Bollettino Unione Matematica Italiana

Fabrizio Caselli

On the Multivariate Robinson-Schensted Correspondence

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **1** (2008), n.3, p. 591–602.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2008_9_1_3_591_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana,
2008.

Bollettino U. M. I.
(9) **I** (2008), 591-602

On the Multivariate Robinson-Schensted Correspondence (*)

FABRIZIO CASELLI

Abstract. – *We show the existence of a multivariate extension of the Robinson-Schensted correspondence. This is inspired by the interpretation of the classical two dimensional case in the invariant theory of (finite) reflection groups.*

1. – Introduction.

The Robinson-Schensted correspondence (see [15, 16]) is a bijection between the symmetric group on n elements and the set of ordered pairs of standard tableaux with n boxes with the same shape. This is based on the row bumping algorithm and was originally introduced by Robinson to study the Littlewood-Richardson rule and by Schensted to study the lengths of increasing subsequences of a word. This algorithm has found applications in the representation theory of the symmetric group, in the theory of symmetric functions and the theory of the plactic monoid. Moreover, it is certainly fascinating from a combinatorial point of view and has inspired a considerable number of papers in the last decades. This correspondence has been generalized to other Weyl groups, by defining ad hoc tableaux, or to semistandard tableaux in the so-called RSK-correspondence, by considering permutations as special matrices with non-negative integer entries.

The invariant theory of finite subgroups generated by reflections has attracted many mathematicians in the last fifty years since their classification in the works of Chevalley [8] and Shepard and Todd [17] with particular attention on the combinatorial aspects of it. The main goal of this communication is to explain the relationship between the Robinson-Schensted correspondence and the theory of invariants of a reflection group. We are naturally carried to consider the Hilbert series of multigraded algebras arising by considering invariant and coinvariant algebra associated to a reflection group. By interpreting this Hilbert series in terms of tableaux and in terms of permutations we deduce the existence of a multivariate extension of the Robinson-Schensted correspondence.

(*) Comunicazione tenuta a Bari il 26 settembre 2007 in occasione del XVIII Congresso dell'Unione Matematica Italiana.

2. – Multimahonian distributions.

Let V be a finite dimensional $\mathbb{C}$-vector space and W be a finite subgroup of the general linear group $GL(V)$ generated by reflections, i.e. elements that fix a hyperplane pointwise. We refer to such a group simply as a *reflection group*. The most significant example of such a group is the symmetric group acting by permuting a fixed linear basis of V. Other important examples are Weyl groups acting on the corresponding root space. Though many of the results we are going to discuss in this communication are valid in the generality of reflection groups, for the sake of clearness we concentrate on the case of the symmetric group. Despite this we preserve the symbol W to denote the symmetric group S_n.

Given a permutation $\sigma \in W$ we denote by

$$\mathrm{Des}\,(\sigma) \stackrel{\mathrm{def}}{=} \{i | \sigma(i) > \sigma(i+1)\}$$

the (right) *descent set* of σ and its *major index* by

$$\mathrm{maj}\,(\sigma) \stackrel{\mathrm{def}}{=} \sum_{i \in \mathrm{Des}\,(\sigma)} i.$$

For example if $\sigma = 35241$ we have $\mathrm{Des}\,(\sigma) = \{2, 4\}$ and $\mathrm{maj}\,(\sigma) = 6$. We recall the following equidistribution result due to MacMahon (see [13]).

THEOREM 2.1. – *We have*

$$W(q) \stackrel{\mathrm{def}}{=} \sum_{\sigma \in W} q^{\mathrm{maj}\,(\sigma)} = \sum_{\sigma \in W} q^{\mathrm{inv}(\sigma)} = \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^i),$$

where $\mathrm{inv}(\sigma) = |\{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$ *is the number of inversions of* σ.

If W is a generic reflection group we can consider the polynomial $W(q) = \sum_{w \in W} q^{\ell(w)}$, where $\ell(w)$ is the length function on W: in this case one obtains an explicit expression for $W(q)$ in terms of the degrees of W.

