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Nonlinear Elliptic Equations with Lower Order Terms and Symmetrization Methods

ANGELO ALVINO - ANNA MERCALDO

Dedicated to the memory of Guido Stampacchia

Abstract. – *We consider the homogeneous Dirichlet problem for nonlinear elliptic equations as*

$$-\operatorname{div}\mathbf{a}(x,\nabla u) = b(x,\nabla u) + \mu\,,$$

where μ is a measure with bounded total variation. We fix structural conditions on functions $\mathbf{a}$, b which ensure existence of solutions. Moreover, if μ is an L^1 function, we prove a uniqueness result under more restrictive hypotheses on the operator.

1. – Introduction.

Let us consider the homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div}\mathbf{a}(x,\nabla u) = b(x,\nabla u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$.

We assume that

$$\mathbf{a} \,:\, (x,z) \in \Omega \times \mathbb{R}^N \longrightarrow \mathbf{a}(x,z) = (a_i(x,z)) \in \mathbb{R}^N$$

and

$$b \,:\, (x,\xi) \in \Omega \times \mathbb{R}^N \longrightarrow b(x,\xi) \in \mathbb{R}$$

are Carathéodory functions such that

$$\mathbf{a}(x,\xi)\cdot\xi \geq |\xi|^p\,, \tag{1.2}$$

$$|\mathbf{a}(x,\xi)| \leq A|\xi|^{p-1}\,, \tag{1.3}$$

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta))\cdot(\xi-\eta) > 0\,, \qquad \xi \neq \eta\,, \tag{1.4}$$

$$|b(x,\xi)| \leq B(x)|\xi|^{p-1}\,, \tag{1.5}$$

where $p \in\,]1,N[$, A is a positive constant and $B \in L^r(\Omega)$ for some $r > N$.

Finally μ denotes a Radon measure with bounded total variation; nevertheless it is an $L^1(\Omega)$ function when we deal with uniqueness results.

These conditions on μ do not allow us to refer to the notion of weak solution. Indeed it works well when the known term belongs to the dual space $W^{-1,p'}$ of $W_0^{1,p}(\Omega)$; in this case (1.2), (1.3), (1.4), (1.5) (see [18] and also [13]) guarantee the existence of a unique $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega (\mathbf{a}(x, \nabla u) \cdot \nabla\varphi)\, dx = \int_\Omega b(x, \nabla u)\varphi\, dx + \int_\Omega \varphi\, d\mu, \quad \forall \varphi \in W_0^{1,p}(\Omega)\,. \tag{1.6}$$

Even the notion of solution in the distributional sense is not admissible; indeed the classical Serrin's example [24] shows that a "local" uniqueness result for Dirichlet problem does not hold.

So, which is the right definition of solution when the datum is a Radon measure? This question goes back to Stampacchia's paper [25] where the Green function of a linear elliptic operator is studied. First he obtains suitable a priori estimates for weak solutions in terms, for example, of the L^1 norm of μ, then, since the problem is linear, by density arguments he gets a solution to the problem when the datum is a Dirac delta-function. Moreover he characterizes the set of test functions to be used in (1.6).

This same procedure cannot be extended to nonlinear cases. The first results in this direction are due to Boccardo and Gallouët. In [7] they prove, for operators without lower order terms, some a priori estimates which are the starting point of a delicate limit procedure under integral sign performed to define a solution known as Solution Obtained as a Limit of Approximations (SOLA for short). Furthermore it is also solution in the distributional sense.

The procedure can be simplified if a continuous dependence from data result is available; this holds (see [10]) changing (1.4) into the following "strong monotonicity" conditions

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) \geq |\xi - \eta|^p\,, \qquad \text{if } p \geq 2\,, \tag{1.7}$$

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) \geq \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}\,, \qquad \text{if } 1 < p < 2\,. \tag{1.8}$$

We point out that (1.7), (1.8) can be easily obtained if $\mathbf{a}(x, \mathbf{0}) = \mathbf{0}$ and the following ellipticity condition

$$\gamma |z|^{p-2} |\xi|^2 \leq \sum_{i,j=1}^N \frac{\partial a_i}{\partial z_j}(x,z)\xi_i\xi_j\,, \quad (x,z,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N\,,$$

with a suitable γ, is satisfied.

It is worth reminding there are other notions of solution, i.e. the entropy solution ([3]) and the renormalized solution ([19], [21]). In both these cases the same limit procedure is performed; however also the question of the choice of test functions to put in (1.6) is stressed. We refer to [11] for a complete list of references on this subject.

