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Convergence to Equilibrium of the Solution of Kac's Kinetic Equation. A Probabilistic View (*)

EUGENIO REGAZZINI

Abstract. – *Let $f(\cdot, t)$ be the probability density function representing the solution of Kac's Boltzmann–like equation at time t, with initial data f_0, and let g_σ be the Gaussian density with zero mean and variance σ^2, σ^2 being the value of the second moment of f_0. Henry McKean Jr. put forward the conjecture that the total variation distance between $f(\cdot, t)$ and g_σ goes to zero, as $t \to +\infty$, with an exponential rate equal to $-1/4$. This lecture aims at explaining the main efforts made to a view to validating this conjecture, and concludes with the theorem stating that this holds true whenever f_0 has finite fourth moment and its Fourier transform φ_0 satisfies $|\varphi_0(\xi)| = o(|\xi|^{-p})$ as $|\xi| \to +\infty$, for some $p > 0$. The first part of the lecture expounds the derivation of the Kac Boltzmann-like equation from the Kac master equation. A detailed description of the probabilistic methods resorted to prove the above-mentioned theorem is then given. The final part mentions further applications of these methods to other kinetic models.*

1. – Motivation and scheme for the lecture.

The focus of this lecture is on quantitative investigations pertaining to the rate of relaxation to equilibrium of solutions of a Boltzmann-like equation known as *Kac's caricature of a Maxwellian gas*. Boltzmann-like equations have probabilistic origins, which are more or less explicitly expressed. Thus, considerable work has been done to analyze the speed of approach to equilibrium of their solutions from a probabilistic stance. Mark Kac and Henry McKean Jr. have been pioneers in this field of studies, as the following passage from [29] clearly shows

> According to Boltzmann's classical investigation, the entropy should increase to its bound $\log[\sigma\sqrt{2\pi e}]$ as $t \uparrow +\infty$, while the solution of Kac's kinetic equation tends to the Maxwellian function g. Entropy *does* increase, the entropy production vanishes only for the Maxwellian function, and the approach to the Maxwellian is usually considered self-evident on this basis...

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> But, while the fact cannot be doubted, no proof of it has been advanced, except by Carleman for a 3-dimensional gas of hard balls. Wild's sum suggests a simpler explanation: *the central limit theorem for Maxwellian molecules*.

This hint at the central limit theorem of probability theory actually constitutes a starting point for this lecture. Indeed, it aims at describing how this fundamental theorem could be applied in order to obtain both precise evaluations of the rate of approach to the Maxwellian distribution and bounds on the error in approximating for fixed time t. The rest of the lecture is divided into the following sections:

2. Kac's master equation and derivation of Kac's equation for the density of the velocity of one molecule (*Kac's equation*, for short).
3. Wild expansion for the solution of Kac's equation and its probabilistic interpretation.
4. Analysis of the convergence of the solution with respect to "weak" metrics such as Kolmogorov's and Monge-Wasserstein's.
5. Analysis of the same problem in the total variation metric.
6. Final remarks.

Section 2 provides motivations for the subsequent exposition. Basic ideas for application of probabilistic methods are presented in Section 3. By means of the Wild expansion it is shown that the solution of Kac's equation represents the probability distribution of a random weighted sum of random variables. Via suitable conditioning, such a sum can be studied as a weighted sum of independent and identically distribuited (i.i.d., for short) random variables. This paves the way for application of classical results related to the central limit theorem. In particular, upper bounds for the Kolmogorov distance between the solution of the Kac equation and the Maxwellian distribution are presented in Section 4 by means of resorting to the Berry-Esseen inequality (see, e.g., [11]) and to some of its more recent refinements. Bounds are also stated for Monge-Wasserstein's distances of order not greater than $(2 + \delta)$, for some δ in $[0, 1]$. The study of the speed of convergence of the total variation distance between the said distributions is deferred to Section 5 by using suitable refinements and modifications of Cramer's asymptotic expansions. Possible extensions of these methods both to multidimensional kinetic models and to inelastic kinetic equations are briefly mentioned in Section 6.

