Bollettino Unione Matematica Italiana

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The Quantitative Isoperimetric Inequality for Planar Convex Domains

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Abstract. – *We prove that among all the convex bounded domains in $\mathbb{R}^2$ having an assigned Fraenkel asymmetry index, there exists only one convex set (up to a similarity) which minimizes the isoperimetric deficit. We show how to construct this set. The result can be read as a sharp improvement of the isoperimetric inequality for convex planar domains.*

1. – Introduction.

The classical isoperimetric inequality in the plane states that, among all the subsets of $\mathbb{R}^2$ of prescribed measure, the disk has the smallest perimeter, namely

$$P(E) \geq 2(\pi|E|)^{\frac{1}{2}}, \qquad \text{with equality if and only if } E \text{ is a disk.}$$

Here $|E|$ and $P(E)$ denote, as usual, the measure and the perimeter of the set E.

It is almost impossible to give exhaustive references concerning the isoperimetric inequality, therefore we refer the reader to some pioneering papers [4, 7, 19, 21, 22], to the original paper by De Giorgi [10] in the general framework of finite perimeter sets in $\mathbb{R}^n$, to the reviews [15, 27] and to the books [8, 9].

If we define the isoperimetric deficit of a set E by

$$\Delta P(E) = \frac{P(E)}{2(\pi|E|)^{\frac{1}{2}}} - 1,$$

the isoperimetric inequality becomes

$$\Delta P(E) \geq 0, \qquad \text{with equality if and only if } E \text{ is a disk.}$$

(*) Comunicazione tenuta a Bari il 26 settembre 2007 in occasione del XVIII Congresso dell'Unione Matematica Italiana.

There exists a special class of inequalities which have been first studied by Bonnesen [5, 6] and imply the classical isoperimetric inequality. Nowadays they are called Bonnesen–style isoperimetric inequalities (see [8, 23]) and in general they can be represented in the form

$$\Delta P(E) \geq F(E),$$

where the function F is nonnegative, vanishes only on the disks, and it measures how much E is far from being a disk. In principle there are many different kinds of functions F that we may look for, but here we are interested in those functions whose dependence on the set E is only through the so–called Fraenkel asymmetry index, i.e.

$$(1.1) \qquad a(E) = \min_{x \in \mathbb{R}^2} \frac{|E \setminus D_R(x)|}{|E|},$$

where $D_R(x)$ is the disk centered at x and having the same measure as E.

In the framework of planar convex domains Hall and Hayman [17] proved a Bonnesen–style inequality equivalent to

$$(1.2) \qquad \Delta P(E) \geq \frac{\pi}{2(4-\pi)} a(E)^2 - c_0 a(E)^3,$$

valid for some unknown nonnegative constant c_0.

For completeness we mention that, in the last two decades, Bonnesen–style inequalities were found in higher dimensions and for more general sets, as well as in the anisotropic case, for example in [11, 12, 13, 14, 16]. In particular, Hall [16] proved that the isoperimeric deficit of any smooth open set is bounded from below by a constant times a certain power of the Fraenkel asymmetry index. The fact that the optimal power is 2 (regardeless the dimension) has been recently proved in [14].

In the present paper we give the sharp Bonnesen–style inequality for planar convex domains. Since there exists an optimal maximal function G such that it holds

$$(1.3) \qquad \Delta P(E) \geq G(a(E)),$$

we show how, for any $0 \leq a_0 < 1$, it is possible to compute $G(a_0)$. In particular, we work out the analytic expression of the *optimal* set: the unique (up to a similarity) convex set E, with asymmetry index $a(E) = a_0$, which achieves the equality sign in (1.3). Our result is based on a symmetrization technique. More precisely, we reshape a given planar convex set keeping, step by step, its measure and its asymmetry index fixed and shortening the perimeter. The procedure finally supplies with the family of *optimal* sets.

This paper is part of a joint work with Angelo Alvino and Vincenzo Ferone; some of the proofs are omitted and can be found in [1].

2. – Main statement.

In order to formulate our main statement we begin by defining the family of optimal sets. We say that a set S belongs to the family $\mathfrak{S}$ if the following properties hold:

- S is 2-symmetric i.e.: symmetric with respect to two orthogonal axes;
- S has a smooth C^1 boundary made of four arcs $\{a_i\}_{1 \leq i \leq 4}$ of circle two of which can eventually degenerate in parallel segments;
- $a(S) = \dfrac{|S \setminus D|}{|D|}$, D being the disk having the same measure of S and centered at the intersection of the axes of symmetry of S;
- whenever a_i is a proper arc of circle (for some $1 \leq i \leq 4$) then it does not cross ∂D, namely either $a_i \subset \overline{D}$ or $a_i \subset \mathbb{R}^2 \setminus D$.