The dual action of W on V^* can be extended to the symmetric algebra $S(V^*)$ of polynomial functions on V. If we fix a basis of V, the symmetric algebra is naturally identified with the algebra of polynomials $\mathbb{C}[X]$. Here and in what follows we use the symbol X to denote an n-tuple of variables $X = (x_1, \ldots, x_n)$. The symmetric group acts on $\mathbb{C}[X]$ by permuting the variables. As customary we denote by $\mathbb{C}[X]^W$ the ring of invariant polynomials (fixed points of the action of W). We also denote by I_+^W the ideal of $\mathbb{C}[X]$ generated by homogeneous polynomials in $\mathbb{C}[X]^W$ of strictly positive degree. The *coinvariant algebra* associated to W is defined as the corresponding quotient algebra

$$R^W \stackrel{\text{def}}{=} \mathbb{C}[X]/I_+^W.$$

The coinvariant algebra has important applications in the theory of representation since it is isomorphic to the group algebra of W and in the topology of the flag variety since it is isomorphic to the its cohomology ring.

We say that a $\mathbb{C}$-algebra A is *multigraded* in $\mathbb{N}^k$ if

$$A = \bigoplus_{I \in \mathbb{N}^k} A_I,$$

where the A_I's are suitable finite dimensional vector subspaces of A such that $A_I \cdot A_J \subseteq A_{I+J}$. If $a \in A_I$ we say that a is homogeneous of (multi)degree I. We can extend this definition to the category of A-modules by saying that an A-module R is multigraded in $\mathbb{N}^k$ if

$$R = \bigoplus_{I \in \mathbb{N}^k} R_I,$$

where R_I is a A-submodule which is a finite dimensional $\mathbb{C}$-vector space and $A_I \cdot R_J \subseteq R_{I+J}$.

If R is a multigraded A-module we can record the dimensions of its homogeneous components via its Hilbert series

$$\operatorname{Hilb}(R)(q_1, \ldots, q_k) \stackrel{\text{def}}{=} \sum (\dim R_{a_1,\ldots,a_k}) q_1^{a_1} \cdots q_k^{a_k}.$$

We note that, since the ideal I_+^W is generated by homogeneous polynomials (by total degree) the coinvariant algebra is graded in $\mathbb{N}$. It turns out that the polynomial $W(q)$ appearing in Theorem 2.1 is the Hilbert series of the coinvariant algebra R^W:

$$W(q) = \operatorname{Hilb}(R^W)(q).$$

This is a crucial example of interplay between the invariant theory of W and the combinatorics of W (by Theorem 2.1). All the other cases considered in this paper are algebraic and combinatorial variations and generalizations of this fundamental fact.

Given an irreducible representation λ of W let $f^\lambda(q)$ be the polynomial in $\mathbb{N}[q]$ whose coefficient of q^i is the multiplicity of the representation λ in the homogeneous component of degree i in R^W, i.e.

$$f^\lambda(q) = \sum \langle \chi^\lambda, \chi(R_k^W) \rangle q^k.$$

In this formula we denote by $\chi(\rho)$ the character of a representation ρ and by $\langle \cdot, \cdot \rangle$ the Hermitian form on the space of class functions on W with respect to which the characters of the irreducible representations form an orthonormal basis. The polynomials $f^\lambda(q)$ are known as the *fake degree polynomials* and have a very simple combinatorial interpretation based on standard tableaux that we are going to describe.

Given a partition λ of n, the *Ferrers diagram of shape* λ is a collection of boxes, arranged in left-justified rows, with λ_i boxes in row i. A *standard tableau* of shape λ is a filling of the Ferrers diagram of shape λ using the numbers from 1 to n, each occurring once, in such way that rows are increasing from left to right and columns are increasing from top to bottom. For example the following picture

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

represents a standard tableau of shape $(3,2,1,1)$. We say that i is a *descent* of a standard tableau T if i appears strictly above $i+1$ in T. We denote by $\mathrm{Des}\,(T)$ the set of descents of T and we let $\mathrm{maj}\,(T)$ be the sum of its descents. Finally we denote by $\lambda(T)$ the shape of T. In the previous example we have $\mathrm{Des}\,(T) = \{1,3,5,6\}$ and so $\mathrm{maj}\,(T) = 15$.

