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A Uniqueness Result for Solutions to Subcritical NLS (*)

LUIS VEGA - NICOLA VISCIGLIA

Abstract. – *We extend in a nonlinear context previous results obtained in* [8], [9], [10]. *In particular we present a precised version of Morawetz type estimates and a uniqueness criterion for solutions to subcritical NLS.*

In this article we investigate some qualitative properties of solutions to the following family of NLS:

$$i\partial_t u + \Delta u - u|u|^a = 0,\ (t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n,\ n \geq 3, \tag{0.1}$$

$$u(0,x) = \varphi(x),$$

under the condition

$$\frac{4}{n} < a < \frac{4}{n-2}. \tag{0.2}$$

Since now on we shall denote by $\dot{H}^s_x$ and H^s_x the homogeneous and inhomogenous Sobolev spaces $\dot{H}^s(\boldsymbol{R}^n)$ and $H^s(\boldsymbol{R}^n)$, and similarly the Lebsegue spaces $L^p(\boldsymbol{R}^n)$ will be denoted by L^p_x. We shall also use the notation $\nabla_x\eta$, $\nabla_\tau\eta$ and $\partial_{|x|}\eta$ to describe respectively the full gradient, the tangential part of the gradient and the radial derivative of a given function $\eta : \boldsymbol{R}^n \to \boldsymbol{R}$.

The main aim in this note is the extension in a nonlinear context of previous results obtained in [8] for the free Schrödinger group $\{e^{it\Delta}\}_{t\in\boldsymbol{R}}$ and stricty related with the so called *"local smoothing"* estimate (see [3], [6] and [7]):

$$\sup_{R\in(0,\infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\nabla_x(e^{it\Delta}f)|^2 \, dxdt \leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}\ \forall f \in \dot{H}^{\frac{1}{2}}_x \tag{0.3}$$

where $C > 0$ is a constant. More precisely in [8] we have studied the asymptotic

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for large R of l.h.s. in (0.3) and we have deduced the following identity:

$$\lim_{R\to\infty}\frac{1}{R}\int_{-\infty}^{\infty}\int_{|x|<R}|\nabla_x(e^{it\Delta}f)|^2dxdt = 2\pi\|f\|^2_{\dot{H}^{\frac{1}{2}}_x}\ \forall f\in\dot{H}^{\frac{1}{2}}_x,$$

that in turn implies the following uniqueness result:

$$\text{if}\ \liminf_{R\to\infty}\frac{1}{R}\int_{-\infty}^{\infty}\int_{|x|<R}|\nabla_x(e^{it\Delta}f)|^2dxdt = 0\ \text{then}\ f\equiv 0. \tag{0.4}$$

One of the aim of this paper is the extension of the uniqueness criterion above for solutions to (0.1).

Let us recall that a basic tool in [8] has been the proof of a family of space–time integral identities that represent a generalization of the ones presented in [5]. Let us also underline that the results in [8] have been generalized in [10] to a family of linear Schrödinger equations perturbed with a short range potential.

Concerning the Cauchy problem (0.1) we recall that it has been extensively studied in the literature, under the condition (0.2), from the following point of view: the global well–posedness and the scattering theory. Assume $n\geq 3$ and (0.2) then the following facts can be proved:

(1) $\forall\varphi\in H^1_x$ $\exists$ a unique solution to (0.1) $u(t,x)\in\mathcal{C}(\boldsymbol{R};H^1_x)$;
(2) there exist $\varphi_\pm\in H^1_x$ such that:

$$\lim_{t\to\pm\infty}\|u(t,.))-e^{it\Delta}\varphi_\pm\|_{H^1_x}=0, \tag{0.5}$$

$$\|\varphi_\pm\|_{L^2_x}\equiv\|\varphi\|_{L^2_x}\equiv\|u(t,.)\|_{L^2_x}\ \forall t\in\boldsymbol{R} \tag{0.6}$$

and

$$\|\nabla_x\varphi_\pm\|^2_{L^2_x}\equiv\int_{\boldsymbol{R}^n}\left(|\nabla_x\varphi|^2+\frac{2}{\alpha+2}|\varphi|^{\alpha+2}\right)dx$$

$$\equiv\int_{\boldsymbol{R}^n}\left(|\nabla_x u(t,.)|^2+\frac{2}{\alpha+2}|u(t,.)|^{\alpha+2}\right)dx\ \forall t\in\boldsymbol{R}. \tag{0.7}$$

