k$ in case $k$ is even, and by the polynomials in* (4) *plus the polynomial of degree $k+1$*

$$[x_1, x_2] \cdots [x_k, x_{k+1}]$$

*in case $k$ is odd.*

2)

$$c_n(U_k) = \sum_{j=0}^{k-1} \frac{n!}{(n-j)!} \theta_j \approx \theta_{k-1} n^{k-1}, \quad n \to \infty,$$

*where $\theta_i = \sum_{j=0}^{i} \frac{(-1)^j}{j!}$, for $i \in \mathbb{N}$.*

The relevance of $U_k$ is shown in the following.

THEOREM 3. – *Let $A$ be a unitary algebra over an infinite field $F$ such that $c_n(A) \approx qn^k$, $n \to \infty$. Then* $\mathrm{Id}(A) \supseteq \mathrm{Id}(U_{k+1})$.

PROOF. – By (2) we have that $c_n(A) = \binom{n}{k} c_k^p(A) + \cdots$ and $c_{k+i}^p(A) = 0$, $i \ge 1$. This says that $\Gamma_{k+i} = \Gamma_{k+i} \cap \mathrm{Id}(A)$, i.e., $\Gamma_{k+i} \subseteq \mathrm{Id}(A)$, $i \ge 0$. Since by the previous theorem $\mathrm{Id}(U_{k+1})$ is generated by $\Gamma_{k+1}$, and eventually by $[x_1, x_2] \cdots [x_k, x_{k+1}] \in \Gamma_{k+2}$, we get that $\mathrm{Id}(U_{k+1}) \subseteq \mathrm{Id}(A)$. □

We now turn to the problem of constructing algebras with 1 realizing the minimal possible value for $q$.

Let $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in U_k$ denote the diagonal just above the main diagonal of $U_k$.

DEFINITION 4. – *For $k \geq 2$ let*

$$N_k = \operatorname{span}\{E, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq U_k$$

*where $E$ denotes the identity $k \times k$ matrix.*

Let also $G_{2k}$ denote the Grassmann algebra with 1 on a $2k$-dimensional vector space over $F$. Recall that

$$G_{2k} = \langle 1, e_1, \ldots, e_{2k} \mid e_i e_j = -e_j e_i \rangle.$$

The following two results characterizing the polynomial identities and the codimensions of $N_k$ and $G_{2k}$, will show that the smallest value of $q$ is realized by $N_{k+1}$ in case $k$ is odd and by $G_k$ in case $k$ is even.

THEOREM 5 [6, Theorem 3.4]. – *Let $k \geq 3$ and let $F$ be an infinite field. Then*

1) *A basis of the identities of $N_k$ is given by the polynomials*

$$[x_1, \ldots, x_k],\ [x_1, x_2][x_3, x_4].$$

2) $c_n(N_k) = 1 + \sum_{j=2}^{k-1} (j-1)\binom{n}{j} \approx \frac{k-2}{(k-1)!}\, n^{k-1}, \quad n \to \infty.$

THEOREM 6 [6, Theorem 3.5]. – *Let $F$ be an infinite field. Then*

1) *A basis of the identities of $G_{2k}$ is given by the polynomials*

$$[x_1, x_2, x_3],\ [x_1, x_2] \cdots [x_{2k+1}, x_{2k+2}].$$

2)

$$c_n(G_{2k}) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!}\, n^{2k}, \quad n \to \infty.$$

In the sequel we shall use the following notation.

DEFINITION 7. – *Let $A$ and $B$ be algebras. We say that $A$ is PI-equivalent to $B$ and we write $A \sim_{PI} B$ if* $\operatorname{Id}(A) = \operatorname{Id}(B)$.

We remark that

– if $k = 2$ then $N_k = U_k \sim_{PI} F$;
– if $k = 3$ then $N_k = U_k \sim_{PI} G_2$;
– if $k = 4$ then $N_k \sim_{PI} U_k$;
– if $k > 4$ then $\operatorname{var}(N_k) \subsetneq \operatorname{var}(U_k)$.

Given polynomials $f_1, \ldots, f_n \in F\langle X\rangle$, let us denote by $\langle f_1, \ldots, f_n\rangle_T$ the T-ideal generated by $f_1, \ldots, f_n$.