For the seek of completeness we give an explicit analytic expression of the family $\mathfrak{S}$. If the symmetry axes are used as reference axes in (ξ, η)-plane, up to a similarity, such a family can be described as a one–parameter family $\{H_\vartheta\}_{\vartheta \in (0, \pi/4)}$ where $|H_\vartheta| = \pi$ and the parameter ϑ denotes the angular variable of the interserction of ∂H_ϑ with ∂D in the first quadrant of (ξ, η)-plane. When $0 < \vartheta \leq \arctan(\pi/4)$ the part of the set H_ϑ which belongs to the first quadrant is described as (see Figure 1)

$$\begin{aligned}
\bar{H}_\vartheta &= \{(\xi, \eta) \in H_\vartheta : \xi \geq 0, \ \eta \geq 0\} \\
&= \left\{ (\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq \frac{\pi}{4} \frac{\cos^2 \vartheta}{\sin \vartheta} + \sin \vartheta, 0 \leq \eta \leq \gamma_1(\xi) \right\},
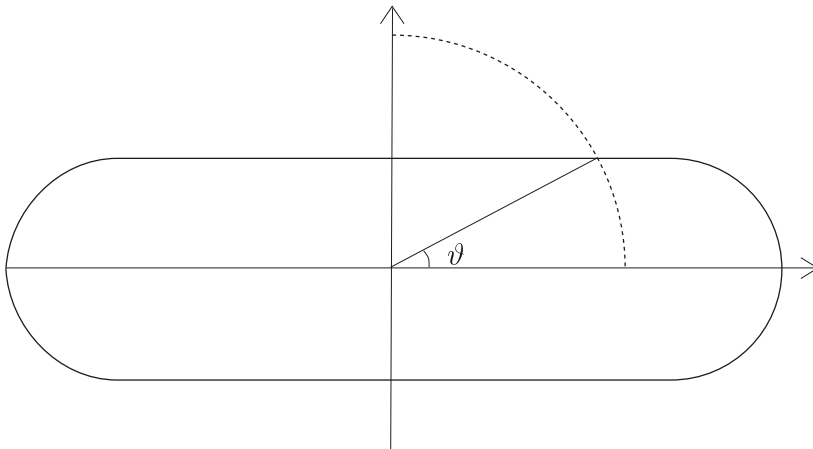\end{aligned}$$

Fig. 1. – The set H_ϑ for $0 < \vartheta \leq \arctan(\pi/4)$.

where

$$
\gamma_1(\xi) = \begin{cases} \sin \vartheta & 0 \leq \xi \leq \dfrac{\pi}{4} \dfrac{\cos^2 \vartheta}{\sin \vartheta} \\[3mm] \sqrt{\sin^2 \vartheta - \left(\xi - \dfrac{\cos^2 \vartheta}{\sin \vartheta}\right)^2} & \dfrac{\pi}{4} \dfrac{\cos^2 \vartheta}{\sin \vartheta} < \xi \leq \dfrac{\pi}{4} \dfrac{\cos^2 \vartheta}{\sin \vartheta} + \sin \vartheta. \end{cases}
$$

In this case we have:

$$
\Delta P(H_\vartheta) = \frac{(1 - \sin \vartheta)^2}{2 \sin \vartheta}
$$

$$
a(H_\vartheta) = \frac{\pi - 2\vartheta - 2 \sin \vartheta \cos \vartheta}{\pi}
$$

We observe explicitly that, when $\vartheta \to 0$, H_ϑ degenerates into a segment and we have:

$$
\lim_{\vartheta \to 0} \Delta P(H_\vartheta) = + \infty
$$

$$
\lim_{\vartheta \to 0} a(H_\vartheta) = 1.
$$

When $\arctan(\pi/4) < \vartheta < \pi/4$ the part of the set H_ϑ which belongs to the first quadrant is bounded by arcs of circumferences which meet at the point
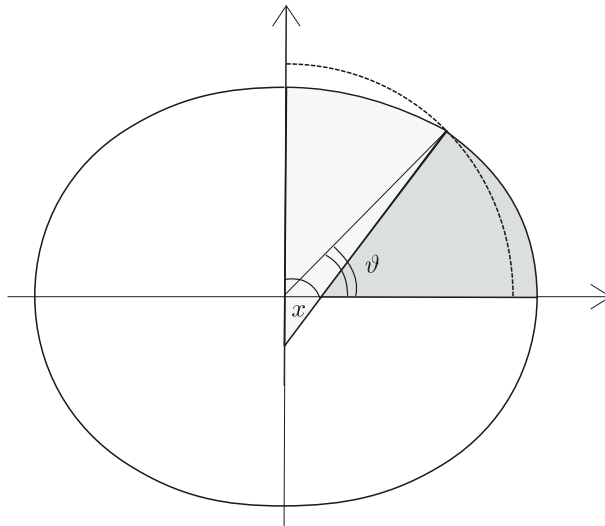

Fig. 2. – The set H_ϑ for $\arctan(\pi/4) < \vartheta \leq \pi/4$.

$(\cos\vartheta, \sin\vartheta)$ in such a way that ∂H_ϑ is of class C^1. This means that the radii of such circumferences passing through the point $(\cos\vartheta, \sin\vartheta)$ belong to the same straight line which forms an angle x with η-axis (see Figure 2).

For this type of domains the isoperimetric deficit is given by:

$$(2.1) \qquad \Delta P(H_\vartheta) = \frac{2}{\pi}\left(\left(\frac{\sin\vartheta}{\cos x}\left(\frac{\pi}{2}-x\right) + \frac{\cos\vartheta}{\sin x}x\right) - \frac{\pi}{2}\right),$$

while the Fraenkel asymmetry is given by:

$$(2.2) \qquad a(H_\vartheta) = \frac{4}{\pi}\left(\frac{\sin^2\vartheta}{\cos^2 x}\left(\frac{\pi}{2}-x-\sin x\cos x\right) - \vartheta + \sin\vartheta\cos\vartheta\right).$$

The condition $|H_\vartheta| = \pi$ gives a relation between x and ϑ:

$$(2.3) \qquad \frac{\sin^2\vartheta}{\cos^2 x}\left(\frac{\pi}{2}-x\right) + \frac{\cos^2\vartheta}{\sin^2 x}x - \frac{(\cos\vartheta - \tan x\sin\vartheta)^2}{\tan x} = \frac{\pi}{2}.$$

As a result of a straightforward but very tedious calculation both $\Delta P(H_\vartheta)$ and $a(H_\vartheta)$ happen to be continuous decreasing functions of ϑ and therefore the following property holds.