For a proof of all those facts see [2] and all the references therein. Let us also notice that due to the conservation laws of the Schrödinger equations it is easy to show that the unique solution $u(t,x)\in\mathcal{C}(\boldsymbol{R};H^1_x)$ to (0.1) is bounded in H^1_x and in particular for every $\varphi\in H^1_x$ there exists a constant $C\equiv C(\varphi)>0$ such that:

$$\|u(t,.)\|_{H^1_x}<C\ \forall t\in\boldsymbol{R}. \tag{0.8}$$

Next we state the main result of the paper.

THEOREM 0.1. – *Assume* $n \geq 3$ *and* (0.2). *Let* $u(t,x) \in \mathcal{C}(\boldsymbol{R}; H_x^1)$ *be the unique global solution to* (0.1) *with* $\varphi \in H_x^1$ *and let* $\varphi_\pm \in H_x^1$ *be the functions introduced in* (0.5), *then we have the following identities:*

$$\text{(0.9)} \qquad \lim_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} \left(|\nabla_x u|^2 + \frac{2}{a+2} |u|^{a+2} \right) dxdt$$

$$= \pi \sum_{\pm} \lim_{t\to\pm\infty} \|u(t,.)\|^2_{\dot{H}_x^{\frac{1}{2}}} = \pi \sum_{\pm} \|\varphi_\pm\|^2_{\dot{H}_x^{\frac{1}{2}}},$$

$$\text{(0.10)} \qquad \lim_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\nabla_\tau u|^2 \, dxdt = 0$$

and

$$\text{(0.11)} \qquad \lim_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |u|^{a+2} \, dxdt = 0.$$

In particular we get the following implication:

$$\text{(0.12)} \qquad \textit{if } \liminf_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 \, dxdt = 0 \textit{ then } u \equiv 0.$$

REMARK 0.1. – In [9] it is proved a version of theorem 0.1 for solutions to (0.1) where $a \equiv \frac{4}{n}$, provided that a smallness assumption is done on the initial data φ. Moreover under those conditions it is shown that the r.h.s. in (0.9) is equivalent to the quantity $\|\varphi\|^2_{\dot{H}_x^{\frac{1}{2}}}$.

The proof of theorem 0.1 follows from a family of Morawetz type identities that in our opinion have their own interest, hence we shall include them in next theorem.

THEOREM 0.2. – *Assume* $n \geq 3$ *and* (0.2). *Let* $u(t,x)$, φ *and* $\varphi_\pm$ *as in theorem* 0.1. *Let* ψ *be a radially symmetric function such that the following limit exists:*

$$\text{(0.13)} \qquad \lim_{|x|\to\infty} \partial_{|x|} \psi = \psi'(\infty) \in [0,\infty),$$

and moreover

$$\nabla_x \psi, D^2\psi, \Delta^2\psi \in L_x^\infty.$$

Then the following identity holds:

$$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^n}\left[\nabla_x\bar{u}D^2\psi\nabla_x u-\Delta^2\psi\frac{|u|^2}{4}+\left(\frac{1}{2}-\frac{1}{a+2}\right)\Delta\psi|u|^{a+2}\right]dxdt \tag{0.14}$$
$$=\pi\psi'(\infty)\sum_{\pm}\lim_{t\to\pm\infty}\|u(t,.)\|^2_{\dot{H}_x^{\frac{1}{2}}}=\pi\psi'(\infty)\sum_{\pm}\|\varphi_\pm\|^2_{\dot{H}_x^{\frac{1}{2}}}.$$

1. – Proof of theorem 0.2.