In [1] we suggest a different approach to problem (1.1) which follows classical symmetrization methods, made famous by Talenti and Maz'ya (see, for instance, [26] and [20]). In this work we extend the method to operators with lower order terms. The main part in the approach consists in choosing suitable test functions, built on the level sets of the solution; the rest of the process is based, as usual, on some isoperimetric inequalities and properties of rearrangements. By this way we obtain a priori estimates, in some sense optimal, which, on the one side, allow us to start the limit procedure mentioned above, and, on the other side, suggest the right set of test functions to put in (1.6).

Moreover, for sake of simplicity, we assume $p > 2 - \dfrac{1}{N}$ to ensure that the gradient of the solution is at least summable (see [7], [8]).

In Section 2 we prove the following existence result.

THEOREM 1.1. – *Let assumptions* (1.3), (1.5), (1.7) *or* (1.8) *hold; if μ is a Radon measure with bounded total variation, then there exists at least a SOLA to* (1.1) *which belongs to $W_0^{1,q}(\Omega)$ with*

$$q < \frac{N(p-1)}{N-1}. \tag{1.9}$$

We explicitly remark that similar existence results have been proved in [12] by simmetrization methods and in [5] for renormalized solutions; however, in both cases, completely different techniques are used.

As far as uniqueness is concerned the presence of lower order terms does not allow us to use heavily the strong monotonicity conditions to get a continuity with respect to data result. However this can be obtained if we strengthen the structural conditions of $\mathbf{a}$, b and impose further restrictions on the index p. In some sense our procedure can be understood as a kind of linearization of the original problem; this allows us to perform an estimate of the rearrangement of the difference of two solutions again via symmetrization methods.

So we get the following uniqueness results where μ is not a measure any more but merely an L^1 function. We remind that such results have been proved for renormalized solutions in [4].

THEOREM 1.2. – *Assume* $p > 2$, $\mu \in L^1$, (1.3),

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) \geq (1 + |\xi| + |\eta|)^{p-2} |\xi - \eta|^2 \tag{1.10}$$

and

$$|\nabla_z b| \leq H |z|^{p-2}, \tag{1.11}$$

with H positive constant; then problem (1.1) *has a unique SOLA*

$$\begin{cases} p < N\,, & \text{if } N = 3, 4\,, \\ p < \dfrac{2(N-2)}{N-3}\,, & \text{if } N \geq 5\,. \end{cases} \tag{1.12}$$

THEOREM 1.3. – *Assume* $p \in \left]2 - \frac{1}{N}, 2\right]$, $\mu \in L^1$, (1.3), (1.8) *and*

$$|\nabla_z b| \leq H(1 + |z|)^{p-2}\,, \tag{1.13}$$

with H positive constant; then problem (1.1) *has a unique SOLA.*

We remark that the monotonicity condition (1.10) holds if $\mathbf{a}(x, \mathbf{0}) = \mathbf{0}$ and the following uniform ellipticity condition

$$\frac{\partial a_i}{\partial z_j}(x, z)\xi_i\xi_j \geq \lambda(1 + |z|)^{p-2}|\xi|^2\,,$$

with a suitable value of λ, is satisfied.

2. – Existence result.

We begin by recalling some properties of rearrangements. If u is a measurable function in Ω and

$$\nu(t) = |\{x \in \Omega : |u(x)| \geq t\}|\,, \quad t \geq 0$$

is its distribution function, then

$$u^*(s) = \sup\{t \geq 0 \,:\, \nu(t) > s\}\,, \quad s \in (0, |\Omega|)\,,$$

is the decreasing rearrangement of u and $u_*(s) = u^*(|\Omega| - s)$ is the increasing rearrangement of u. If ω_N is the measure of the unit ball of $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered in zero with the same measure as Ω,

$$u^\#(x) = u^*(\omega_N|x|^N)\,, \quad u_\#(x) = u_*(\omega_N|x|^N)\,, \qquad x \in \Omega^\#\,,$$

denote the spherically decreasing and spherically increasing rearrangements of u respectively. We recall the well-known Hardy-Littlewood inequality [15] which is crucial in the following

$$\int_{\Omega^\#} u^\#(x)v_\#(x)dx \leq \int_{\Omega} |u(x)v(x)|dx \leq \int_{\Omega^\#} u^\#(x)v^\#(x)dx\,. \tag{2.1}$$

For any $q \in]1, \infty[$ we set

$$\|u\|_{q,r} = \begin{cases} \left(\omega_N^{\frac{r}{q}-1} \int\limits_{\Omega^{\#}} \left[|x|^{\frac{N}{q}} u^{\#}(x) \right]^r \dfrac{dx}{|x|^N} \right)^{\frac{1}{r}}, & \text{if } r \in]0, +\infty[\,, \\ \sup\limits_{s>0} u^*(s)\, s^{1/q}\,, & \text{if } r = +\infty\,. \end{cases}$$