2. – Kac's master equation and derivation of Kac's equation for the density of the velocity of one molecule.

We start by describing the Kac model. As emerges from [24] and [25], Kac was motivated by the desire to find an appropriate methodology for the study of relaxation to equilibrium for kinetic models connected with the Boltzmann equation. Kac's stance was that one should be able to get quantitative results about a many-particle evolution equation and, as a result, this could lead to analogous statements in the case of the one-particle (i.e., the Boltzmann-like) equation. The said deduction, in turn, ought to be made possible by a connection between many-particles and one-particle model, which Kac himself stated by showing that the basic Boltzmann assumption of independence (Stasszhlansatz) *propagates in time*. Let us now get down to detail.

For the sake of simplicity, Kac considered also an n-particle system in one dimension. Assuming the positions are in equilibrium, he analyzed the velocities $(v_1, \dots, v_n)$ under the sole restriction that the total energy $v_1^2 + \cdots + v_n^2 = n\sigma^2$ is conserved (hence the restriction to the sphere). Particles are supposed to exchange energy as follows. At the times of a Poisson process with rate $n\lambda$, a subset $\{i,j\}$ is assumed to have probability $\binom{n}{2}^{-1}$. Moreover, the initial velocities v_i and v_j change into the post-collisional velocities according to

$$(v_i, v_j) \to (v_i \cos\theta + v_j \sin\theta, -v_i \sin\theta + v_j \cos\theta)$$

with θ uniformly distributed on $[0, 2\pi)$. So, if for every $\{i,j\}$ one defines $R^\theta_{i,j}$ to be the clockwise rotation given by

$$R^\theta_{i,j} = \begin{array}{c} \begin{array}{cccccccc} & & & i & & j & & \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots & & \vdots \\ \vdots & & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} \qquad (1 \le i < j \le n)$$

where all the diagonal entries are 1 except for the (i,i) and the (j,j) entries that are $c := \cos\theta$, and all the off-diagonal entries are 0 except for the (i,j) and (j,i) entries that are equal to $s := \sin\theta$ and $-s$, respectively, one can write the operator

$$H_t := \exp\{-n\lambda t(I - Q)\}$$

on L^2 of the n-sphere with

$$Qf(V) := \frac{1}{2\pi\binom{n}{2}} \sum_{1 \le i < \le n} \int_0^{2\pi} f(R^\theta_{i,j}) d\theta \quad (V := (v_1, \dots, v_n)).$$

With this operator one can define a Markov process on the sphere, in the standard way. If an initial probability density $\ell_n(V,0)$ is given, with respect to the Haar measure on the sphere, then a density $\ell_n(V,t)$ of the process at time t is given by

$$\ell_n(V,t) = H_t \ell_n(V,0).$$

$\{H_t\}$ is a semigroup and the differential equation associated with it (the Kolmogorov backward equation for the Markov process) yields the so-called *Kac's master equation*

$$\begin{aligned} \frac{\partial}{\partial t}\ell_n(V,t) &= -\, n\lambda(I-Q)\ell_n(V,t) \\ &= \frac{n\lambda}{2\pi\binom{n}{2}} \sum_{1\le i<j\le\le n} \int_0^{2\pi} \Big\{\ell_n(R^{\theta}_{i,j}V,t) - \ell_n(V,t)\Big\} d\theta. \end{aligned} \tag{1}$$

In order to obtain a (non-linear) Boltzmann-like equation from (linear) equation (1) Kac focused on the marginal densities of the first coordinate v_1 and of the first two coordinates (v_1,v_2), indicated by $f_n^{(1)}$ and $f_n^{(2)}$ respectively. Assuming each term of the sequence of initial densities $(\ell_n(\cdot,0))_{n\ge 2}$ is symmetric in the argument $(v_1,\dots,v_n)$ and varies with n so that the marginals approximately factor and, therefore,

$$f_n^{(2)}(v_1,v_2,0) \sim f_n^{(1)}(v_1,0) f_n^{(1)}(v_2,0)$$

as $n\to+\infty$, for every (v_1,v_2), Kac proved the already mentioned propagation in time of the factorization property – a phenomenon commonly known as *propagation of chaos* –