We first state a proposition whose proof can be found in [8].

PROPOSITION 1.1. – *Let* $\varphi\in H^1_x$, $v(t,x)\equiv e^{it\Delta}\varphi$ *and* ψ *as in theorem* 0.2, *then:*

$$\lim_{t\to\pm\infty}\mathcal{I}m\int_{\boldsymbol{R}^n}\bar{v}(t,.)\nabla_x v(t,.)\cdot\nabla_x\psi\,dx=\pm 2\pi\psi'(\infty)\|\varphi\|^2_{\dot{H}_x^{\frac{1}{2}}}. \tag{1.1}$$

PROOF OF THEOREM 0.2. – First notice that if $\varphi_\pm$ are the functions introduced in (0.5), then

$$\lim_{t\to\pm\infty}\|u(t,.)-e^{it\Delta}\varphi_\pm\|_{\dot{H}_x^{\frac{1}{2}}}=0, \tag{1.2}$$

where we have used the embedding $H^1_x\subset\dot{H}_x^{\frac{1}{2}}$. On the other hand we have the following identity:

$$\|e^{it\Delta}\varphi_\pm\|_{\dot{H}_x^{\frac{1}{2}}}\equiv\|\varphi_\pm\|_{\dot{H}_x^{\frac{1}{2}}}\ \forall t\in\boldsymbol{R}, \tag{1.3}$$

that in conjunction with (1.2) gives

$$\lim_{t\to\pm\infty}\|u(t,.)\|^2_{\dot{H}_x^{\frac{1}{2}}}=\|\varphi_\pm\|^2_{\dot{H}_x^{\frac{1}{2}}}. \tag{1.4}$$

Hence the proof of theorem 0.2 will be complete provided that we show that l.h.s. and r.h.s. in (0.14) are equal. Following [1] we multiply (0.1) by the quantity

$$\nabla_x\bar{u}\cdot\nabla_x\psi+\frac{1}{2}\bar{u}\,\Delta\psi, \tag{1.5}$$

and we integrate on the strip $(-T,T)\times\boldsymbol{R}^n$. In this way we get the following family of identities:

$$\int_{-T}^{T}\int_{\boldsymbol{R}^n}\left(\nabla_x\bar{u}D^2\psi\nabla_x u-\Delta^2\psi\frac{|u|^2}{4}+\left(\frac{1}{2}-\frac{1}{a+2}\right)\Delta\psi|u|^{a+2}\right)dxdt \tag{1.6}$$
$$=\frac{1}{2}\mathcal{I}m\sum_{\pm}\pm\int_{\boldsymbol{R}^n}\bar{u}(\pm T,.)\nabla_x u(\pm T,.)\cdot\nabla_x\psi\,dx,$$

(for more details on this computation see [1] and [8]). Next we introduce the functions

$$v_{\pm}(t,.) \equiv e^{it\Delta}\varphi_{\pm},$$

where $\varphi_{\pm}$ are defined in (0.5). It is not difficult to verify that

$$\lim_{T\to\infty} \mathcal{I}m \int_{\mathbf{R}^n} \bar{u}(\pm T,.)\nabla_x u(\pm T,.)\cdot\nabla_x\psi\, dx \tag{1.7}$$

$$= \lim_{T\to\infty} \int_{\mathbf{R}^n} \bar{v}_{\pm}(\pm T,.)\nabla_x v_{\pm}(\pm T,.)\cdot\nabla_x\psi\, dx = \pm 2\pi\psi'(\infty)\|\varphi_{\pm}\|^2_{\dot{H}^{\frac{1}{2}}_x},$$

where at the last step we have used proposition 1.1. The proof of (0.14) can be completed by combining (1.6) with (1.7). □

Next we shall deduce some consequences of (0.14) that we shall need along the proof of theorem 0.1.