Recall that by [17], $\mathrm{Id}(UT_2) = \langle [x_1, x_2][x_3, x_4]\rangle_T$ and by [15], $\mathrm{Id}(G) = \langle [x_1, x_2, x_3]\rangle_T$. Hence, for any $k \geq 1$, $N_k \in \mathrm{var}(UT_2)$ and $G_{2k} \in \mathrm{var}(G)$.

## 4. – Characterizing $N_k$ and $G_{2k}$.

Recall that if $\mathcal{V}$ is a variety of algebras then $c_n(\mathcal{V}) = c_n(A)$, where $\mathcal{V} = \mathrm{var}(A)$ and the growth of $\mathcal{V}$ is the growth of the codimensions of $\mathcal{V}$. We start by making the following.

DEFINITION 8. – *A variety $\mathcal{V}$ is minimal of polynomial growth if $c_n(\mathcal{V}) \approx qn^k$ for some $k \geq 1, q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n(\mathcal{U}) \approx q'n^t$ with $t < k$.*

We shall prove that $G_{2k}$ and $N_k$ generate minimal subvarieties of the variety generated by $G$ or $UT_2$.

We start with the following.

LEMMA 9 [16, Lemma 4.3]. – *Let $A \in \mathrm{var}(UT_2)$ be an algebra with 1. If $c_k^p(A) = 0$, for some $k \geq 2$, then $c_m^p(A) = 0$ for all $m \geq k$.*

Recall that if $A = F + J$ is a finite dimensional algebra over $F$, where $J$ is the Jacobson radical of $A$, then $J$ can be decomposed into the direct sum of $F$-bimodules (see for instance [10]), i.e., $J = J_{00} + J_{01} + J_{10} + J_{11}$ where for $i \in \{0, 1\}$, $J_{ik}$ is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, $J_{ik}$ is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where $\delta_{kl}$ is the Kronecker delta and $J_{11} = FN$ for some nilpotent subalgebra $N$ of $A$ commuting with $F$.

LEMMA 10. – *Let $A = F + J$ be an algebra with $J = J_{10} + J_{01} + J_{11} + J_{00}$. If $A$ satisfies the identity $[x_1, \ldots, x_r] \equiv 0$, for some $r \geq 2$, then $J_{10} = J_{01} = 0$ and $A = (F + J_{11}) \oplus J_{00}$, a direct sum of algebras.*

PROOF. – The proof is obvious since for instance $J_{01} = [J_{01}, \underbrace{F, \cdots, F}_{r-1}] = 0$. □

Now we are going to prove that $N_k$ and $G_{2k}$ generate minimal varieties.

THEOREM 11. – *For any $k \geq 3$, $N_k$ generates a minimal variety.*

PROOF. – Suppose that the algebra $A \in \mathrm{var}(N_k)$ generates a subvariety of $\mathrm{var}(N_k)$ and $c_n(A) \approx qn^{k-1}$, for some $q > 0$. We shall prove that in this case $A \sim_{PI} N_k$ and this will complete the proof.

By [11, Theorem 7.2.12] we may assume that

$$A = A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are finite dimensional algebras such that $\dim A_i/J(A_i) \leq 1$ and $J(A_i)$ denotes the Jacobson radical of $A_i$, $1 \leq i \leq m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent algebra. Since

$$c_n(A) \leq c_n(A_1) + \cdots + c_n(A_m),$$

then there exists $A_i$ such that $c_n(A_i) \approx bn^{k-1}$, for some $b > 0$. Hence

$$\mathrm{var}(N_k) \supseteq \mathrm{var}(A) \supseteq \mathrm{var}(F + J(A_i)) \supseteq \mathrm{var}(F + J_{11}(A_i))$$

and $c_n(F + J(A_i)) \approx bn^{k-1}$, for some $b > 0$. By Lemma 10, since $F + J(A_i)$ satisfies the identity $[x_1, \ldots, x_k] \equiv 0$, $F + J(A_i) = (F + J_{11}(A_i)) \oplus J_{00}(A_i)$ and $c_n(F + J(A_i)) = c_n(F + J_{11}(A_i))$, for $n$ large enough. Hence, in order to prove that $A \sim_{PI} N_k$, it is enough to show that $F + J_{11}(A_i) \sim_{PI} N_k$. Hence, without loss of generality, we may assume that $A$ is a unitary algebra.