It is known that irreducible representations of the symmetric group W are indexed by partitions of n. We therefore use the same symbol λ to denote a partition or the corresponding Specht module. The following theorem is attributed to Lusztig (unpublished) and to Kraśkiewicz and Weyman ([12]).

THEOREM 2.2. – *Let λ be a partition of n. We have*

$$f^\lambda(q) = \sum_{\{T:\lambda(T)=\lambda\}} q^{\mathrm{maj}\,(T)}.$$

An analogous result holds for Weyl groups of type B and D and is due to Stembridge ([19]).

Since R^W is isomorphic to the group algebra $\mathbb{C}W$ as a W-module we deduce the following identity

$$W(q) = \sum_\lambda f^\lambda(1) f^\lambda(q).$$

Considering the following natural generalization

$$W(q,t) \stackrel{\text{def}}{=} \sum_\lambda f^\lambda(t) f^\lambda(q)$$

we fall again on another Hilbert polynomial. In fact we consider $W \times W$ as a finite reflection group in $GL(V \oplus V)$ and we denote by ΔW the diagonal subgroup of $W \times W$. The symmetric algebra on $V \oplus V$ can be identified with the polynomial algebra $\mathbb{C}[X, Y]$ in the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. The following result is due to Barcelo, Reiner and Stanton ([4]).

THEOREM 2.3. – *We have*

$$W(q,t) = \mathrm{Hilb}\,(\mathbb{C}[X,Y])^{\Delta W}/I_+^{W\times W}).$$

Following [4] we refer to the polynomial $W(q,t)$ as the *bimahonian distribution* of W.

We recall that the algebra $(\mathbb{C}[X,Y])^{\Delta W}$ is a Cohen-Macauley algebra (being the invariant algebra of a finite reflection group). In particular it is a free module on the subalgebra $\mathbb{C}[X,Y]^{W\times W}$. It follows another interpretation for the polynomial $W(q,t)$:

$$W(q,t) = \frac{\mathrm{Hilb}\,(\mathbb{C}[X,Y]^{\Delta W})}{\mathrm{Hilb}\,(\mathbb{C}[X,Y]^{W\times W})}.$$

From the study of the two Hilbert series appearing in the previous formula, by means of the theory of bipartite partitions of Gordon [11] and the results of Garsia and Gessel appearing in [10], one can prove the following combinatorial interpretation for their quotient.

THEOREM 2.4. – *We have*

$$W(q,t) = \sum_{\sigma\in W} q^{\mathrm{maj}\,(\sigma)} t^{\mathrm{maj}\,(\sigma^{-1})}.$$

Also in this case, analogous results have been obtained for Weyl groups of type B by Adin and Roichman [2] and for Weyl groups of type D by Biagioli and the present author [6]. We can summarize the previous results in the following way:

$$W(q,t) = \sum_{\{S,T:\lambda(S)=\lambda(T)\}} q^{\mathrm{maj}\,(S)} t^{\mathrm{maj}\,(T)} \tag{1}$$

$$(\text{Lusztig, Kraskiewicz and Weyman}) = \sum_{\lambda} f^\lambda(q) f^\lambda(t) \tag{2}$$

$$(\text{Barcelo, Reiner, Stanton}) = \mathrm{Hilb}\left(\mathbb{C}[X,Y]^{\Delta W}/I_+^{W\times W}\right) \tag{3}$$

$$(\text{Cohen} - \text{Macauleyness}) = \frac{\mathrm{Hilb}\,((\mathbb{C}[X,Y])^{\Delta W})}{\mathrm{Hilb}\,(\mathbb{C}[X,Y]^{W\times W})} \tag{4}$$

$$(\text{Garsia} - \text{Gessel}) = \sum_{\sigma\in S_n} q^{\mathrm{maj}\,(\sigma)} t^{\mathrm{maj}\,(\sigma^{-1})} \tag{5}$$

The equality between the first and the last row of the previous sequence of equalities follows also from the Robinson-Schensted correspondence. We do not give an explicit description of such correspondence (and for this we refer the reader to [18, § 7.11]) but we simply state its main properties.