PROPOSITION 2.1. – *For each $0 \le t < 1$, up to a similarity, there exists one and only one set $S_t \in \mathfrak{S}$ such that $a(S_t) = t$. Moreover, whenever $0 \le t_1 < t_2 < 1$, then $\Delta P(S_{t_1}) < \Delta P(S_{t_2})$.*

Therefore we can parametrize the family $\mathfrak{S}$ by the Fraenkel asymmetry index or the isoperimetric deficit.

The statement of our main result follows.

THEOREM 2.1. – *Every convex set $\Omega \in \mathbb{R}^2$ satisfies*

$$(2.4) \qquad \Delta P(\Omega) \ge \Delta P(S_{a(\Omega)}),$$

where the equality holds if and only if $\Omega \in \mathfrak{S}$.

By using the monotonicity of $\Delta P(H_\vartheta)$ and $a(H_\vartheta)$ with respect to ϑ we deduce the monotonicity of the function G in (1.3) but, regretfully, it seems impossible to write its expression in an elementary form. Nevertheless inequality (1.2) can be carried out by an asymptotic expantion of $a(H_\vartheta)$ and $\Delta P(H_\vartheta)$ in the limit as $\vartheta \to \frac{\pi}{4}$. In particular it is quite easy to deduce from (2.1), (2.2) and (2.3) that

$$\lim_{\vartheta \to \frac{\pi}{4}} \frac{\Delta P(H_\vartheta)}{a(H_\vartheta)^2} = \frac{\pi}{2(4-\pi)},$$

and that the constant c_0 in (1.2) has to be positive.

Finally, a different understanding of Theorem (2.1) is

COROLLARY 2.1. − *For any given $0 \leq \Delta P_0 < +\infty$ there exists $0 \leq \bar{t} < 1$ such that, for all convex sets $\Omega \subset \mathbb{R}^2$ with $\Delta P(\Omega) = \Delta P_0$, we have*

$$a(\Omega) \leq a(S_{\bar{t}})$$

when

$$\Delta P(S_{\bar{t}}) = \Delta P_0.$$

Therefore, taking into account the meaning of the asymmetry index, our result provides a sharp way to control how much a convex set differs from the disk having the same measure in terms of its isoperimetric deficit.

3. − Rearrangements and circular symmetrization.

Given a coordinate system (ξ, η) in $\mathbb{R}^2$, as usual the polar coordinates will be denoted by (r, θ) with $r \geq 0$ and $\theta \in [0, 2\pi]$, such that $\theta = 0$ points in the positive direction of the ξ−axis. According to Kawohl [20], a compact domain $\Omega \subset \mathbb{R}^2$ in polar coordinates can be Steiner symmetrized with respect to the angular coordinate θ to obtain a set Ω^*, that is symmetric in θ and that he calls the circular symmetrized of Ω. If we deal with a smooth and starshaped set with respect to the origin O, then the circular symmetrization can be also obtained by the Schwarz symmetrization of its radial function $\rho(\theta)$, namely the function which represents in polar coordinates the boundary of such a set. From well−known properties of the symmetrization we get $|\Omega^*| = |\Omega|$ and $|\partial\Omega^*| \leq |\partial\Omega|$.

Our goal is to introduce a new kind of symmetrization which somehow generalizes the circular one. Let us consider a smooth and bounded set $\Omega \subset \mathbb{R}^2$, starshaped with respect to the origin O, and two directions v_1 and v_2. We fix a coordinate system such that in polar coordinates $\theta = 0$ points in the direction of v_1 and let us denote by $\bar{\theta} \in [0, 2\pi]$ the direction of v_2. Thereafter we consider the restriction $\rho_1(\theta)$ of $\rho(\theta)$ to the set $[0, \bar{\theta}]$ and the restriction $\rho_2(\theta)$ of $\rho(\theta)$ to the set $[\bar{\theta}, 2\pi]$, and we define

$$\bar{\rho}(\theta) = \begin{cases} \rho_1^\sharp\left(\theta - \dfrac{\bar{\theta}}{2}\right) & \theta \in [0, \bar{\theta}[, \\[3mm] \rho_2^\sharp\left(\theta - \pi - \dfrac{\bar{\theta}}{2}\right) & \theta \in [\bar{\theta}, 2\pi[, \end{cases}$$

$$\underline{\rho}(\theta) = \begin{cases} \rho_{1\sharp}\left(\theta - \dfrac{\bar{\theta}}{2}\right) & \theta \in [0, \bar{\theta}[, \\[3mm] \rho_{2\sharp}\left(\theta - \pi - \dfrac{\bar{\theta}}{2}\right) & \theta \in [\bar{\theta}, 2\pi[. \end{cases}$$

Here by $f^\sharp$ and $f_\sharp$ we denote, respectively, the symmetric decreasing and increasing rearrangements of f (see [2, 3, 18, 26]). Roughly speaking, we are actually considering the rearrangement of ρ obtained by the symmetric rearrangements of its restrictions ρ_1 and ρ_2.