PROPOSITION 1.2. – *Assume* $n \geq 3$, (0.2) *and* φ, $u(t,x)$ *as in theorem* 0.1, *then*

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \Delta\psi_R(x)|u|^{\alpha+2}\, dxdt = 0$$

where $\psi_R \equiv R\psi\left(\frac{x}{R}\right)$ *and* $|\Delta\psi(x)| \leq \frac{C}{1+|x|}$. *In particular*

$$\lim_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |u|^{\alpha+2} = 0 \tag{1.8}$$

PROOF. – Notice that it is sufficient to show:

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \frac{|u|^{\alpha+2}}{R+|x|}\, dxdt = 0.$$

In turn this fact will follow by the dominated convergence theorem provided that we can show

$$\int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \frac{|u|^{\alpha+2}}{1+|x|}\, dxdt < \infty. \tag{1.9}$$

In order to prove (1.9) we notice that an explicit computation shows that the function $\psi(x) \equiv \sqrt{1+|x|}$ is a convex function and moreover:

$$-\Delta^2\psi \geq 0 \text{ and } \Delta\psi \geq \frac{c}{1+|x|}.$$

Hence it is sufficient to choose $\psi \equiv \sqrt{1+|x|}$ in (0.14) in order to deduce (1.9). □

PROPOSITION 1.3. – *Assume* $n \geq 4$, (0.2) *and* φ, $u(t,x)$ *as in theorem* 0.1, *then*

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \Delta^2 \psi_R(x) |u|^2 \, dxdt = 0,$$

where $\psi_R \equiv R\psi\left(\frac{x}{R}\right)$ *and* $|\Delta^2 \psi(x)| \leq \frac{C}{(1+|x|)^3}$.

PROOF. – It is sufficient to show that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \frac{|u|^2}{R^3 + |x|^3} \, dxdt = 0.$$

Notice that this fact will follow by combining the dominated convergence theorem with the following estimate:

$$\int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \frac{|u|^2}{1 + |x|^3} \, dxdt < \infty, \tag{1.10}$$

whose proof is similar to the proof of (1.9). In fact notice that the function $\psi \equiv \sqrt{1+|x|}$ is a convex function such that

$$-\Delta^2 \psi \geq \frac{c}{1+|x|^3} \quad \text{on } \mathbf{R}^n \text{ for any } n \geq 4. \tag{1.11}$$

Hence it is sufficient to choose $\psi \equiv \sqrt{1+|x|}$ in (0.14), in order to deduce (1.10). □

2. – Strichartz estimates for NLS in dimension $n = 3$.

The main goal in this section is the proof of a partial substitute of proposition 1.3 that works in dimension $n = 3$. The basic tool that we use is the end–point Strichartz estimate whose proof can be found in [4].

PROPOSITION 2.1. – *Assume* $n = 3$, (0.2) *and let* φ, $u(t,x)$ *as in theorem* 0.1, *then*

$$\lim_{R\to\infty} \frac{1}{R^3} \int_{-\infty}^{\infty} \int_{|x|<R} |u|^2 \, dxdt = 0. \tag{2.1}$$

We shall need the following lemma.

LEMMA 2.1. – *Let* $n = 3$, (0.2) *and* $\varphi, u(t,x)$ *as in theorem* 0.1. *Then*

$$u(t,x) \in L^2(\mathbf{R}; W^{1,6}(\mathbf{R}^3)). \tag{2.2}$$

PROOF. – Notice that by combining (0.5) with the Sobolev embedding $H^1(\mathbf{R}^3) \subset L^6(\mathbf{R}^3)$, we deduce that

$$\lim_{t\to\pm\infty} \|u(t,.) - e^{it\Delta}\varphi_\pm\|_{L^6(\mathbf{R}^3)} = 0. \tag{2.3}$$