$$f_n^{(2)}(v_1,v_2,t) \sim f_n^{(1)}(v_1,t) f_n^{(1)}(v_2,t).$$

Substituting this in (1) with $\lambda = 1/2$ and indicating the one-dimensional limiting density (as $n\to+\infty$) by $f(\cdot,t)$, one obtains the *Kac equation*

$$\begin{aligned} \frac{\partial f}{\partial t}(v,t) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} &[f(v\cos\theta - w\sin\theta,t)\cdot f(v\sin\theta + w\cos\theta,t) \\ &-f(v,t)f(w,t)]\,dw d\theta \quad (v\in\mathbb{R}, t>0) \end{aligned} \tag{2}$$

with initial datum given by some specific probability density function on $\mathbb{R}: f_0(\cdot) = f(\cdot,0^+)$.

This is the desired one-dimensional Boltzmann-like equation derived from the Kac model that is encapsulated in (1). Such a derivation explains the way Kac expected to deduce quantitative results on the limiting behaviour of solutions of (2) from quantitative results on the linear master equation. In fact, $H_t\ell_n(V,0)$ converges to the uniform density u on the n-sphere (a.e., as $t\to+\infty$) and the question is just how fast this relaxation actually occurs. After defining the quantity

$$\lambda_n := \sup\left\{\int_{\mathbb{R}} f(x)(Qf)(x)dx : \int f^2(x)dx = 1, \int f(x)dx = 0\right\}$$

and observing that

$$\Delta_n := \frac{n}{2}(1-\lambda_n)$$

represents the spectral gap for the generator of the semigroup (see the right-hand side of (1) with $\lambda = 1/2$), Kac conjectured that $C := \limsup \Delta_n$ must be strictly positive. If this was true, then one could write, as an easy consequence of the spectral theorem,

$$\left(\int_{\mathbb{R}} |H_t\ell_n(V,0) - u(V)|^2 dV\right)^{1/2} \leq \exp\{-Ct\}\left(\int |\ell_n(V,0) - u(V)|^{1/2} dV\right)^{1/2}. \tag{3}$$

This, in turn, would imply that the L^1 distance (on $\mathbb{R}^n$) between $\ell_n(\cdot,t)$ and u had an upper bound that went to zero exponentially, at a rate equal to $-C$, as $t \to +\infty$.

The first statements which bear the Kac conjecture out are contained in [13] and [23]. In the latter paper, the author supplies a lower bound of the form c^*/n for $(1-\lambda_n)$ without adding, however, any information on the value of c^*. More recently, in [7] Carlen, Carvalho and Loss have specified that

$$\Delta_n = \frac{1}{4} \cdot \frac{n+2}{n+1}.$$

Hence, one can say that the rate of convergence to zero in the right-hand side of (3) is $(-1/4)$.

At this stage, one wonders whether an analogous rate holds for the one-dimensional model, as a consequence of the described connections between models (1) and (2). So far, bounds for (2) have been obtained, independently of the relationship between (1) and (2), and independently as well of the above-mentioned remarkable Carlen, Carvalho and Loss's statement.

It is easy to see that the Gaussian density (Maxwellian density in the kinetic-theoretical literature)

$$x \mapsto g_\sigma(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \qquad (x \in \mathbb{R}, \sigma > 0)$$

is a steady-state solution of the Kac equation (2) when σ^2 coincides with the second moment of f_0. In Section 4 we will state that the Gaussian distribution is the sole non-degenerate admissible limit distribution, and that such a limit is actually attainable if and only if $0 < \sigma^2 := \int_{\mathbb{R}} x^2 f_0(x)dx < +\infty$. Thus, our problem lies in verifying whether there is some constant $\tilde{c}$, depending only on f_0, such that

$$\int_{\mathbb{R}} |f(x,t) - g_\sigma(x)|dx \le \tilde{c} \exp\left\{-\frac{t}{4}\right\} \tag{4}$$

holds true for every $t \ge 0$ and any initial data in some fair class of probability density functions on $\mathbb{R}$.