Let $\Omega^{v_1 v_2}$ and $\Omega_{v_1 v_2}$ be the two open sets defined respectively by the interior points of $\{(r, \theta) : 0 \leq r \leq \bar{\rho}(\theta)\}$ and $\{(r, \theta) : 0 \leq r \leq \underline{\rho}(\theta)\}$. Obviously $|\Omega^{v_1 v_2}| = |\Omega_{v_1 v_2}| = |\Omega|$ but in general $\bar{\rho}$ and $\underline{\rho}$ are discontinuous at both 0 and $\bar{\theta}$, a condition which may increase the perimeter of the symmetrized set. The rest of the section is devoted to establish sufficient conditions such that either $|\partial \Omega^{v_1 v_2}| \leq |\partial \Omega|$ or $|\partial \Omega_{v_1 v_2}| \leq |\partial \Omega|$.

We first introduce the following definition

DEFINITION 3.1 (The set $\Lambda(a, b)$). – *We say that a function g belongs to $\Lambda(a, b)$ if*

- *g is a nonnegative Lipschitz continuous function on $[a, b]$;*
- *$\#\{g = t\} \geq 2$ whenever $\min\limits_{x \in [a,b]} g(x) < t < \max\limits_{x \in [a,b]} g(x)$.*

The main result of this section follows.

LEMMA 3.1. – *Let Ω be an open starshaped set with respect to O, having a Lipschitz continuous radial function $\rho(\theta)$. Let v_1, v_2 and $\bar{\theta}$ be defined as before. If the restriction of $\rho(\theta)$ to $(0, \bar{\theta})$ and $(\bar{\theta}, 2\pi)$ belong, respectively, to $\Lambda(0, \bar{\theta})$ and $\Lambda(\bar{\theta}, 2\pi)$, and $\min\limits_{\theta \in (0,\bar{\theta})} \rho = \min\limits_{\theta \in (\bar{\theta}, 2\pi)} \rho$ $\left(\max\limits_{\theta \in (0,\bar{\theta})} \rho(\theta) = \max\limits_{\theta \in (\bar{\theta}, 2\pi)} \rho(\theta) \right)$, then $\bar{\rho}(\theta)$ $(\underline{\rho}(\theta))$ is Lipschitz continuous, and $|\partial \Omega^{v_1 v_2}| \leq |\partial \Omega|$ $(|\partial \Omega_{v_1 v_2}| \leq |\partial \Omega|)$.*

Lemma 3.1 is a consequence of a well known result involving symmetric rearragements [25].

LEMMA 3.2. – *If a function g belongs to $\Lambda(a, b)$ then*

$$\int\limits_a^b g(s)^2 ds = \int\limits_{-\frac{a+b}{2}}^{\frac{a+b}{2}} g^\sharp(s)^2 ds = \int\limits_{-\frac{a+b}{2}}^{\frac{a+b}{2}} g_\sharp(s)^2 ds,$$

$$\int\limits_a^b \sqrt{g(s)^2 + g'(s)^2}\, ds \geq \int\limits_{-\frac{a+b}{2}}^{\frac{a+b}{2}} \sqrt{g^\sharp(s)^2 + g^{\sharp\prime}(s)^2}\, ds$$

and

$$\int\limits_a^b \sqrt{g(s)^2 + g'(s)^2}\, ds \geq \int\limits_{-\frac{a+b}{2}}^{\frac{a+b}{2}} \sqrt{g_\sharp(s)^2 + g_\sharp{}'(s)^2}\, ds.$$

4. – Preliminary results.

Let Ω be an open, bounded, convex subset of $\mathbb{R}^2$, and let D be a circle of radius $R = \left(\dfrac{|\Omega|}{\pi}\right)^{\frac{1}{2}}$ that achieves the index of asymmetry of Ω, i.e.:

$$(4.1) \qquad\qquad |\Omega \setminus D| = \min_{x \in \mathbb{R}^2} |\Omega \setminus D_R(x)|.$$

We refer to the last condition as the *optimality condition* for D with respect to Ω. From now on we shall use as the origin of the coordinate system in $\mathbb{R}^2$ the center O of D. Since Ω is starshaped with respect to O (it is easy to check that $O \in \Omega$, and a convex set is starshaped with respect to any internal point) there exists a one to one correspondence $\Phi : \partial D \to \partial \Omega$ such that, for any $x \in \partial D$, we have $\dfrac{x}{|x|} = \dfrac{\Phi(x)}{|\Phi(x)|}$. Throughout the paper we shall use the following notation (see Figure 3):

- $A = \partial D \cap \Omega$;
- $B = \partial D \setminus \overline{\Omega}$;
- $G = \partial D \cap \partial \Omega = \partial D \setminus (A \cup B)$;
- a_i ($i \in \mathfrak{I} \equiv \{1, 2, \} \subseteq \mathbb{N}$) are the connected components of A;
- β_k ($k \in \mathfrak{N} \equiv \{1, 2, \} \subseteq \mathbb{N}$) are the connected components of B;
- L denotes the arclength of a curve in $\mathbb{R}^2$;
- T_v is the line containing O and orthogonal to the direction v. As T_v splits $\mathbb{R}^2$ in two we denote by P_v^+ and P_v^- the two open half planes (v belongs to P_v^+);
- Π_v is the projection operator over T_v.
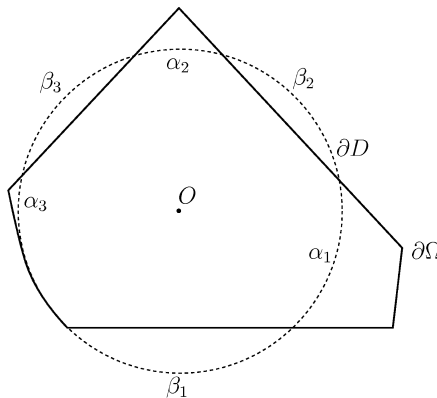