On the other hand by combining a density argument, with the dispersive estimate:

$$\|e^{it\Delta}\varphi\|_{L^6(\mathbf{R}^3)} \leq \frac{C}{|t|}\|\varphi\|_{L^{\frac{6}{5}}(\mathbf{R}^3)},$$

and with the a–priori bound:

$$\|e^{it\Delta}\varphi\|_{L^6(\mathbf{R}^3)} \leq C\|\varphi\|_{H^1(\mathbf{R}^3)}$$

(whose proof follows by combining the conservation of the Sobolev norm for the free evolution with the Sobolev embedding), we get:

$$\lim_{t\to\pm\infty} \|e^{it\Delta}\varphi\|_{L^6(\mathbf{R}^3)} = 0 \ \forall\varphi \in H^1(\mathbf{R}^3).$$

By combining this fact with (2.3) we get

$$\lim_{t\to\pm\infty} \|u(t,.)\|_{L^6(\mathbf{R}^3)} = 0$$

that in conjunction with the conservation of the charge $\|u(t,.)\|_{L^2(\mathbf{R}^3)} \equiv const$ gives:

$$\lim_{t\to\pm\infty} \|u(t,.)\|_{L^p(\mathbf{R}^3)} = 0 \ \forall\ 2 < p \leq 6. \tag{2.4}$$

In particular, since we are assuming (0.2), we can apply (2.4) in order to deduce that:

$$\lim_{T\to\infty} \sup_{t\in(T,\infty)} \|u(t,.)\|_{L^{\frac{3}{2}a}(\mathbf{R}^3)} = 0 \tag{2.5}$$

and

$$\lim_{T\to\infty} \sup_{t\in(-\infty,-T)} \|u(t,.)\|_{L^{\frac{3}{2}a}(\mathbf{R}^3)} = 0. \tag{2.6}$$

Next we shall prove (2.2). Due to the end–point Strichartz estimate (see [4]) it is easy to show that

$$u(t,x) \in L^2_{loc}(\mathbf{R}; W^{1,6}(\mathbf{R}^3)).$$

Hence it is sufficient to show that $u(t,x) \in L^2((T,\infty); W^{1,6}(\mathbf{R}^3))$ and $u(t,x) \in L^2((-\infty,-T); W^{1,6}(\mathbf{R}^3))$ for $T > 0$ large enough.

Notice that since $u(t,x)$ solves (0.1) we can use the end–point Strichartz estimate in order to deduce:

$$\|u\|_{L^2((T,\infty);W^{1,6}(\mathbf{R}^3))} \leq C\Big(\|u(T,.)\|_{H^1(\mathbf{R}^3)} + \|u|u|^a\|_{L^2((T,\infty);W^{1,\frac{6}{5}}(\mathbf{R}^3))}\Big) \tag{2.7}$$

for every $T > 0$. On the other hand an elementary computation implies:

$$\|u(t,.)|u(t,.)|^{a}\|_{W^{1,\frac{6}{5}}(\boldsymbol{R}^3)} \leq C\|u(t,.)\|^{a}_{L^{\frac{3}{2}a}(\boldsymbol{R}^3)}\|u(t,.)\|_{W^{1,6}(\boldsymbol{R}^3)} \quad \forall t \in \boldsymbol{R}$$

and hence due to the Hölder inequality we get:

$$\|u|u|^{a}\|_{L^2((T,\infty);W^{1,\frac{6}{5}}(\boldsymbol{R}^3))}$$

$$\leq C\Bigg(\sup_{t\in(T,\infty)}\|u(t,.)\|^{a}_{L^{\frac{3}{2}a}(\boldsymbol{R}^3)}\Bigg)\|u(t,.)\|_{L^2((T,\infty);W^{1,6}(\boldsymbol{R}^3))}.$$