The Lorentz space $L^{q,r}(\Omega)$ is the collection of all funtions u such that $\|u\|_{q,r}$ is finite. These spaces are in some sense refinements of the Lebesgue spaces. Indeed (see [16], [22]) $L^{q,q}(\Omega) = L^q(\Omega)$ and $L^{q,\infty}(\Omega)$ is the Marcinkiewicz space L^q-weak; moreover, fixed the first index q, they become larger as the second index r increases, while, if Ω is bounded, we have

$$q < q_1 \Longrightarrow L^{q_1,r}(\Omega) \subset L^q(\Omega)\,. \tag{2.2}$$

In order to prove the existence of a SOLA to problem (1.1) we begin, as usual, by assuming that the datum in (1.1) is a smooth function we denote by f.

THEOREM 2.1. – *Let assumptions* (1.2), (1.3), (1.4), (1.5) *hold; if u is a weak solution to problem* (1.1) *with datum $f \in C^\infty(\Omega)$ and q satisfies condition* (1.9) *we have*

$$\| |\nabla u| \|_{L^q} \le C \|f\|_{L^1}^{\frac{1}{p-1}}\,. \tag{2.3}$$

The constant C depends on N, p, q, $|\Omega|$ and $\|B\|_{L^r}$.

PROOF. Let ν be the distribution function of u. Inserting in (1.6) the test function

$$\varphi(x) = \text{sign}\,[u(x)] \int\limits_0^{|u(x)|} [\nu(t)]^a\, dt\,,$$

with

$$a > \frac{N-p}{N(p-1)}\,, \tag{2.4}$$

we have

$$\int\limits_\Omega [\nu(|u(x)|)]^a (\mathbf{a}(x, \nabla u) \cdot \nabla u)\, dx = \int\limits_\Omega b(x, \nabla u) \varphi\, dx + \int\limits_\Omega f \varphi\, dx\,.$$

By (1.2) and (1.5) we obtain

$$\int\limits_\Omega [\nu(|u(x)|)]^a |\nabla u|^p\, dx \le \int\limits_\Omega B |\nabla u|^{p-1} |\varphi|\, dx + \int\limits_\Omega |f| |\varphi|\, dx\,. \tag{2.5}$$

Now we use some arguments of [1]. By Hardy-Littlewood inequality (2.1) we get

$$\int_\Omega [\mu(|u(x)|)]^a |\nabla u|^p \, dx \geq C \| |\nabla u| \|^p_{\frac{p}{1+a},p} \tag{2.6}$$

where, henceforth, we denote by C a constant whose value changes from line to line. Fixed q, by (2.4) we can choose a such that $q < \frac{p}{1+a}$; then, from the embeddings (2.2), we obtain

$$C \, \| |\nabla u| \|_{L^q} \leq \| |\nabla u| \|_{\frac{p}{1+a},p} \, . \tag{2.7}$$

Collecting (2.5), (2.6), (2.7) we have

$$\| |\nabla u| \|^p_{L^q} \leq C \left[\int_\Omega B |\nabla u|^{p-1} |\varphi| \, dx + \int_\Omega |f||\varphi| \, dx \right] . \tag{2.8}$$

To estimate the right-hand side of (2.8) we have to evaluate the L^∞ norm of φ. To this aim we use the following estimate of the decreasing rearrangement of u

$$u^*(s) \leq C \|f\|^{\frac{1}{p-1}}_{L^1} \, s^{-\frac{N-p}{N(p-1)}} \, , \tag{2.9}$$

proved in [20] and [27] for linear operators, in [6] for nonlinear ones as immediate consequence of sharp comparison results between rearrangements of solutions to elliptic problems and solutions to suitable problems with radially symmetric data. By (2.9) and (2.4) we get

$$\|\varphi\|_{L^\infty} = \int_0^{+\infty} [v(t)]^a \, dt = a \int_0^{|\Omega|} s^{a-1} u^*(s) \, ds \leq C \|f\|^{\frac{1}{p-1}}_{L^1} \, . \tag{2.10}$$

If we choose q such that

$$\frac{r(p-1)}{r-1} < q < \frac{N(p-1)}{N-1} \tag{2.11}$$

we have

$$\frac{1}{r} + \frac{p-1}{q} < 1$$

and then

$$\int_\Omega B |\nabla u|^{p-1} \, dx \leq C \|B\|_{L^r} \| |\nabla u| \|^{p-1}_{L^q} \, . \tag{2.12}$$

By (2.8), (2.10) and (2.12), we obtain

$$\||\nabla u|\|_{L^q}^p \leq C\|f\|_{L^1}^{\frac{1}{p-1}}\left\{\||\nabla u|\|_{L^q}^{p-1} + \|f\|_{L^1}\right\}$$

and then, via Young inequality, (2.3). □

A priori estimate (2.3) alone does not allow us to conclude the approximation process which yields a SOLA. We need also to estimate the gradient of the difference of two different solutions. This can be done by replacing (1.2), (1.4) with the strong monotonicity property (1.7) or (1.8).