Fig. 3. – The set Ω and the disk D. We split ∂D in three subsets: $A = \partial D \cap \Omega$, $B = \partial D \setminus \overline{\Omega}$ and $G = \partial D \cap \partial \Omega = \partial D \setminus (A \cup B)$

Let us now define $h_{a_i} = \max\{|\Phi(x)| : x \in a_i\}$ and $d_{\beta_k} = \min\{|\Phi(x)| : x \in \beta_k\}$; we shall assume without loss of generality that, for every $i \in \mathfrak{I}$ and $k \in \mathfrak{N}$,

$$(4.2.\text{i}) \qquad h_{a_i} = \max_{j \geq i} h_{a_j},$$

and

$$(4.2.\text{ii}) \qquad d_{\beta_k} = \min_{j \geq k} d_{\beta_j};$$

somehow we assume that a_i and β_k are ordered according to a "descending height order" in the radial direction. The existence of a maximum and a minimum in (4.2.i) and (4.2.ii) is a consequence of the fact that Ω is convex. Indeed, for any $C > 0$ there exists only a finite number of indexes j such that $h_{a_j} \geq R + C$ and a finite number of j such that $d_{\beta_j} \leq R - C$.

By using the convexity of Ω and the optimality condition of D with respect to Ω it is possible to prove the following Lemma.

LEMMA 4.1. – *Let v be any direction in $\mathbb{R}^2$ then*

$$L(\Pi_v(A \cap P_v^+)) \leq L(\Pi_v(A \cap P_v^-)) + L(\Pi_v(G \cap P_v^-)),$$

and

$$L(\Pi_v(B \cap P_v^+)) \leq L(\Pi_v(B \cap P_v^-)) + L(\Pi_v(G \cap P_v^-)).$$

The previous relations between the lenght of the projections of A, B and G over an arbitrary plane lead to a useful relation between the archlengths.

LEMMA 4.2. – *Let*

$$I = \max\left\{ \max_k L(\beta_k), \frac{1}{2} \sum_k L(\beta_k) \right\}$$

and

$$J = \max\left\{ \max_i L(a_i), \frac{1}{2} \sum_i L(a_i) \right\},$$

then $I + J \leq \pi R$.

5. – Proof of the main result.

We are ready to give a sketch of the proof of Theorem (2.1) by providing a symmetrization algorithm. The characterization of the equality case in (2.4) is omitted and we refer the interested reader to [1].

For the reader convenience we divide the proof in three steps. At each step the procedure preserves the Fraenkel asymmetry while it not increases the isoperimetric deficit. In Step–1 we reduce a given bounded convex set to a new set (not necessarily convex) having two orthogonal axes of symmetry. In Step–2 we reshape the set in order to have a set with the boundary made only of arcs of circle but not necessarily convex neither smooth. Finally in Step–3 we show that among all the sets of fixed asymmetry index given in Step–2, the optimal one (the set having the smallest possible isoperimetric deficit) is the unique (up to a similarity) smooth set.

5.1 – *Step–1 (Reduction to a 2-symmetric set)*.

Let us show how to reduce any given convex set Ω to one having two orthogonal axes of symmetry, exactly the same measure and Fraenkel asymmetry index of Ω but a perimeter not greater than $P(\Omega)$.

We consider now two directions v_1 and v_2 and the two half–lines originating from O and containing them which split the plane in two angles that we denote by $\mathcal{A}_1$ and $\mathcal{A}_2$. We fix v_1 and v_2 according to the occurrence of one of the following cases:

A1 for all $i \in \mathfrak{I}$ it holds $L(a_i) \leq \sum\limits_{j \neq i} L(a_j)$;

A2 there exists $\hat{i} \in \mathfrak{I}$ such that $L(a_{\hat{i}}) > \sum\limits_{j \neq i} L(a_j)$.

In the first case we fix v_1 and v_2 such that $a_1 \subset \mathcal{A}_1$, $a_2 \subset \mathcal{A}_2$ and $L(A \cap \mathcal{A}_1) = L(A \cap \mathcal{A}_2)$. In the second case we choose v_1 and v_2 such that $a_{\hat{i}} = A \cap \mathcal{A}_1$.

In a similar way if μ_1 and μ_2 are two directions and the two half–lines originating from the origin and containing them split the plane in two angles denoted by $\mathcal{B}_1$ and $\mathcal{B}_2$, we can always fix μ_1 and μ_2 according to the occurrence of one of the following cases:

B1 for all $k \in \mathfrak{K}$ it holds $L(\beta_k) \leq \sum\limits_{j \neq k} L(\beta_j)$;

B2 there exists $\hat{k} \in \mathfrak{K}$ such that $L(\beta_{\hat{k}}) > \sum\limits_{j \neq k} L(\beta_j)$.

In the first case we fix μ_1 and μ_2 such that $\beta_1 \subset \mathcal{B}_1$, $\beta_2 \subset \mathcal{B}_2$ and $L(B \cap \mathcal{B}_1) = L(B \cap \mathcal{B}_2)$. In the second case we choose μ_1 and μ_2 such that $\beta_{\hat{k}} = B \cap \mathcal{B}_1$.