By combining (2.7) with (2.5) we deduce that if we choose $T \equiv T(\varepsilon) > 0$ large enough then we get:

$$\|u\|_{L^2((T,\infty);W^{1,6}(\boldsymbol{R}^3))} \leq C\big(C(\varphi) + \varepsilon\|u(t,.)\|_{L^2((T,\infty);W^{1,6}(\boldsymbol{R}^3))}\big),$$

where we have used (0.8). In particular if we choose $\varepsilon > 0$ small in such a way that $C\varepsilon < \frac{1}{2}$ then we deduce $\|u\|_{L^2((T,\infty);W^{1,6}(\boldsymbol{R}^3))} < \infty$. In a similar way we can show $\|u\|_{L^2((-\infty,-T);W^{1,6}(\boldsymbol{R}^3))} < \infty$ for a suitable $T > 0$ and the proof of (2.2) is complete. □

PROOF OF PROPOSITION 2.1. – Due to lemma 2.1 and to the Sobolev embedding

$$W^{1,6}(\boldsymbol{R}^3) \subset L^\infty(\boldsymbol{R}^3)$$

we deduce that if $u(t,x)$ is as in the assumptions, then

$$u(t,x) \in L^2(\boldsymbol{R}; L^\infty(\boldsymbol{R}^3)). \tag{2.8}$$

Next notice that for every $T > 0$ we have:

$$\int_T^\infty \int_{|x|<R} |u|^2 dxdt \leq CR^3 \int_T^\infty \sup_{|x|<R} |u(t,.)|^2 dt$$

$$\leq CR^3\|u\|^2_{L^2((T,\infty);L^\infty(\boldsymbol{R}^3))}.$$

By combining this fact with (2.8) we get the following implication:

$$\forall \varepsilon > 0 \text{ there exists } T_1(\varepsilon) > 0 \text{ s.t. } \limsup_{R\to\infty} \frac{1}{R^3} \int_{T_1(\varepsilon)}^\infty \int_{|x|<R} |u|^2 \, dxdt \leq \varepsilon.$$

Of course by a similar argument we can prove that:

$$\forall \varepsilon > 0 \text{ there exists } T_2(\varepsilon) > 0 \text{ s.t. } \limsup_{R\to\infty} \frac{1}{R^3} \int_{-\infty}^{-T_2(\varepsilon)} \int_{|x|<R} |u|^2 \, dxdt \leq \varepsilon.$$

In particular, if we choose $T(\varepsilon) = \max\{T_1(\varepsilon), T_2(\varepsilon)\}$, then we get:

(2.9) $$\forall \varepsilon > 0 \text{ there exists } T(\varepsilon) > 0 \text{ s.t.}$$

$$\limsup_{R\to\infty} \frac{1}{R^3} \int_{\boldsymbol{R}\setminus(-T(\varepsilon);T(\varepsilon))} \int_{|x|<R} |u|^2 \, dxdt \leq \varepsilon.$$

Hence the proof of (2.1) will follow from the following fact:

(2.10) $$\forall T > 0 \text{ we have } \limsup_{R\to\infty} \frac{1}{R^3} \int_{-T}^{T} \int_{|x|<R} |u|^2 \, dxdt = 0.$$

Notice that by using the Hölder inequality we get:

$$\int_{|x|<R} |u(t,.)|^2 \, dx \leq CR^2 \|u(t)\|^2_{L^6(\boldsymbol{R}^3)},$$

and this implies:

(2.11) $$\frac{1}{R^3} \int_{-T}^{T} \int_{|x|<R} |u|^2 \, dxdt$$
$$\leq \frac{C}{R} \int_{-T}^{T} \|u(t)\|^2_{L^6(\boldsymbol{R}^3)} \, dt \leq \frac{2CT}{R} \|u\|^2_{L^\infty(\boldsymbol{R};L^6(\boldsymbol{R}^3))}.$$

On the other hand by combining (0.8) with the embedding $H^1(\boldsymbol{R}^3) \subset L^6(\boldsymbol{R}^3)$, we deduce that $u(t,x) \in L^\infty(\boldsymbol{R}; L^6(\boldsymbol{R}^3))$. Hence (2.10) follows from (2.11). □

3. – Proof of theorem 0.1.