THEOREM 2.5 (Robinson-Schensted correspondence). – *There exists a (explicit) map* $\sigma \mapsto (Q(\sigma), P(\sigma))$ *(where* $\sigma \in S_n$*) with the following properties*

1. $Q(\sigma)$ *and* $P(\sigma)$ *are standard tableaux with* n *boxes;*
2. $Q(\sigma)$ *and* $P(\sigma)$ *have the same shape;*
3. $\mathrm{Des}(\sigma) = \mathrm{Des}(Q(\sigma))$ *and* $\mathrm{Des}(\sigma^{-1}) = \mathrm{Des}(P(\sigma))$*;*
4. *The map* $\sigma \mapsto (Q(\sigma), P(\sigma))$ *is a bijection between* S_n *and the set of ordered pairs of standard tableaux of the same shape of size* n*.*

The main goal of this communication is to see the sequence of equalities (1)–(5) as a 2-dimensional case of a more general result. To illustrate the proper multivariate generalization we need a further ingredient coming from the representation theory. Given k irreducible representations $\lambda^{(1)}, \ldots, \lambda^{(k)}$ of a finite group W we let

$$\begin{aligned} d_{\lambda^{(1)},\ldots,\lambda^{(k)}} &\stackrel{\mathrm{def}}{=} \frac{1}{|W|}\sum_{w\in W} \chi^{\lambda^{(1)}}(w)\cdots\chi^{\lambda^{(k)}}(w) \\ &= \langle \chi^{\lambda^{(1)}}\cdots\chi^{\lambda^{(k-1)}}, \chi^{\lambda^{(k)}} \rangle. \end{aligned}$$

In other words $d_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ is the multiplicity of $\lambda^{(k)}$ in the (reducible) representation $\lambda^{(1)} \otimes \cdots \otimes \lambda^{(k-1)}$. These numbers have been deeply studied in the literature (see, i.e. [5, 9, 14]) though they do not have an explicit description such as a combinatorial interpretation.

Some further notation is necessary. We let $\mathbb{C}[X_1, \ldots, X_k]$ be the algebra of polynomials in the nk variables $x_{i,j}$, with $i = 1, \ldots, k$ and $j = 1, \ldots, n$, i.e. we use the capital variable X_i for the n-tuple of variables $x_{i,1}, \ldots, x_{i,n}$. The group W^k is considered as a finite reflection subgroup of $GL(V^{\oplus k})$ and the group ΔW is the diagonal subgroup of W^k. We note that the algebra $\mathbb{C}[X_1, \ldots, X_k]$ is multigraded in $\mathbb{N}^k$: a monomial is homogeneous of multidegree $(a_1, \ldots, a_k)$ if the total degree in the variables $x_{i,1}, \ldots, x_{i,n}$ is a_i for all $i = 1, \ldots, k$. This multidegree is inherited also by the algebra $\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}/I_+^{W^k}$ since the multidegree is preserved by the action of ΔW and the ideal $I_+^{W^k}$ is generated by homogeneous polynomials. The following is a generalization of Theorem 2.3.

THEOREM 2.6. – *We have*

$$\mathrm{Hilb}\left(\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}/I_+^{W^k}\right) = \sum_{\lambda^{(1)},\ldots,\lambda^{(k)}} d_{\lambda^{(1)},\ldots,\lambda^{(k)}} f^{\lambda^{(1)}}(q_1)\cdots f^{\lambda^{(k)}}(q_k).$$

Note that for $k=2$ we obtain Theorem 2.3 since $d_{\lambda^{(1)},\lambda^{(2)}} = \delta_{\lambda^{(1)},\lambda^{(2)}}$, the latter being the Kronecker symbol. By Theorem 2.6 and the appropriate generalizations of the other results of the bidimensional case, we may write the following multivariate extension of (1)-(5).