Thereafter we consider $\Omega^+ = D \cup \Omega$ and $\Omega^- = D \cap \Omega$ and their circular symmetrizations that we denote respectively $\Omega^{v_1 v_2}$ and $\Omega_{\mu_1 \mu_2}$ (see Figure 4(b) and Figure 4(c)).

REMARK 5.1. – Let us use the same notation as in section 3, and suppose that the coordinate axes are chosen in such a way that v_1 and v_2 correspond in polar
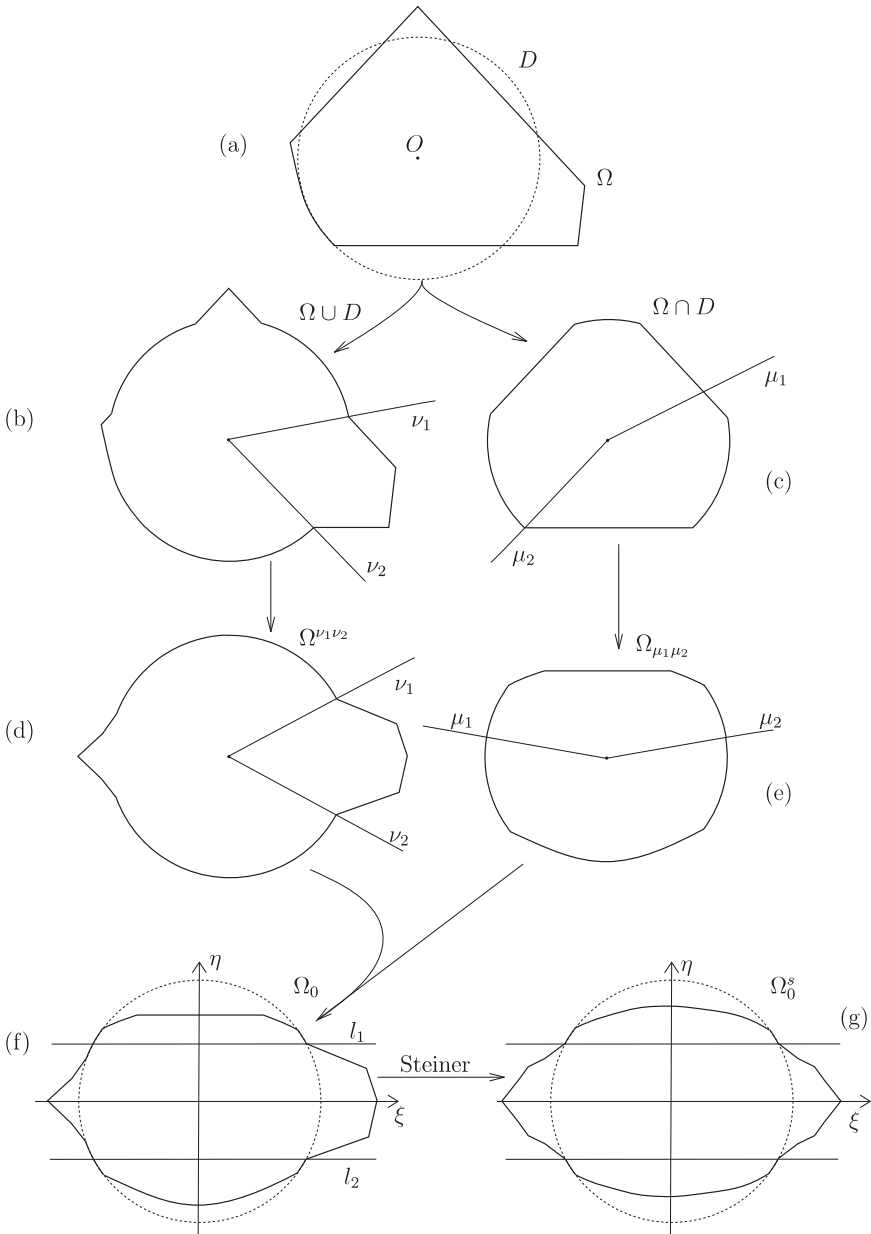
Fig. 4. – The symmetrizzation procedure carring Ω to Ω_0^s.

coordinates respectively to $\theta = 0$ and $\theta = \bar{\theta}$. The radial function of the set Ω^+ is $\max\{\rho(\theta), R\}$, and from the definition of ν_1 and ν_2 we notice that $\max\{\rho(\theta), R\}$ belongs to both $\Lambda(0, \bar{\theta})$ and $\Lambda(\bar{\theta}, 2\pi)$. Similarly, the radial function of the set Ω^- is $\min\{\rho(\theta), R\}$. In such a case we assume that μ_1 and μ_2 correspond in polar coordinates respectively to 0 and $\bar{\theta}$, then the definition of μ_1 and μ_2 implies that $\min\{\rho(\theta), R\}$ belongs to both $\Lambda(0, \bar{\theta})$ and $\Lambda(\bar{\theta}, 2\pi)$.

By definition the set $\Omega^{\nu_1 \nu_2}$ is symmetric with respect to the axis which lays in the direction $\nu_1 + \nu_2$, while the set $\Omega^{\mu_1 \mu_2}$ is symmetric with respect to the axis which lays in the direction $\mu_1 + \mu_2$. We can rotate the two sets around the point O until their symmetry axes happen to be mutually orthogonal (see Figure 4(d) and Figure 4(e)). From now on we use these orthogonal axes as a reference system (ξ, η).