PROOF OF THEOREM 0.1 FOR $n \geq 4$. – Notice that (0.11) follows from proposition 1.2. Next recall the following identity:

(3.1) $$\nabla_x \bar{u} D^2_x \psi \nabla_x u = \partial^2_{|x|}\psi |\partial_{|x|} u|^2 + \frac{\partial_{|x|}\psi}{|x|} |\nabla_\tau u|^2,$$

where ψ is a radially symmetric function. By using this identity and by choosing in (0.14) the function $\psi \equiv \sqrt{1+|x|^2}$, then it is easy to deduce that

(3.2) $$\int_{-\infty}^{\infty} \int_{|x|>1} \frac{|\nabla_\tau u|^2}{|x|} \, dxdt < \infty,$$

and in particular:

$$\lim_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\nabla_\tau u|^2 \, dxdt = 0. \tag{3.3}$$

Due to this fact, (1.4) and (1.8) we deduce that it is sufficient to prove the following identity:

$$\lim_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 \, dxdt = \pi \sum_{\pm} \|\varphi_\pm\|^2_{\dot{H}^{\frac{1}{2}}_x}, \tag{3.4}$$

in order to deduce (0.9).

For any $k \in \boldsymbol{N}$ we fix a function $h_k(r) \in C_0^\infty(\boldsymbol{R}; [0,1])$ such that:

$$h_k(r) = 1 \; \forall r \in \boldsymbol{R} \text{ s.t. } |r| < 1, h_k(r) = 0 \; \forall r \in \boldsymbol{R} \text{ s.t. } |r| > \frac{k+1}{k}, \tag{3.5}$$
$$h_k(r) = h_k(-r) \; \forall r \in \boldsymbol{R}.$$

Let us introduce the functions $\psi_k(r), H_k(r) \in C^\infty(\boldsymbol{R})$:

$$\psi_k(r) = \int_0^r (r-s) h_k(s) ds \quad \text{and} \quad H_k(r) = \int_0^r h_k(s) ds. \tag{3.6}$$

Notice that

$$\psi_k''(r) = h_k(r), \psi_k'(r) = H_k(r) \forall r \in \boldsymbol{R} \text{ and } \lim_{r\to\infty} \partial_r \psi_k(r) = \int_0^\infty h_k(s) ds. \tag{3.7}$$

Moreover an elementary computation shows that:

$$\Delta \psi_k \leq \frac{C}{1+|x|} \quad \forall x \in \boldsymbol{R}^n$$

and

$$\Delta^2 \psi_k = \frac{C}{|x|^3} \quad \forall x \in \boldsymbol{R}^n \text{ s.t. } |x| \geq 2 \text{ and } n \geq 4,$$

where Δ^2 is the bilaplacian operator. Thus the functions $\phi \equiv \psi_k$ satisfy the assumptions of proposition 1.2 and 1.3.

In the sequel we shall need the rescaled functions

$$\psi_{k,R}(x) \equiv R \psi_k\left(\frac{x}{R}\right) \forall x \in \boldsymbol{R}^n, k \in \boldsymbol{N} \text{ and } R > 0, \tag{3.8}$$

where ψ_k is defined in (3.6). Notice that by combining (3.1) with (0.14), where we choose $\psi = \psi_{k,R}$ and recalling (3.7) we get:

$$\int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \Big(\partial^2_{|x|} \psi_{k,R} |\partial_{|x|} u|^2 + \frac{\partial_{|x|} \psi_{k,R}}{|x|} |\nabla_\tau u|^2 - \frac{1}{4} |u|^2 \Delta^2 \psi_{k,R} + \left(\frac{1}{2} - \frac{1}{\alpha+2} \right) |u|^{\alpha+2} \Delta \psi_{k,R} \Big) dx dt = \pi \left(\int_0^\infty h_k(s)\, ds \right) \sum_{\pm} \|\varphi_\pm\|^2_{\dot{H}^{\frac{1}{2}}_x} \ \forall k \in \mathbf{N}, R > 0. \tag{3.9}$$