Taking into consideration the difficulties inherent in following the Kac approach to prove or disprove (4), McKean tried to get evidence from linearizing Kac's equation (2) about g_σ. See [29]. He found that the spectral gap for the linearized form coincides with $1/4$. Being unable to extend this fact to (2), in order to obtain (4) McKean indicated an alternative route which, as it will be shown in Section 5, brings to a successful conclusion.

3. – Wild expansion for the solution of Kac's equation and its probabilistic interpretation.

Applying the Fourier transformation to both sides of (2) and setting $\varphi(y,t) := \int_{\mathbb{R}} e^{iyx} f(x,t)dx$ for every y in $\mathbb{R}$, (2) becomes

$$\frac{\partial \varphi}{\partial t}(\xi,t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\xi\cos\theta,t)\varphi(\xi\sin\theta,t)d\theta - \varphi(\xi,t) \qquad (\xi \in \mathbb{R}, t > 0) \tag{5}$$

with initial condition

$$\varphi_0(\xi) := \int_{\mathbb{R}} e^{i\xi x} f_0(x)dx \qquad (\xi \in \mathbb{R}). \tag{6}$$

Problem (5)-(6) could be seen as a new, slightly more general problem, if compared to (2). Indeed, φ_0 and $\varphi(\cdot,t)$ could be viewed as Fourier-Stieltjes transforms of not necessarily absolutely continuous probability measures μ_0 and $\mu(\cdot,t)$, respectively. With reference to (2), one has $\mu_0(A) = \int_A f_0(x)dx$ and $\mu(A,t) = \int_A f(x,t)dt$, for any A in the Borel class on $\mathbb{R}$, $\mathscr{B}(\mathbb{R})$. So, in the following, we will say that $\mu(\cdot,t)$ is a solution of (5) provided that its Fourier-Stieltjes transform $\varphi(\cdot,t)$ satisfies (5) and φ_0 is the analogous transform for μ_0.

One can prove that (5)-(6) has a unique solution in the class of the Fourier-Stieltjes transforms of all probability laws on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Cf., e.g., [29], [34] and [14].

Now, from [37], the solution admits the following series expansion

$$\varphi(\xi,t) = \sum_{n\ge 1} e^{-t}(1-e^{-t})^{n-1}\hat{q}_n(\xi;\varphi_0) \tag{7}$$

where

$$
(8) \qquad \begin{cases} \hat{q}_1(\xi;\varphi_0) & := & \varphi_0(\xi) \\ \hat{q}_n(\xi;\varphi_0) & = & \dfrac{1}{n-1}\displaystyle\sum_{k=1}^{n-1} \hat{q}_k(\xi;\varphi_0) * \hat{q}_{n-k}(\xi;\varphi_0) \quad (n = 2,3,\dots) \end{cases}
$$

are valid, for every ξ in $\mathbb{R}$, with $*$ defined by

$$
(9) \qquad (\hat{g} * \hat{h})(\xi) := \frac{1}{2\pi}\int_0^{2\pi} \hat{g}(\xi\cos\theta)\hat{h}(\xi\sin\theta)d\theta.
$$

Then, $\hat{g} * \hat{h}$ gives the Fourier-Stieltjes transform of the so-called *Wild convolution*, i.e.

$$
\frac{1}{2\pi}\int_0^{2\pi} \mathscr{L}(V\cos\theta + W\sin\theta)d\theta
$$

where $\mathscr{L}(Y)$ denotes the probability distribution of Y, and V, W are independent random variables with characteristic functions $\hat{g}$ and $\hat{h}$ respectively.