We merge together the sets $\Omega^{\nu_1 \nu_2}$ and $\Omega_{\mu_1 \mu_2}$ by considering the set $\Omega_0 = (\Omega^{\nu_1 \nu_2} \setminus D) \cup \Omega_{\mu_1 \mu_2}$ (see Figure 4(f)). The following result holds.

LEMMA 5.1. $-$ Ω_0 *has the following properties:*

1. Ω_0 *is starshaped;*
2. $|\Omega_0| = |\Omega|$;
3. $|\Omega_0 \setminus D| = |\Omega \setminus D|$;
4. $|\partial \Omega_0| \leq |\partial \Omega|$.

After the above operation, the set Ω_0 is such that the part of the boundary of D given by $\partial D \setminus \partial \Omega_0$ consists at most of four arcs which we denote by a_1, a_2, b_1, b_2 such that $L(a_1) = L(A \cap \mathcal{A}_1)$, $L(a_2) = L(A \cap \mathcal{A}_2)$, $L(b_1) = L(B \cap \mathcal{B}_1)$ and $L(b_2) = L(B \cap \mathcal{B}_2)$. Without loss of generality, we can assume that a_1 belongs to the halfplane $\xi \geq 0$. If we are not considering the trivial case $\Omega = D$, then $a_1 \neq \emptyset$ and $b_1 \neq \emptyset$. One of the following occurrence certainly happens:

(o1) $a_2 \neq \emptyset$ and $b_2 \neq \emptyset$;
(o2) $a_2 = \emptyset$ and $b_2 \neq \emptyset$;
(o3) $a_2 = \emptyset$ and $b_2 = \emptyset$;
(o4) $a_2 \neq \emptyset$ and $b_2 = \emptyset$.

In any case

$$a_1 \cup a_2 = \partial D \cap \Omega_0, \qquad L(a_1) \geq L(a_2)$$

$$b_1 \cup b_2 = \partial D \setminus \overline{\Omega}_0, \qquad L(b_1) \geq L(b_2).$$

Each one of the arcs a_1, a_2 is symmetric with respect to the ξ axis, while b_1 and b_2 are symmetric with respect to the η axis. Let $\bar{d} = R \sin\left(\dfrac{L(a_1)}{2R}\right)$ and let l_1 and l_2

be two lines orthogonal to η and having the same distance $\bar{d}$ from the origin O (see Figure 4(f)). We denote by Σ_{in} the open subset of $\mathbb{R}^2$ between these two lines and by Σ_{out} the open set $\mathbb{R}^2 \setminus \overline{\Sigma}_{in}$. Moreover we denote by $\mathcal{C}$ the smallest angle with vertex in the origin, containing a_1, and by $-\mathcal{C}$ its symmetric with respect to the η-axis. The following result holds.

LEMMA 5.2. – *The lines l_1 and l_2 intersect ∂D in four points which belong to $\partial\Omega_0$. Moreover the sets $\Omega_0 \setminus \overline{D}$ and $D \setminus \overline{\Omega}_0$ are respectively subset of Σ_{in} and Σ_{out}.*

We denote by Ω_0^s the Steiner symmetrization of Ω_0 with respect to both axes ξ and η (see Figure 4(g)). Lemma (5.1)–(5.2) are both fulfilled when replacing Ω_0 by Ω_0^s with the same lines l_1 and l_2 defined above. The set Ω_0^s is the desired 2-symmetric set which concludes this Step.

REMARK 5.2. – We came out with the set Ω_0^s by making a lots of efforts since, as we claimed in the beginning of the section, we were looking for an algorithm which leaves the asymmetry index unchanged during the process of symmetrization. Indeed, by using Lemma 5.1 and Lemma 5.2, it is possible to show that the disk D is still the optimal one, namely

$$a(\Omega_0^s) = \frac{|\Omega_0^s \setminus D|}{|\Omega_0^s|}.$$

The importance of the last condition is evident, nevertheless, in order to get to the next Step, we do not need to prove it explicitly.

5.2 – Step–2. Reduction to a 2-symmetric set with the boundary made of arcs

DEFINITION 5.1 (The family $\mathfrak{Z}$). – *We say that E belongs to the family $\mathfrak{Z}(a_0, m)$ (for some $m \geq 0$ and $0 \leq a_0 < 1$) if, up to a rototranslation;*

(a1) *E is star shaped with respect to O;*
(a2) *E is 2-symmetric;*
(a3) *$|E| = m$;*
(a4) *$|E \setminus D| = a_0|E|$, where D is the disk having center at O and radius $(|E|/\pi)^{\frac{1}{2}}$;*
(a5) *using the same notation of the previous paragraph there exist two lines l_1 and l_2, orthogonal to η axis and having the same distance from O, such that they intersect ∂D in four points which also belong to ∂E. Moreover the sets $E \setminus D$ and $D \setminus E$ are respectively subset of Σ_{in} and Σ_{out}.*

REMARK 5.3. − It is important to observe that conditions (a1)–(a5) yield

$$a(E) = \frac{|E \setminus D|}{|\Omega_0^s|},$$

but it is not necessary to prove this assertion.