THEOREM 2.2. – *Let assumptions* (1.3), (1.5), (1.7) *or* (1.8) *hold; if* u, v *are weak solutions to problem* (1.1) *with data* $f, g \in C^\infty(\Omega)$ *respectively,* q *satisfies condition* (1.9) *and*

$$m \leq q^* = \frac{Nq}{N-q}\,, \tag{2.13}$$

then

$$\||\nabla(u-v)|\|_{L^q} \leq C(\|f\|_{L^1} + \|g\|_{L^1})^{\frac{1}{p}}\|u-v\|_{L^m}^{\frac{1}{p}}\,, \qquad \text{if } p \geq 2\,, \tag{2.14}$$

$$\||\nabla(u-v)|\|_{L^q} \leq C(\|f\|_{L^1} + \|g\|_{L^1})^{\frac{1}{2(p-1)}}\|u-v\|_{L^m}^{\frac{1}{2}}\,, \qquad \text{if } p < 2\,. \tag{2.15}$$

The constant C depends on N, p, q, $|\Omega|$, $\|B\|_{L^r}$.

PROOF. – Let ν be the distribution function of $(u-v)$. We use in (1.6), with data f and g, the test function

$$\Phi(x) = \text{sign}\,[(u-v)(x)] \int_0^{|u-v|(x)} [\nu(t)]^a\,dt\,,$$

with a as in (2.4). Subtracting and using (1.5) we have

$$\int_\Omega [\nu(|u-v|(x))]^a[(\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla v))\cdot\nabla(u-v)]\,dx \leq \int_\Omega B(|\nabla u|^{p-1} + |\nabla v|^{p-1})|\Phi|\,dx + \int_\Omega |f-g||\Phi|\,dx. \tag{2.16}$$

Now we proceed in a different way whether $p \geq 2$ or $p < 2$.

Let $p \geq 2$. To estimate from below the left-hand side of (2.16), we can proceed as in the proof of (2.8) replacing when necessary (1.2) with the strong monotonicity condition (1.7). So we obtain

$$(2.17)\quad |||\nabla(u-v)|||^p_{L^q} \le C\left(\int_\Omega B(|\nabla u|^{p-1}+|\nabla v|^{p-1})|\Phi|\,dx + \int_\Omega |f-g||\Phi|\,dx\right).$$

As for the L^∞ norm of Φ we have

$$\sup_\Omega |\Phi(x)| = \int_0^{+\infty} [\nu(t)]^a\,dt = a\int_0^{|\Omega|} s^{a-1}(u-v)^*(s)\,ds = a\|u-v\|_{1/a,1}\,.$$

Fixed m as in (2.13), we choose once again a such that $1/a < m$; by (2.2), after all, we get

$$(2.18)\qquad \|\Phi\|_{L^\infty} \le C\|u-v\|_{L^m}\,.$$

Moreover, if q verifies (2.11), then (2.12) holds both for u and for v. So, from (2.17) and (2.18), we have

$$(2.19)\qquad |||\nabla(u-v)|||^p_{L^q} \le C\|u-v\|_{L^m}\left(|||\nabla u|||^{p-1}_{L^q} + |||\nabla v|||^{p-1}_{L^q} + \|f-g\|_{L^1}\right)$$

and then, by (2.3), (2.14).

Let $p<2$. Setting

$$G = \frac{|\nabla(u-v)|^{\frac{2}{p}}}{(|\nabla u|+|\nabla v|)^{\frac{2-p}{p}}}\,,$$

we have

$$\int_\Omega [\nu(|u-v|(x))]^a[(\mathbf{a}(x,\nabla u)-\mathbf{a}(x,\nabla v))\cdot\nabla(u-v)]\,dx$$

(by (1.8))

$$\ge \int_\Omega [\nu(|u-v|(x))]^a[G(x)]^p\,dx$$

(by (2.1))

$$\ge C\int_{\Omega^\#}\left[G^\#(x)\right]^p|x|^{Na}\,dx = C\|G\|^p_{\frac{p}{1+a},p}\,.$$