COROLLARY 2.1. – *We have*

$$W(q_1,\ldots,q_k) \stackrel{\text{def}}{=} \sum_{\{T_1,\ldots,T_k\}} d_{\lambda(T_1),\ldots,\lambda(T_k)} q_1^{\operatorname{maj}(T_1)} \ldots q_k^{\operatorname{maj}(T_k)} \tag{6}$$

$$= \sum_{\{\lambda^{(1)},\ldots,\lambda^{(k)}\}} d_{\lambda^{(1)},\ldots,\lambda^{(k)}} f^{\lambda^{(1)}}(q_1)\cdots f^{\lambda^{(k)}}(q_k) \tag{7}$$

$$= \operatorname{Hilb}\left(\mathbb{C}[X_1,\ldots,X_k]^{\Delta W}/I_+^{W^k}\right) \tag{8}$$

$$= \frac{\operatorname{Hilb}(\mathbb{C}[X_1,\ldots,X_k]^{\Delta W})}{\operatorname{Hilb}(\mathbb{C}[X_1,\ldots,X_k]^{W^k})} \tag{9}$$

$$= \sum_{\substack{\sigma_1,\ldots,\sigma_k:\\ \sigma_1\cdots\sigma_k=1}} q_1^{\operatorname{maj}(\sigma_1)} \cdots q_k^{\operatorname{maj}(\sigma_k)}, \tag{10}$$

where the sum in (6) *is on all k-tuples of standard tableaux of size n, the sum in* (7) *is on all k-tuples of partitions of n, and the sum in* (10) *is on all k-tuples of permutations in* S_n *whose product is the identity element.*

We refer to the polynomial $W(q_1,\ldots,q_k)$ appearing in Corollary 2.1 as the *multimahonian distribution*. Considering the first and the last line of Corollary 2.1 we can deduce the existence of a multivariate Robinson-Schensted correspondence.

COROLLARY 2.2. – *There exists a map Q that associates to any k-tuple of permutations in* S_n, $(\sigma_1,\ldots,\sigma_k)$ *whose ordered product is the identity, a k-tuple of standard tableaux with n boxes satisfying the following two conditions*

1. *For every k-tuple of standard tableaux* $(T_1,\ldots,T_k)$,

   $$\#Q^{-1}(T_1,\ldots,T_k) = d_{\lambda(T_1),\ldots,\lambda(T_k)}.$$

   In particular $\#Q^{-1}(T_1,\ldots,T_k)$ *depends only on the shapes of the tableaux* $T_1,\ldots,T_k$;
2. *If* $Q(\sigma_1,\ldots,\sigma_k) = (T_1,\ldots,T_k)$ *then* $\operatorname{maj}(T_i) = \operatorname{maj}(\sigma_i)$ *for all* $i = 1,\ldots,k$.

3. – Refined multimahonian distributions.

From a combinatorial point of view the results appearing in the previous section can be slightly refined. This is essentially due to a further decomposition of the homogeneous pieces of the coinvariant algebra that can be described in terms of descents of permutations and descents of tableaux. This decomposition has been originally obtained in a work of Adin, Brenti and Roichman [1] for Weyl groups of type A and B (see also [7] for Weyl groups of type D and [3] for all complex reflection groups).

If M is a monomial in $\mathbb{C}[X]$ we denote by $\lambda(M)$ its *exponent partition*, i.e. the partition obtained by rearranging the exponents of M. We say that a polynomial is homogeneous of degree λ if it is the sum of monomials whose exponent partition is λ. We note that the exponent partition is not well-defined in the coinvariant algebra. For example, for $n = 3$ the monomials x_1^2 and x_2x_3 are in the same class in the coinvariant algebra (since $x_1^2 - x_2x_3 = x_1(x_1 + x_2 + x_3) - (x_1x_2 + x_1x_3 + x_2x_3)$), though they have distinct exponent partitions.