On the other hand (3.2) in conjunction with propositions 1.2 and 1.3 gives:

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \Big(\partial_{|x|} \psi_{k,R} \frac{|\nabla_\tau u|^2}{|x|} - \frac{1}{4} \Delta^2 \psi_{k,R} |u|^2 + \left(\frac{1}{2} - \frac{1}{\alpha+2} \right) \Delta \psi_{k,R} |u|^{\alpha+2} \Big)\ dx dt = 0$$

for every $k \in \mathbf{N}$. By combining this fact with (3.9) we deduce:

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \partial^2_{|x|} \psi_{k,R} |\partial_{|x|} u|^2 dx dt = \pi \left(\int_0^\infty h_k(s) ds \right) \sum_{\pm} \|\varphi_\pm\|^2_{\dot{H}^{\frac{1}{2}}_x} \ \forall k \in \mathbf{N}. \tag{3.10}$$

On the other hand, due to the properties of h_k (see (3.5)), we get

$$\frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 dx dt \leq \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \partial^2_{|x|} \psi_{k,R} |\partial_{|x|} u|^2 dt dx = \frac{1}{R} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} h_k\left(\frac{x}{R}\right) |\partial_{|x|} u|^2 dt dx \leq \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<\frac{k+1}{k}R} |\partial_{|x|} u|^2 dx dt$$

that due to (3.10) implies:

$$\limsup_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 dx dt \leq \pi \left(\int_0^\infty h_k(s) ds \right) \sum_{\pm} \|\varphi_\pm\|^2_{\dot{H}^{\frac{1}{2}}} \leq \frac{k+1}{k} \liminf_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 dx dt \ \forall k \in \mathbf{N}. \tag{3.11}$$

Since $k \in \boldsymbol{N}$ is arbitrary and since the following identity is trivially satisfied:

$$\lim_{k\to\infty} \int_0^\infty h_k(s)ds = 1,$$

we can deduce (3.4) from (3.11).

Finally we shall prove (0.12). Assume that

$$\liminf_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|}u|^2 \, dxdt = 0,$$

then by (0.9), (0.10) and (0.11) we get $\varphi_\pm \equiv 0$, and in particular due to the identity $\|\varphi_\pm\|_{L^2_x} = \|\varphi\|_{L^2_x}$ we get $\varphi \equiv 0$. □

PROOF OF THEOREM 0.1 FOR $n = 3$. – Let ψ_k and $\psi_{k,R}$ be the radially symmetric functions on $\boldsymbol{R}^3$ corresponding to the ones introduced in the proof of theorem 0.1 for $n \geq 4$.

Notice that the point where the proof of theorem 0.1 given for $n \geq 4$ fails in dimension $n = 3$, is that it is unclear whether or not the following fact is true for $n = 3$:

$$\lim_{R\to\infty} \int \int_{|x|<R} \Delta^2\psi_{k,R}|u|^2 \, dxdt = 0. \tag{3.12}$$

On the other hand an elementary computation in dimension $n = 3$ implies:

$$\Delta^2\psi_k \equiv 0 \; \forall x \in \boldsymbol{R}^3 \setminus \{|x| > 2\} \text{ and } \forall k \in \boldsymbol{N},$$

and hence the proof of (3.12) for $n = 3$ follows from proposition 2.1.
Once (3.12) is proved in dimension $n = 3$, then it is easy to verify that proof of theorem 0.1 for $n \geq 4$ still works for $n = 3$. □

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