Therefore, proceeding as in the previous case, we obtain, instead of (2.19),

$$(2.20)\qquad \|G\|^p_{L^q} \le C\|u-v\|_{L^m}\left(|||\nabla u|||^{p-1}_{L^q} + |||\nabla v|||^{p-1}_{L^q} + \|f-g\|_{L^1}\right).$$

Since

$$\int_\Omega |\nabla(u-v)|^q\,dx \le \left(\int_\Omega |G|^q\,dx\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u| + |\nabla v|)^q\,dx\right)^{1-\frac{p}{2}},$$

by (2.20) and (2.3) we obtain (2.15). □

Proof of Theorem (1.1). – Let $\{f_n\}$ be a sequence of C^∞ functions weakly-$*$ converging in the sense of the measures to μ. If $u_n \in W_0^{1,p}(\Omega)$ solves problem (1.1) with datum f_n then

$$\int_\Omega (\mathbf{a}(x,\nabla u_n)\cdot\nabla\varphi)\,dx = \int_\Omega b(x,\nabla u_n)\varphi\,dx + \int_\Omega f_n\varphi\,dx, \quad \forall\varphi\in C_0^\infty(\Omega)\,. \tag{2.21}$$

By (2.3) the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ if q satisfies (1.9); hence it is compact in L^m if $m < q^*$. Therefore, there exists a subsequence, denoted again by $\{u_n\}$, converging in L^m and then, by (2.14), in $W_0^{1,q}(\Omega)$ to a function u.

Now we start the limit procedure in (2.21) to prove u is a SOLA to (1.1).

Obviously we have

$$\lim_{n\to\infty}\int_\Omega f_n\varphi\,dx = \int_\Omega \varphi\,d\mu\,, \quad \forall\varphi\in C_0^\infty(\Omega)\,.$$

For the other terms we use properties of Nemytskii operators (see [17]).

If $p \ge 2$, the sequence $\{\nabla u_n\}$ converge in $L^{p-1}(\Omega)$ to ∇u. So, by (1.3), we have

$$\mathbf{a}(x,\nabla u_n) \xrightarrow{(L^1)^N} \mathbf{a}(x,\nabla u)\,.$$

If $p<2$, setting $\mathbf{A}(x, z_i|z_i|^{p-2}) = \mathbf{a}(x,z)$, from (1.3) we obtain $|\mathbf{A}(x,w)| \le C|w|$. Therefore the operator $w \to \mathbf{A}(x,w)$ is continuous from $(L^1)^N$ into itself. Since

$$\left|\frac{\partial u_n}{\partial x_i}\left|\frac{\partial u_n}{\partial x_i}\right|^{p-2} - \frac{\partial u_m}{\partial x_i}\left|\frac{\partial u_m}{\partial x_i}\right|^{p-2}\right| \le 2^{2-p}\left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\right|^{p-1},$$

by (2.15), we obtain

$$\frac{\partial u_n}{\partial x_i}\left|\frac{\partial u_n}{\partial x_i}\right|^{p-2} \xrightarrow{L^1} \frac{\partial u}{\partial x_i}\left|\frac{\partial u}{\partial x_i}\right|^{p-2}.$$

Hence

$$\mathbf{A}\left(x, \frac{\partial u_n}{\partial x_i}\left|\frac{\partial u_n}{\partial x_i}\right|^{p-2}\right) = \mathbf{a}(x,\nabla u_n)$$

converges to $\mathbf{a}(x,\nabla u)$ in $(L^1)^N$.

Similarly we prove that

$$b(x, \nabla u_n) \xrightarrow{L^1} b(x, \nabla u).$$

Therefore we can pass to the limit in (2.21) and conclude that u is a SOLA.

$\square$

3. – Uniqueness results.

The assumptions of Theorem 1.1 cannot ensure the uniqueness too. To obtain such a result it seems necessary to modify the hypotheses on the structure of the operator as it is suggested by the following arguments.

Let $u(\cdot, \varepsilon)$ be solutions to the problems

$$\begin{cases} -\operatorname{div} \mathbf{a}(x, \nabla_x u(x,\varepsilon)) = b(x, \nabla_x u(x,\varepsilon)) + (f + \varepsilon g) & \text{in } \Omega, \\ u(\cdot, \varepsilon) = 0 & \text{on } \partial\Omega, \end{cases}$$