We recall the definition of the *dominance order* on the set of partitions of n. We write $\lambda \lhd \mu$, and we say that λ is smaller than μ in the dominance order, if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i. We let $R_\lambda^{(1)}$ be the subspace of R^W consisting of elements that can be represented as the sum of monomials with exponent partition smaller than or equal to λ in dominance order. We also denote by $R_\lambda^{(2)}$ the subspace of R^W consisting of elements that can be represented as the sum of monomials with exponent partition strictly smaller than λ in dominance order. The subspaces $R_\lambda^{(1)}$ and $R_\lambda^{(2)}$ are also W-submodules of R^W and we denote their quotient by

$$R_\lambda \stackrel{\text{def}}{=} R_\lambda^{(1)}/R_\lambda^{(2)}.$$

The W-modules R_λ provide a further decomposition of the homogeneous components of the coinvariant algebra R^W.

THEOREM 3.1. – *There exists an isomorphism of W-modules*

$$\varphi : R_k^W \xrightarrow{\cong} \bigoplus_{|\lambda|=k} R_\lambda,$$

such that $\varphi^{-1}(R_\lambda)$ can be represented by homogeneous polynomials of degree λ.

We can use this result to define a multidegree on the coinvariant algebra: we simply say that an element in R_k^W is homogeneous of degree λ if its image under the isomorphism φ is in R_λ. We can therefore define the Hilbert polynomial of R^W with respect to this degree by

$$\text{Hilb}\,(R^W)(q_1, \ldots, q_n) = \sum_\lambda (\dim R_\lambda) q_1^{\lambda_1} \cdots q_n^{\lambda_n}.$$

Putting $q = q_1 = q_2 = \cdots q_n$ we clearly obtain the Hilbert polynomial with respect to the total degree.

It turns out that if $\dim R_\lambda \neq 0$ then $\lambda_{i+1} - \lambda_i = 0, 1$ for all i and in this case

$$\dim R_\lambda = |\{\sigma \in S_n : i \in \operatorname{Des}(\sigma) \text{ if and only if } \lambda_{i+1} - \lambda_i = 1\}|.$$

By means of this decomposition of the coinvariant algebra we can also decompose the algebra

$$\frac{\mathbb{C}[X_1, \ldots, X_k]}{I_+^{W^k}}$$

and its subalgebra

$$\left(\frac{\mathbb{C}[X_1, \ldots, X_k]}{I_+^{W^k}}\right)^{\Delta W} \cong \frac{\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}}{I_+^{W^k}}$$

in homogeneous components whose degrees are k-tuples of partitions with at most n parts. This is because we have the canonical isomorphism

$$\frac{\mathbb{C}[X_1, \ldots, X_k]}{I_+^{W^k}} \cong (R^W)^{\otimes k}.$$

We can therefore consider its Hilbert polynomial

$$\operatorname{Hilb}\left(\frac{\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}}{I_+^{W^k}}\right) \overset{\text{def}}{=} \sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} \dim\left(\frac{\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}}{I_+^{W^k}}\right)_{\lambda^{(1)}, \ldots, \lambda^{(k)}} Q_1^{\lambda^{(1)}} \cdots Q_k^{\lambda^{(k)}}.$$

In this formula the symbol Q_i stands for the n-tuple of variables $q_{i,1}, \ldots, q_{i,n}$.

We define the *refined fake degree polynomial* by $f^\lambda(q_1, \ldots, q_k)$ as the polynomial whose coefficient of $q_1^{\mu_1} \cdots q_k^{\mu_k}$ is the multiplicity of the representation λ in R_μ. The following is the main result of this communication.

THEOREM 3.2. – *We have*

$$\operatorname{Hilb}\left(\frac{\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}}{I_+^{W^k}}\right)(Q_1, \ldots, Q_k) = \sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} d_{\lambda^{(1)}, \ldots, \lambda^{(k)}} f^{\lambda^{(1)}}(Q_1) \cdots f^{\lambda^{(k)}}(Q_k).$$

Given a tableau or a permutation S we define a partition $\mu(S)$ by putting

$$(\mu(S))_i = |\operatorname{Des}(S) \cap \{i, \ldots, n\}|.$$

The following result appearing in [1] describes explicitly the decomposition into irreducibles of the W-modules R_λ.