By (2) and Theorem 5

$$c_n(N_k) = \sum_{i=0}^{k-1} \binom{n}{i} c_i^p(N_k) = \sum_{i=2}^{k-1} \binom{n}{i}(i-1) + 1.$$

For $i = 2, \ldots, k-1$, let $f = [x_2, \underbrace{x_1, \ldots, x_1}_{i-1}]$ be an highest weight vector corresponding to the partition $\lambda = (i-1, 1) \vdash i$.

It is clear that $f$ is not an identity of $N_k$, so, for $2 \leq i \leq k-1$, $\chi_{(i-1,1)}$ participates in the $i$-th proper cocharacter $\chi_i^p(N_k)$ of $N_k$ with non-zero multiplicity. Hence for $2 \leq i \leq k-1$, since by Theorem 5 $c_i^p(N_k) = i-1$, we have $\chi_i^p(N_k) = \chi_{(i-1,1)}$.

Now, since $c_n(A) \approx qn^{k-1}$ then

$$c_n(A) = \sum_{i=0}^{k-1} \binom{n}{i} c_i^p(A)$$

and by Lemma 9, $c_i^p(A) \neq 0$ for all $2 \leq i \leq k-1$.

Recall that since $\mathrm{Id}(A) \supseteq \mathrm{Id}(N_k)$, $\Gamma_i/(\Gamma_i \cap \mathrm{Id}(A))$ is isomorphic to a quotient module of $\Gamma_i/(\Gamma_i \cap \mathrm{Id}(N_k))$. Hence if $\chi_i^p(A) = \sum_{\lambda \vdash i} m_\lambda \chi_\lambda$ and $\chi_i^p(N_k) = \sum_{\lambda \vdash i} m'_\lambda \chi_\lambda$ we must have $m_\lambda \leq m'_\lambda$ for all $\lambda \vdash i$. Since by the above for all $2 \leq i \leq k-1$, $\lambda \vdash i$, $\chi_i^p(N_k) = \chi_{(i-1,1)}$ and $c_i^p(A) \neq 0$ we obtain that also $\chi_i^p(A) = \chi_{(i-1,1)}$. Hence

$$c_n(A) = \sum_{i=0}^{k-1} \binom{n}{i} c_i^p(A) = 1 + \sum_{i=2}^{k-1} \binom{n}{i}(i-1) = c_n(N_k).$$

Thus $A$ and $N_k$ have the same sequence of codimensions and, since $\mathrm{Id}\,(N_k) \subseteq \mathrm{Id}\,(A)$ we get the equality $\mathrm{Id}\,(A) = \mathrm{Id}\,(N_k)$. □

THEOREM 12. – *For any $k \geq 1$, $G_{2k}$ generates a minimal variety.*

PROOF. – Let $A \in \mathrm{var}\,(G_{2k})$ and suppose that $c_n(A) \approx qn^{2k}$, for some $q > 0$. We shall prove that $A \sim_{PI} G_{2k}$. As in the proof of the previous theorem we may assume that $A$ is unitary and, since $A \in \mathrm{var}\,(G_{2k})$, by Theorem 6

$$c_n(A) = \sum_{i=0}^{2k} \binom{n}{i} c_i^p(A) = \sum_{i=0}^{k} \binom{n}{2i} c_{2i}^p(A)$$

where, by Lemma 9, $c_{2i}^p(A) \neq 0$ for all $i = 0, \ldots, k$, and $c_{2i}^p(A) \leq c_{2i}^p(G_{2k}) = 1$. It follows that $c_n(A) = c_n(G_{2k})$ for all $n$ and so, $A \sim_{PI} G_{2k}$. □

## 5. – Algebras without 1 of polynomial codimension growth.

Let $UT_k = UT_k(F)$ be the algebra of $k \times k$ upper triangular matrices over $F$. Given $A \subseteq UT_k$, we shall denote by $A^*$ the subalgebra of $UT_k$ obtained by flipping $A$ along its secondary diagonal.

Notice that given a polynomial $f \in F\langle X \rangle$ if we denote by $f^*$ the polynomial obtained by reversing the order of the variables in each monomial of $f$, then $f$ is a polynomial identity of $A$ if and only if $f^*$ is a polynomial identity of $A^*$.