where f, g are L^1 functions. Setting $u = u(\cdot, 0)$ and $v = u_\varepsilon(\cdot, 0)$, function v solves the linear problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \left(\frac{\partial a_i}{\partial z_j}(x, \nabla u) \frac{\partial v}{\partial x_j} \right)_{x_i} = \sum_{i=1}^{N} \frac{\partial b}{\partial z_i}(x, \nabla u) \frac{\partial v}{\partial x_i} + g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Any a priori estimate of a norm of v in terms of the L^1 norm of g is essentially equivalent to a result of continuity with respect to L^1 data for solutions of (1.1). For such an estimate to hold, it is enough to assume that the matrix $\left(\frac{\partial a_i}{\partial z_j}\right)$ is elliptic and the coefficients b_{z_i} have an index of summability greater than N. If $p > 2$ this happens when $\mathbf{a}$ satisfies (1.10) and b (1.11) with $p < 2 + \frac{1}{N-2}$. This last condition on p, plus (2.3), ensures that b_{z_i} has the required summability. However it can be weakened if one takes into account how the degenerancy of the matrix $\left(\frac{\partial a_i}{\partial z_j}\right)$ is linked to the structure of the coefficients of the first order terms. In this framework we prove the following results which give estimates of the rearrangements of the difference of two solutions in terms of L^1 norm of the difference of the known terms. The procedure is adapted from [27] (see also [20]) and uses a trick of [2].

LEMMA 3.1. – *Under the assumptions of Theorem* 1.2, *if u, v are weak solutions to problem* (1.1) *with data f, g respectively, we have*

$$(u - v)^*(s) \le C \|f - g\|_{L^1} s^{-(N-2)/N}. \tag{3.1}$$

The constant C depends on N, $|\Omega|$, p, H and, moreover, on $\|f\|_{L^1}, \|g\|_{L^1}$; however it is bounded when f, g belong to bounded subsets of L^1.

LEMMA 3.2. – *Under the assumptions of Theorem* 1.3, *if u, v are weak solutions to problem* (1.1) *with data f, g respectively, we have*

$$(u-v)^*(s) \le C\|f-g\|_{L^1} s^{-(N-2)/N-\beta(2-p)}, \tag{3.2}$$

for some $\beta > \dfrac{N-1}{N(p-1)}$, with C as in the previous lemma.

PROOF OF LEMMA (3.1). – Set $w = u - v$ and $h = f - g$. For any fixed $t, k > 0$ we consider the function

$$\phi = \begin{cases} k \text{ sign } w & \text{if } |w| > t+k \\ w - t \text{ sign } w & \text{if } t < |w| \le t+k \\ 0 & \text{otherwise}. \end{cases}$$

Using ϕ as a test function in (1.6) with data f and g respectively, subtracting and dividing by k, we have

$$\begin{aligned} \frac{1}{k} \int\limits_{t<|w|\le t+k} &[\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla v)] \cdot \nabla w \, dx \\ &= \int\limits_{|w|>t+k} [b(x,\nabla u) - b(x,\nabla v)] \text{sign } w \, dx + \int\limits_{|w|>t+k} h \text{ sign } w \, dx \\ &+ \frac{1}{k} \int\limits_{t<|w|\le t+k} [b(x,\nabla u) - b(x,\nabla v)](w - t \text{ sign } w) \, dx \\ &+ \frac{1}{k} \int\limits_{t<|w|\le t+k} h(w - t \text{ sign } w) \, dx = I_1 + I_2 + I_3 + I_4 . \end{aligned} \tag{3.3}$$

By (1.10) the limit as k goes to zero of the left-hand side in (3.3) can be estimated from below by

$$-\frac{d}{dt} \int\limits_{|w|>t} \rho(x) |\nabla w|^2 \, dx$$

where

$$\rho(x) = (1 + |\nabla u| + |\nabla v|)^{p-2} . \tag{3.4}$$

On the right-hand side, we estimate I_1 by (1.11) and I_2 by the Hardy-Littlewood inequality (2.1); since the terms I_3, I_4 go to zero as k vanishes, we have

$$-\frac{d}{dt}\int_{|w|>t}\rho(x)|\nabla w|^2\,dx \le H\int_{|w|>t}\rho(x)|\nabla w|\,dx + \int_0^{\nu(t)} h^*(\sigma)\,d\sigma \tag{3.5}$$

where ν denotes the distribution function of w.