The set Ω_0^s obtained in Step−1 belongs to the family $\mathfrak{Z}(a(\Omega), |\Omega|)$ when the lines l_1 and l_2 are defined as in the previous paragraph, and $\partial D \setminus \partial \Omega_0^s$ is made of four arcs d_1, d_2, e_1, e_2 such that:

$$L(d_1) = L(d_2),$$
$$L(e_1) = L(e_2),$$
$$d_1 \cup d_2 \subset \overline{\Sigma}_{in},$$
$$e_1 \cup e_2 \subset \overline{\Sigma}_{out}.$$

Let us consider the four points given by the intersection of l_1 and l_2 with ∂D. They are placed on the vertex of a rectangle and we denote them by P_i, $i = 1, ..., 4$, assuming that starting from the upper left corner they are placed in the clockwise ordered. It is trivial to check that $P_i \in \partial \Omega_0^s$ for all $1 \le i \le 4$. We look for a set belonging to $\mathfrak{Z}(a(\Omega), |\Omega|)$, having minimum perimeter, with the further constraint that in (a5) of Definition 5.1 the lines l_1 and l_2 are fixed. This implies in particular that $\{P_1, P_2, P_3, P_4\}$ belong to its boundary. Because of the symmetry the problem can be solved by finding the optimal shape of the boundary, namely by looking for two curves g_1 and g_2 having the smallest possible length and satisfying the following properties:

(b1) g_1 connects P_1 and P_2;
(b2) $g_1 \subset \overline{\Sigma}_{out} \cap D$;
(b3) the region between $\partial D \cap \Sigma_{out}$ and g_1 has measure equal to $\frac{1}{2}a(\Omega)|\Omega|$;
(b4) g_2 connects P_2 and P_3;
(b5) $g_2 \subset \overline{\Sigma}_{in} \setminus D$;
(b6) the region between $\partial D \cap \Sigma_{in}$ and g_2 has measure equal to $\frac{1}{2}a(\Omega)|\Omega|$.

The set $\Omega^a \in \mathfrak{Z}(a(\Omega), |\Omega|)$ having g_1 and g_2 as part of the boundary (the rest of the boundary can be deduced using its symmetry) has the desired properties. A simple variational argument says that g_1 has to be an arc of circumference. As regards g_2, the same argument can be applied if $a(\Omega)$ is small enough, namely,

$$(5.1) \qquad a(\Omega) \le \frac{1}{|\Omega|}\left(\pi d^2 + 2d\sqrt{\frac{|\Omega|}{\pi} - d^2} - 2\left(\frac{|\Omega|}{\pi}\right)^{\frac{1}{2}} \arcsin\frac{d}{r}\right).$$

If condition (5.1) is not satisfied, we can find g_2 using an argument contained in [24, Th 3.32]. Indeed $\Omega^a \cap \Sigma_{in}$ is the convex hull of two balls of radius $\dfrac{d}{2}$. In other words g_2 is the only C^1 curve given by the union of an arc of circumference with two segments k_1 and k_2 laying respectively on l_1 and l_2.

5.3 − Step−3. Reduction to a smooth convex set.

In this paraagraph we give a sketch of the last Step of the symmetrization procedure: among all the sets in $\Im(a(\Omega), |\Omega|)$ the set $S_{a(\Omega)}$ achieves the minimum perimeter.

We begin by proving that, for any $m \geq 0$ and $0 \leq a_0 < 1$, there exists a set, the *optimal* set, having minimum perimeter among all the sets belonging to $\Im(a_0, m)$. If we denote by $\Im^a(a_0, m)$ the family $\{E \in \Im(a_0, m) : E = E^a\}$, a trivial consequence of the previous paragraphs is that we achieve our goal if we can prove that there exists a set having minimum perimeter in $\Im^a(a_0, m)$.

The symmetry allows us to restrict our analysis to the part of the plane (ξ, η) with $\xi, \eta \geq 0$; we observe that any element in $\Im^a(a_0, m)$ is uniquely determined by the point $P \equiv \left(\xi_0, \sqrt{\dfrac{m}{\pi} - \xi_0^2} \right)$ intersection of its boundary with ∂D. A straightforward calculation shows that $\Im^a(a_0, m)$ is a one parameter family of sets. Indeed, if $0 < \bar{\xi} < \left(\dfrac{m}{\pi} \right)^{\frac{1}{2}}$ is chosen such that

$$(5.2) \qquad a_0 = \frac{1}{m} \left(2 \left(\frac{m}{\pi} \right)^{\frac{1}{2}} \arcsin \left(\bar{\xi} \left(\frac{\pi}{m} \right)^{\frac{1}{2}} \right) - 2 \bar{\xi} \sqrt{\frac{m}{\pi} - \bar{\xi}^2} \right),$$

for any given $\bar{\xi} \leq \xi_0 < \left(\dfrac{m}{\pi} \right)^{\frac{1}{2}}$, there exists only one set $E_{\xi_0} \in \Im^a(a_0, m)$ whose boundary passes through the point $P \equiv \left(\xi_0, \sqrt{\dfrac{m}{\pi} - \xi_0^2} \right)$ and vice versa. Since the isoperimetric deficit $\Delta P(E_{\xi_0})$ changes continuously with respect to ξ_0 and

$$\lim_{\xi_0 \to \left(\frac{m}{\pi} \right)^{\frac{1}{2}}} \Delta P(E_{\xi_0}) = +\infty,$$

we can conclude that, among the sets belonging to $\Im^a(a_0, m)$, there exists at least one having minimum perimeter.

Now, it is not difficult to see that S_{a_0} belongs to $\Im^a(a_0, \pi)$ and that, up to a similarity, it is the unique smooth set (with C^1 boundary) in $\Im^a(a_0, m)$. It is finally possible to provide a simple variational argument to show that a non smooth set in $\Im^a(a_0, m)$ can not be the *optimal* set.

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