For $i = 1, \ldots, k$, let $A_k^{(i)}$ denote the subalgebra of $UT_k$ having zero entries on the main diagonal except eventually the $(i, i)$-position, i.e.,

$$A_k^{(i)} = \mathrm{span}\{e_{ii}, e_{pq} \mid 1 \leq p < q \leq k\}.$$

The polynomial identities and the codimensions of the above algebras have been determined in [12].

We shall denote by $y, z$ variables of $X$.

THEOREM 13. – *For all $n > 1$,*

1) *the T-ideal $Id(A_k^{(i)})$ is generated by the polynomial*

$$x_1 \cdots x_{i-1}[y, z]x_i \cdots x_{k-1}.$$

2) $c_n(A_k^{(i)}) = n(n-1)\cdots(n-k+2) \approx n^{k-1}$.

*Moreover, if $A = A_k^{(1)} \oplus \cdots \oplus A_k^{(k)}$ then $c_n(A) \approx kn^{k-1}$.*

This says that for every $q \geq 1$ there exists an algebra $A$ such that $c_n(A) \approx qn^{q-1}$.

DEFINITION 14. – *For $k \geq 2$ let*

$$A_{k,1} = A_{k,1}(F) = span\{e_{11}, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq A_k^{(1)}.$$

Clearly $A_{2,1} = A_2^{(1)}$, $A_{2,1}^* = A_2^{(2)}$ and by the previous theorem, $\mathrm{Id}(A_{2,1}) = \langle [x_1, x_2]x_3\rangle_T$, $\mathrm{Id}(A_{2,1}^*) = \langle x_3[x_1, x_2]\rangle_T$ and $c_n(A_{2,1}) = c_n(A_{2,1}^*) = n$.

Next we describe explicitly the identities of $A_{k,1}$ and $A_{k,1}^*$ for any $k \geq 3$.

LEMMA 15 [16, Lemma 3.1]. – *If $k \geq 3$, then*

1) $\mathrm{Id}(A_{k,1}) = \langle [x_1, x_2][x_3, x_4], [x_1, x_2]x_3 \ldots x_{k+1}\rangle_T$.

2) $c_n(A_{k,1}) = \sum_{l=0}^{k-2} \binom{n}{l}(n - l - 1) + 1 \approx qn^{k-1}$, *where $q \in \mathbb{Q}$ is a non-zero constant.*

*Hence* $\mathrm{Id}(A_{k,1}^*) = \langle [x_1, x_2][x_3, x_4], x_3 \ldots x_{k+1}[x_1, x_2]\rangle_T$ *and* $c_n(A_{k,1}^*) = c_n(A_{k,1})$.

SKETCH OF PROOF. – Let $Q = \langle [x_1, x_2][x_3, x_4], [x_1, x_2]x_3 \ldots x_{k+1}\rangle_T$. Since $A_{k,1} \subseteq A_k^{(1)}$ and $\mathrm{Id}\,(A_k^{(1)}) = \langle [x_1, x_2]x_3 \ldots x_{k+1}\rangle_T$ it follows that $[x_1, x_2]x_3 \ldots x_{k+1} \in \mathrm{Id}\,(A_{k,1})$. Moreover, since $[A_{k,1}, A_{k,1}] \subseteq \mathrm{span}\{e_{12}, e_{13}, \ldots, e_{1k}\}$, we have that $[x_1, x_2][x_3, x_4] \in \mathrm{Id}\,(A_{k,1})$ and so, $Q \subseteq \mathrm{Id}\,(A_{k,1})$.

The following polynomials

$$(5) \qquad x_1 \cdots x_n,\ x_{i_1} \cdots x_{i_t}[x_i, x_j]x_{j_1} \cdots x_{j_l}$$

where $t + l = n - 2$, $l < k - 1$, $i > j < i_1 < \cdots < i_t$, $j_1 < \cdots < j_l$, span $V_n$ (mod $V_n \cap Q$) and are linearly independent (mod $V_n \cap \mathrm{Id}\,(A_{k,1})$).

Hence, since

$$V_n \cap \mathrm{Id}\,(A_{k,1}) \supseteq V_n \cap Q$$

it follows that $\mathrm{Id}\,(A_{k,1}) = Q$, and the elements in (5) are a basis of (mod $V_n \cap \mathrm{Id}\,(A_{k,1})$). Thus by counting we obtain

$$c_n(A_{k,1}) = \dim \frac{V_n}{V_n \cap Id(A_{k,1})} = \sum_{l=0}^{k-2} \binom{n}{l}(n - l - 1) + 1 \approx \frac{1}{(k-2)!}n^{k-1}.$$

Notice that $\mathrm{Id}(A_{k,1}^*) = \langle [x_1, x_2][x_3, x_4], x_3 \ldots x_{k+1}[x_1, x_2]\rangle_T$ and $c_n(A_{k,1}^*) = c_n(A_{k,1})$. □

We remark that $A_{k,1}, A_{k,1}^* \in \mathrm{var}\,(UT_2)$.