By coarea formula and Schwarz inequality we obtain

$$\begin{aligned}\int_{|w|>t}\rho(x)|\nabla w|\,dx &= \int_t^{+\infty}\left(-\frac{d}{d\tau}\int_{|w|>\tau}\rho(x)|\nabla w|\,dx\right)d\tau \\ &\le \int_t^{+\infty}\left(-\frac{d}{d\tau}\int_{|w|>\tau}\rho(x)|\nabla w|^2\,dx\right)^{\frac12}\left(-\frac{d}{d\tau}\int_{|w|>\tau}\rho(x)\right)^{\frac12}d\tau\,.\end{aligned} \tag{3.6}$$

Following [2] we define a function $\underline{\rho}$ such that

$$\underline{\rho}(\nu(t))|\nu'(t)| = -\frac{d}{dt}\int_{|w|>t}\rho(x)\,dx\,.$$

A significant property tells that $\underline{\rho}$ is weak limit of functions with the same rearrangement as ρ, therefore any Lebesgue or Lorentz norm of $\underline{\rho}$ can be estimated from above with the same norm of ρ. An analogous result is also in [23] where the directional derivative of the map $u \to u^*$, known as relative rearrangement, is introduced (see also [14] and references therein).

Therefore (3.6) becomes

$$\int_{|w|>t}\rho(x)|\nabla w|\,dx \le \int_t^{+\infty}\underline{\rho}^{\frac12}(\nu(\tau))|\nu'(\tau)|^{\frac12}\left(-\frac{d}{d\tau}\int_{|w|>\tau}\rho(x)|\nabla w|^2\,dx\right)^{\frac12}d\tau\,. \tag{3.7}$$

Set $K_N = N\omega_N^{\frac1N}$; by using the isoperimetric and Schwarz inequalities, since $\rho \ge 1$, it follows

$$K_N\nu(t)^{1-\frac1N} \le -\frac{d}{dt}\int_{|w|>t}|\nabla w|\,dx \le \left(-\frac{d}{dt}\int_{|w|>t}\rho(x)|\nabla w|^2\,dx\right)^{\frac12}|\nu'(t)|^{\frac12} \tag{3.8}$$

from which

$$1 \le \frac{1}{K_N\nu(t)^{1-\frac1N}}\left(-\frac{d}{dt}\int_{|w|>t}\rho(x)|\nabla w|^2\,dx\right)^{\frac12}|\nu'(t)|^{\frac12}\,. \tag{3.9}$$

By (3.5), (3.7), (3.9), via Gronwall Lemma, we obtain

$$-\frac{d}{dt}\int_{|w|>t}\rho(x)|\nabla w|^2\,dx \le \int_0^{\nu(t)} h^*(\sigma)\,d\sigma$$

$$+\frac{H}{K_N}\int_t^{+\infty}\frac{[\underline{\rho}(\nu(\tau))]^{\frac12}}{\nu(\tau)^{1-\frac1N}}\left(-\frac{d}{d\tau}\int_{|w|>\tau}\rho(x)|\nabla w|^2\,dx\right)|\nu'(\tau)|\,d\tau$$

$$\le \int_t^{+\infty} h^*(\nu(\tau))\exp\left(\frac{H}{K_N}\int_t^{\tau}\frac{[\underline{\rho}(\nu(z))]^{\frac12}}{\nu(z)^{1-\frac1N}}|\nu'(z)|\,dz\right)|\nu'(\tau)|\,d\tau\,.$$

Therefore

$$-\frac{d}{dt}\int_{|w|>t}\rho(x)|\nabla w|^2\,dx \le \int_0^{\nu(t)}\exp\left(\frac{H}{K_N}\int_r^{\nu(t)}\sigma^{-1+\frac1N}[\underline{\rho}(\sigma)]^{\frac12}\,d\sigma\right)h^*(r)\,dr\,. \tag{3.10}$$

Assume that

$$\int_0^{|\Omega|}\sigma^{-1+\frac1N}[\underline{\rho}(\sigma)]^{\frac12}d\sigma < +\infty\,. \tag{3.11}$$

From (3.9), (3.10) and (3.11), in a standard way (see again [27]), we get the following differential inequality

$$-\frac{dw^*}{ds}(s) \le C\|h\|_{L^1}s^{-2+2/N}$$

which easily yields (3.1).

Now we have to exibit a condition on p which ensures (3.11). It is satisfied if $\underline{\rho}$, and then ρ because of the above mentioned property of $\underline{\rho}$, belongs to L^β with $\beta > N/2$. By (2.3) this happens if p satisfies conditions (1.12). □

The proof of Lemma 3.2 is similar to that of Lemma 3.1.