For $n = 2$, (8) yields

$$
(10) \qquad \hat{q}_2(F;\varphi_0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi_0(\xi\cos\theta)\varphi_0(\xi\sin\theta)d\theta
$$

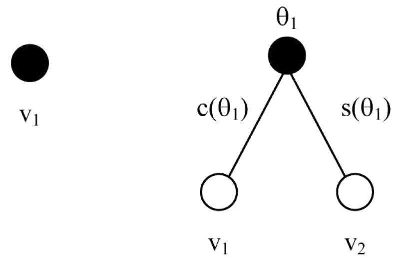
and, for $n = 3$,

$$
\begin{aligned}
(11) \qquad & \hat{q}_3(\xi;\varphi_0) \\
&= \frac{1}{(2\pi)^2}\int_{[0,2\pi)^2} \varphi_0(\xi\cos\theta_1)\varphi_0(\xi\cos\theta_2\sin\theta_1)\varphi_0(\xi\sin\theta_2\sin\theta_1)d\theta_1 d\theta_2 \\
&+ \frac{1}{(2\pi)^2}\int_{[0,2\pi)^2} \varphi_0(\xi\cos\theta_2\cos\theta_1)\varphi_0(\xi\sin\theta_2\cos\theta_1)\varphi_0(\xi\sin\theta_1)d\theta_1 d\theta_2.
\end{aligned}
$$

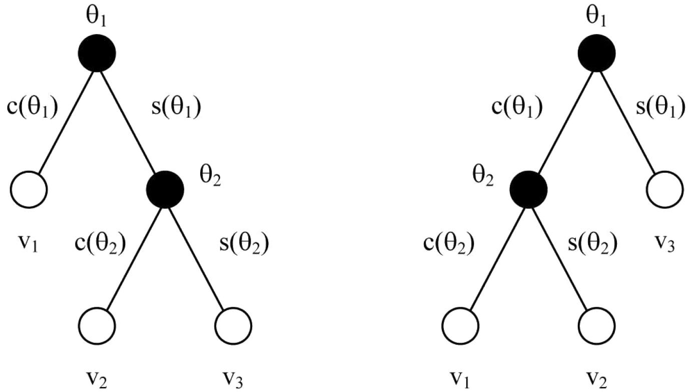
In extending these computations to any n, McKean proposed to make use of certain tree graphs, commonly referred to as *McKean trees*. They are characterized by the fact that each node has either 0 or 2 children: a "left" and a "right" child, respectively. So, for each n, the set $G(n)$ of all trees with n leaves has cardinality equal to the *Catalan number*

$$
C_n = \frac{1}{n}\binom{2n-2}{n-1}.
$$

See Section 15 of volume 1 of [12]. For $n = 1$, as well for $n = 2$, there is one McKean tree, i.e.



whereas, for $n = 3$, there are exactly two trees as in the figure below



Shaded circles stand for nodes, whilst unshaded ones stand for leaves and $c = \cos$, $s = \sin$. Nodes are labelled with $\theta_1, \theta_2, \dots$ according to a level-left to right order, while leaves are labelled by $v_1, v_2, \dots$ following a left to right order. Finally, arcs are identified by circular functions $\cos(\,\cdot\,)$ and $\sin(\,\cdot\,)$ according to the following criterion: $\cos(\theta_j)$ or $\sin(\theta_j)$ for an arc coming out from the node θ_j, depending on whether such an arc is "left" or "right". The path that connects each leaf to the root, in a specific tree, turns out to be completely described by a finite sequence of circular functions whose *product* will be indicated by $\beta_\ell(\gamma)$, γ being the "name" of the tree and ℓ the index of the label (v_ℓ) associated with the leaf taken into consideration. The number of factors (arcs) is called *depth* of this leaf, and will be denoted by δ_ℓ.