THEOREM 16 [16, Theorem 4.3]. *For any $k \geq 2$, $A_{k,1}$ and $A_{k,1}^*$ generate minimal varieties.*

## 6. – Classifying varieties of slow growth.

In this section we classify, up to PI-equivalence, all algebras generating varieties of at most linear growth and, in the unitary case, of at most cubic growth. Throughout this section $F$ is a field of characteristic zero.

THEOREM 17 [6, Theorem 3.6]. – *Let $A$ be an $F$-algebra with 1. If $c_n(A) \approx qn^k$, for some $q \geq 1$, $k \leq 3$, then either $A \sim_{PI} F$ or $A \sim_{PI} N_3$ or $A \sim_{PI} N_4$.*

REMARK 18. – If $A$ satisfies the hypotheses of the above theorem then $A \in \mathrm{var}\,(UT_2)$.

The following corollary follows easily.

COROLLARY 19. – *Let $A$ be an $F$-algebra with 1. If $c_n(A) \approx qn^k$, for some $q \geq 1$, $k \leq 3$, then either $c_n(A) = 1$ or $c_n(A) = \frac{n(n-1)}{2} + 1$ or $c_n(A) = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} + 1$. Hence either $q = 1$ or $q = \frac{1}{2}$ or $q = \frac{1}{3}$.*

Notice that if $A$ is an algebra with 1 then $A$ cannot have linear growth of the codimensions.

In [5] the authors gave a complete list of finite dimensional algebras generating varieties of at most linear growth. In what follows we state their results in our notation. We denote by $M = F(e_{11} + e_{33}) + F(e_{12}) + F(e_{13}) + F(e_{23}) \subseteq UT_3$ an algebra of upper triangular matrixes.

THEOREM 20 [5, Theorem 22]. – *Let $A$ be an $F$-algebra. Then the following conditions are equivalent:*

1) *$c_n(A) \leq kn$ for all $n \geq 1$, for some constant $k$.*

2) *$A$ is PI-equivalent to either $N$ or $C \oplus N$ or $A_{2,1} \oplus N$ or $A^*_{2,1} \oplus N$ or $A_{2,1} \oplus A^*_{2,1} \oplus N$ where $N$ is a nilpotent algebra and $C$ is a commutative non-nilpotent algebra.*

3) *$N_3, A_{3,1}, A^*_{3,1}, A^{(2)}_3, M \notin \mathrm{var}(A)$.*

Notice that the previous theorem allows us to classify all possible linearly bounded codimension sequences.

COROLLARY 21. – *Let $A$ be an $F$-algebra such that $c_n(A) \leq kn$ for all $n \geq 0$. Then there exists $n_0$ such that for all $n > n_0$ we must have either $c_n(A) = 0$ or $c_n(A) = 1$ or $c_n(A) = n$ or $c_n(A) = 2n - 1$.*

Since for any $k \geq 2$, $A_{k,1}, A^*_{k,1} \notin \mathrm{var}\,(G)$, we immediately obtain the following consequence.

COROLLARY 22. – *Let $A \in \mathrm{var}\,(G)$ and $c_n(A) \leq kn$ for all $n \geq 1$, for some constant $k$. Then $A$ is PI-equivalent to either $N$ or $C \oplus N$, where $N$ is a nilpotent algebra and $C$ is a commutative non-nilpotent algebra.*

## 7. – Classifying the subvarieties of $var(G)$ and $var(UT_2)$.

In this section we classify, up to PI-equivalence, all the algebras contained in the variety generated by the Grassmann algebra $G$ or the algebra $UT_2$.

As a consequence we shall see that $G_{2k}, N_k, A_{k,1}$ and $A^*_{k,1}$ generate the only minimal subvarieties of the variety generated by $G$ or $UT_2$.

We start by classifying the subvarieties of $G$.