PROOF OF LEMMA 3.2. – Starting from (3.3) if, in the proof of (3.5), (1.10) is replaced by (1.8) and (1.11) by (1.13), we get

$$-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx \le H\int_{|w|>t}\rho(x)|\nabla w|\,dx + \int_0^{\nu(t)}h^*(\sigma)\,d\sigma\,. \tag{3.12}$$

Instead of (3.6) we easily get

$$\int_{|w|>t} \rho(x)|\nabla w|\,dx \le \int_t^{+\infty}\left(-\frac{d}{d\tau}\int_{|w|>\tau}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\right)^{\frac12}|\nu'(\tau)|^{\frac12}\,d\tau\,. \tag{3.13}$$

Setting

$$\overline{\rho}(\nu(t))|\nu'(t)| = -\frac{d}{dt}\int_{|w|>t}\frac{1}{\rho(x)}\,dx\,,$$

proceeding as for (3.8), we obtain

$$K_N\nu(t)^{1-\frac1N} \le \left(-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\right)^{\frac12}[\overline{\rho}(\nu(t))]^{\frac12}|\nu'(t)|^{\frac12}$$

and then

$$1 \le \frac{1}{K_N\nu(t)^{1-\frac1N}}\left(-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\right)^{\frac12}[\overline{\rho}(\nu(t))]^{\frac12}|\nu'(t)|^{\frac12}\,. \tag{3.14}$$

From (3.12), (3.13), (3.14), via Gronwall Lemma, we firstly get

$$\begin{aligned}
&-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx \le \int_0^{\nu(t)} h^*(\sigma)\,d\sigma\\
&+\frac{H}{K_N}\int_t^{+\infty}\frac{[\overline{\rho}(\nu(\tau))]^{\frac12}}{\nu(\tau)^{1-\frac1N}}\left(-\frac{d}{d\tau}\int_{|w|>\tau}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx\right)|\nu'(\tau)|\,d\tau\\
&\le \int_t^{+\infty} h^*(\nu(\tau))\exp\left(\frac{H}{K_N}\int_t^{\tau}\frac{[\overline{\rho}(\nu(z))]^{\frac12}}{\nu(z)^{1-\frac1N}}|\nu'(z)|\,dz\right)|\nu'(\tau)|\,d\tau\,,
\end{aligned}$$

then

$$-\frac{d}{dt}\int_{|w|>t}(|\nabla u|+|\nabla v|)^{p-2}|\nabla w|^2\,dx \le \int_0^{\nu(t)}\exp\left(\frac{H}{K_N}\int_r^{\nu(t)}\frac{[\overline{\rho}(\sigma)]^{\frac12}}{\sigma^{-1+\frac1N}}\,d\sigma\right)h^*(r)\,dr\,. \tag{3.15}$$

Assume

$$\int_0^{|\Omega|}\sigma^{-1+\frac1N}[\overline{\rho}(\sigma)]^{\frac12}d\sigma < +\infty\,. \tag{3.16}$$

As in the previous case, by (3.14) and (3.15), we have

$$-\frac{dw^*}{ds}(s) \leq C\|h\|_{L^1} s^{-2+2/N-\beta(2-p)}$$

for some β; then we get (3.2).

Now we have to require the sharp conditions on p such that (3.16) holds true. Since $\overline{\rho}$ has the same summability of ρ^{-1}, by (2.3), if $p > 2 - \frac{1}{N}$, it is easy to show that (3.16) is satisfied . □

REMARK 3.1. – We point out that (3.1) allows us to estimate the L^m norm of $u - v$ in terms of the L^1 norm of $f - g$ for $m < \frac{N}{N-2}$, even if $u - v$ belongs to L^m for all $m < \frac{N(p-1)}{N-p}$.

By Lemmas 3.1 and 3.2 we have

$$\|u - v\|_{L^1} \leq C\|f - g\|_{L^1} . \tag{3.17}$$

From (2.14), (2.15) and (3.17) we easily deduce the following result of continuity from the L^1 data which also gives the uniqueness results Theorems 1.2, 1.3.

THEOREM 3.1. – *Let us assume that conditions on* **a**, b *and* p *of Theorems* 1.2 *or* 1.3 *hold true. Then if* u, v *are weak solutions to problem* (1.1) *with data* f, g *respectively, we have*

$$\begin{cases} |||\nabla(u - v)|||_{L^q} \leq C\|f - g\|_{L^1}^{\frac{1}{p}}, & \text{if } p \geq 2 \\ |||\nabla(u - v)|||_{L^q} \leq C\|f - g\|_{L^1}^{\frac{1}{2}}, & \text{if } 1 < p < 2 \end{cases}$$

with C *as in Lemmas* 3.1 *and* 3.2.

REMARK 3.2. – Let us observe that the set of p for which a uniqueness result holds is included, when $N \geq 5$, in the interval of the values for which we have existence. This is consequence of the linearization process. We think that this gap could disappear if one tries to get uniqueness, as in [9], directly from a comparison result, i.e. if one is able to show that $f \leq g \Rightarrow u \leq v$ where u, v are SOLAs corresponding to data f, g.

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