THEOREM 3.3. – *The multiplicity of the representation μ in R_λ is*

$$|\{T \text{ tableau} : \lambda(T) = \lambda \text{ and } \mu(T) = \mu\}|$$

and so

$$f^\lambda(q_1, \ldots, q_n) = \sum_{\{T : \lambda(T) = \lambda\}} Q^{\mu(T)}.$$

Thanks to this result we can deduce the following sequence of identities

$$\sum_{\lambda^{(1)},\dots,\lambda^{(k)}} d_{\lambda^{(1)},\dots,\lambda^{(k)}} f^{\lambda^{(1)}}(Q_1)\cdots f^{\lambda^{(k)}}(Q_k)$$

$$= \sum_{T_1,\dots,T_k} d_{\lambda(T_1),\dots,\lambda(T_k)} Q_1^{\mu(T_1)}\cdots Q_k^{\mu(T_k)}$$

$$= \mathrm{Hilb}\,\frac{\mathbb{C}[X_1,\dots,X_k]^{\Delta W}}{(\mathbb{C}[X_1,\dots,X_k]_+^W)}(Q_1,\dots,Q_k)$$

What happens from the point of view of permutations? The algebra of polynomials in nk variables $\mathbb{C}[X_1\dots,X_k]$ is also multigraded by k-tuples of partitions with at most n parts: we just say that a monomial is homogeneous of multidegree $(\lambda^{(1)},\dots,\lambda^{(k)})$ if its exponent partition with respect to the variables $x_{i,1},\dots x_{i,n}$ is $\lambda^{(i)}$ for all i. On this algebra we consider the actions of W^k and its diagonal subgroup ΔW, which are compatible with our multidegree. Using the results on multipartite partitions in [10] we can prove the following formula for the quotient of the Hilbert polynomials associated to the invariant algebras of ΔW and W^k.

THEOREM 3.4. – *We have*

$$\frac{\mathrm{Hilb}\,(\mathbb{C}[X_1\dots,X_k]^{\Delta W})(Q_1,\dots,Q_k)}{\mathrm{Hilb}\,(\mathbb{C}[X_1\dots,X_k]^{W^k})(Q_1,\dots,Q_k)} = \sum_{\sigma_1\cdots\sigma_k=1} Q_1^{\mu(\sigma_1)}\cdots Q_k^{\mu(\sigma_k)}$$

Putting these results together we obtain the following sequence of (conjecturally) equivalent interpretations for what we call the *refined multimahonian distribution*:

$$W(Q_1,\dots,Q_k) = \sum_{T_1,\dots,T_k} d_{\lambda(T_1),\dots,\lambda(T_k)} Q_1^{\mu(T_1)},\dots,Q_k^{\mu(T_k)}$$

$$= \sum_{\lambda^{(1)},\dots,\lambda^{(k)}} d_{\lambda^{(1)},\dots,\lambda^{(k)}} f^{\lambda^{(1)}}(Q_1)\cdots f^{\lambda^{(k)}}(Q_k)$$

$$= \mathrm{Hilb}\left(\frac{\mathbb{C}[X_1,\dots,X_k]^{\Delta W}}{(\mathbb{C}[X_1,\dots,X_k]_+^W)}\right)(Q_1,\dots,Q_k)$$

$$\stackrel{?}{=} \frac{\mathrm{Hilb}\,(\mathbb{C}[X_1\dots,X_k]^{\Delta W})(Q_1,\dots,Q_k)}{\mathrm{Hilb}\,(\mathbb{C}[X_1\dots,X_k]^{W^k})(Q_1,\dots,Q_k)}$$

$$= \sum_{\sigma_1\cdots\sigma_k=1} Q_1^{\mu(\sigma_1)}\cdots Q_k^{\mu(\sigma_k)}$$

The equality marked by $\stackrel{?}{=}$ is still a conjecture since I do not have a complete proof of it. The following conjecture is a straightforward consequence.