THEOREM 23. – *Let $A \in \mathrm{var}(G)$. Then either $A \sim_{PI} G$ or $A \sim_{PI} G_{2k} \oplus N$, for some $k \geq 1$, or $A \sim_{PI} N$ or $A \sim_{PI} C \oplus N$, where $N$ is a nilpotent algebra and $C$ is a commutative non-nilpotent algebra.*

PROOF. – If $A \sim_{PI} G$ there is nothing to prove. Now let $A$ generate a proper subvariety of $\mathrm{var}(G)$. Since $\mathrm{var}(G)$ has almost polynomial growth, $\mathrm{var}(A)$ has polynomial growth and let $c_n(A) \approx qn^r$ for some $r \geq 0$. If $r \leq 1$ then by the previous corollary, either $A \sim_{PI} N$ or $A \sim_{PI} C \oplus N$ and we are done. Therefore we may assume that $r > 1$. Since $[x_1, x_2, x_3] \equiv 0$ is an identity of $A$, as in the proof of Theorem 11, we may assume that

$$A = A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are finite dimensional algebras such that either $A_i \cong (F + J_{11}) \oplus J_{00}$ or $A_i$ is a nilpotent algebra. Hence

$$A = A_1 \oplus \cdots \oplus A_n = B \oplus N,$$

where $B$ is a unitary algebra, $N$ is a nilpotent algebra and, for $n$ large enough,

$$c_n(A) = c_n(B) = \sum_{i=0}^{r} \binom{n}{i} c_i^p(B).$$

Since $[x_1, x_2, x_3] \in \mathrm{Id}(B)$ then $c_{2j+1}^p(B) = 0$, for all $j \geq 1$. Hence $r = 2k$, for some $k \geq 1$, and

$$c_n(B) = \sum_{i=0}^{k} \binom{n}{2i} c_{2i}^p(B).$$

In particular we get that $\Gamma_{2k+2} \subseteq \mathrm{Id}(B)$. This implies that $B \in \mathrm{var}(G_{2k})$ and, since $G_{2k}$ generates a minimal variety and $c_n(G_{2k}) \approx q'n^{2k}$ we obtain that $B \sim_{PI} G_{2k}$, and, so, $A \sim_{PI} G_{2k} \oplus N$. □

Notice that the previous theorem allows us to classify all codimension sequences of the algebras lying in the variety generated by $G$. We can also classify all algebras generating minimal varieties.

COROLLARY 24. – *Let $A \in \mathrm{var}(G)$ be such that $\mathrm{var}(A) \subsetneq \mathrm{var}(G)$. Then there* *exists* $n_0$ *such that for all* $n > n_0$ *we must have either* $c_n(A) = 0$ *or* $c_n(A) = 1$ *or* $c_n(A) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}$, $k = 1, 2, \ldots$.

COROLLARY 25. – *An algebra* $A \in \mathrm{var}(G)$ *generates a minimal variety if and only if* $A \sim_{PI} G_{2k}$, *for some* $k \geq 1$.

PROOF. – The proof follows from Theorem 12 and the previous theorem. □

THEOREM 26 [16, Theorem 5.4 ]. – *If* $A \in \mathrm{var}(UT_2)$ *then* $A$ *is PI-equivalent to one of the following algebras:*

$$UT_2, N, N_t \oplus N, N_t \oplus A_{k,1} \oplus N, N_t \oplus A^*_{r,1} \oplus N, N_t \oplus A_{k,1} \oplus A^*_{r,1} \oplus N,$$

*where* $N$ *is a nilpotent algebra and* $k, r, t \geq 2$.

It is worth noticing that the previous theorem allows us to classify all algebras generating minimal varieties.

COROLLARY 27. – *Let* $A \in \mathrm{var}(UT_2)$. *Then* $A$ *generates a minimal variety if and only if either* $A \sim_{PI} N_t$ *or* $A \sim_{PI} A_{k,1}$ *or* $A \sim_{PI} A^*_{k,1}$, *for some* $k \geq 2, t > 2$.

PROOF. – If $A$ is PI-equivalent to one of the algebras $N_t$, $A_{k,1}$, $A^*_{k,1}$, $t > 2, k \geq 2$, then by Lemma 11 and Lemma 16, $A$ generates a minimal variety. The converse follows immediately by the previous theorem. □

The previous theorem allows to classify all codimension sequences of the algebras belonging to the variety generated by $UT_2$.

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