CONJECTURE 3.1 – *There exists a map Q that associates to any k-tuple $(\sigma_1, \ldots, \sigma_k)$ of permutations in S_n whose ordered product is the identity, a k-tuple of standard tableaux with n boxes satisfying the following two conditions:*

1. *For every k-tuple of tableaux $(T_1, \ldots, T_k)$,*

$$\#Q^{-1}(T_1, \ldots, T_k) = d_{\lambda(T_1), \ldots, \lambda(T_k)}.$$

In particular it depends only on the shapes of the tableaux $T_1, \ldots, T_k$;

2. *If $Q(\sigma_1, \ldots, \sigma_k) = (T_1, \ldots, T_k)$ then* $\mathrm{Des}\,(T_i) = \mathrm{Des}\,(\sigma_i)$ *for all $i = 1, \ldots, k$.*

The classical Robinson-Schensted correspondence provides a bijective proof of this Corollary in the case $k = 2$.

We also think that the correspondence Q of Conjecture 3.1 should be well-behaved with respect to cyclic permutations of the arguments in the sense that if $Q(\sigma_1, \ldots, \sigma_k) = (T_1, \ldots, T_k)$ then $Q(\sigma_2, \ldots, \sigma_k, \sigma_1) = (T_2, \ldots, T_k, T_1)$.

REFERENCES

[1] R. ADIN - F. BRENTI - Y. ROICHMAN, *Descent representations and multivariate statistics*, Trans. Amer. Math. Soc., **357**, no. 8 (2005), 3051-3082.

[2] R. ADIN - Y. ROICHMAN, *The flag major index and group actions on polynomial rings*, Europ. J. Combin., **22** (2001), 431-446.

[3] E. BAGNO - R. BIAGIOLI, *Colored-descent representations of complex reflection groups $G(r, p, n)$*. Israel J. Math., **160** (2007), 317-347.

[4] H. BARCELO - V. REINER - D. STANTON, *Bimahonian distributions*, preprint (arXiv CO/0703479v1), 2007.

[5] C. BESSENRODT - A. KLESHCHEV, *On Kronecker products of complex representations of the symmetric and alternating groups*, Pacific J. Math., **190**, no. 2 (1999), 201-223.

[6] R. BIAGIOLI - F. CASELLI, *Invariant algebras and major indices for classical Weyl groups*, Proc. London Math. Soc. (3) **88**, no. 3 (2004), 603-631.

[7] R. BIAGIOLI - F. CASELLI, *A descent basis for the coinvariant algebra of type D.* J. Algebra, **275**, no. 2 (2004), 517-539.

[8] C. CHEVALLEY, *Invariants of finite groups generated by reflections*, Amer. J. Math., **77** (1955), 778-782

[9] Y. DVIR, *On the Kronecker product of S_n characters, 1* J. Algebra, **154**, no. 1 (1993), 125-140.

[10] A. GARSIA - I. GESSEL, *Permutation statistics and partitions*, Adv. Math., **31** (1979) 288-305.

[11] B. GORDON, *Two theorems on multipartite partitions*, J. London Math. Soc., **38** (1963), 459-464.

[12] W.KRAŚKIEWICZ - J. WEYMAN, *Algebra of coinvariants and the action of a Coxeter element*, Bayreuth. Math. Schr. no. 63 (2001), 265-284.

[13] P. A. MACMAHON, *Combinatory analysis*, vol. 1, Cambridge University Press, London, 1915.

[14] A. REGEV, *On the height of the Kronecker product of S_n characters*, Israel J. Math., **42**, no. 1-2 (1982), 60-64.

[15] G. de B. Robinson, On the representations of the symmetric group, Amer. J. Math., **60**, no. 3 (1938), 745-760.
[16] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math., **13** (1961), 179-191.
[17] G. C. Shephard - J. A. Todd, *Finite unitary reflection groups*, Canadian J. Math., **6** (1954) 274-304.
[18] R. P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, Cambridge, 1999.
[19] J.R. Stembridge, *On the eigenvalues of representations of reflection groups and wreath products*. Pacific J. Math., **140**, no. 2 (1989), 353-396.

Dipartimento di Matematica,
Università di Bologna, Piazza di Porta San Donato 5, Bologna 40126, Italy
e-mail: caselli@dm.unibo.it

Received January 23, 2008 and in revised form March 6, 2008