After describing McKean trees, define u^* to be the Daniell-Kolmogorov product measure on $([0, 2\pi)^\infty, \mathscr{B}[0, 2\pi)^\infty)$ which makes the coordinates θ_k i.i.d. with common uniform distribution on $[0, 2\pi)$. Then, consistently with (10) and (11),

$$\hat{q}_n(\xi; \varphi_0) = \sum_{\gamma \in G(n)} p_n(\gamma) \int_{[0,2\pi)^\infty} \prod_{\ell=1}^{n} \varphi_0(\beta_\ell(\gamma)\xi) u^*(d\theta) \tag{12}$$

with $p_n(\gamma) > 0$ for each γ in $G(n)$ and $\sum_{\gamma \in G(n)} p_n(\gamma) = 1$. In view of (8), the coefficient

p_n can be determined recursively as follows. For the unique element γ of $G(1)$, set $p_1(\gamma) = 1$. Next, for each γ in $G(n)$ with $n \geq 2$, erase the root to obtain two trees: a "left" tree and a "right" one γ_ℓ and γ_r respectively. It is easy to see that

$$p_n(\gamma) = \frac{1}{n-1} p_k(\gamma_\ell) p_{n-k}(\gamma_r) \tag{13}$$

holds true when k stands for the number, in $\{1, 2, \ldots, n-1\}$, of the leaves of γ_ℓ.

At this stage, plugging (12) in (7) gives

$$\varphi(\xi, t) = \sum_{n \geq 1} \sum_{\gamma \in G(n)} e^{-t}(1 - e^{-t})^{n-1} p_n(\gamma) \int_{[0,2\pi)^\infty} \prod_{\ell=1}^{n} \varphi_0(\beta_\ell(\gamma)\xi) u^*(d\theta) \tag{14}$$

i.e.: $\varphi(\xi, t)$ turns out to be characteristic function of a random weighted sum of random variables. This fact is crucial for future developments and, therefore, we pause over the definition of this random sum.

Put $\mathbb{G} := \bigcup_{n \geq 1} \mathbb{G}(n)$ and with each $t > 0$ associate a copy Ω_t of the product space

$$\mathbb{N} \times \mathbb{G} \times [0, 2\pi)^\infty \times \mathbb{R}^\infty$$

with $\mathbb{N} := \{1, 2, \ldots\}$. Equip Ω_t with its natural product topology and denote the Borel σ-field on Ω_t by $\mathscr{F}_t$. Designate the coordinate random variables of Ω_t by

$$\nu_t, \tau_t, \theta_t := (\theta_{t,n})_{n \geq 1}, v_t := (v_{t,n})_{n \geq 1}$$

and form the probability space $(\Omega_t, \mathscr{F}_t, P_t)$ by specifying

$$P_t\{\nu_t = n, \tau_t = \gamma, \theta_t \in A, v_t \in B\}$$

$$= \begin{cases} 0 & \gamma \notin \mathbb{G}(n) \\ e^{-t}(1 - e^{-t})^{n-1} p_n(\gamma) u^*(A) \mu_0^*(B) & \gamma \in \mathbb{G}(n) \end{cases}$$

for every n in $\mathbb{N}$, γ in $\mathbb{G}$, A in $\mathscr{B}[0, 2\pi)^\infty$ and B in $\mathscr{B}(\mathbb{R}^\infty)$, μ_0^* being the probability distribution of $(v_{t,n})_{n \geq 1}$ that makes the random variables $v_{t,n}$ i.i.d. with common probability distribution μ_0. Throughout the rest of the paper, E_t will denote expectation with respect to P_t. Moreover, it is understood that $\theta_{i,t}$ ($v_{i,t}$, respectively) will replace θ_i as labels of nodes (v_i, respectively, as labels of leaves) and that, consequently, arcs will be labelled by $c(\theta_{i,t})$ or $s(\theta_{i,t})$ in place of $c(\theta_i)$ or $s(\theta_i)$, respectively, whenever $\beta_\ell(\tau_t)$ will be used in the place of $\beta_\ell(\gamma)$. Now, we are in a position to make precise the McKean probabilistic interpretation of the solution of (5), in a form recently given in [20].

THEOREM 3.1. – *For each $t > 0$, one gets*

$$\varphi(\xi, t) = E_t\left(\exp\left\{i\xi \sum_{\ell=1}^{\nu_t} \beta_\ell(\tau_t) v_{t,\ell}\right\}\right) \qquad (\xi \in \mathbb{R}).$$

This is tantamount to saying that $\mu(\cdot,t)$, the solution of (5) *with initial datum μ_0, turns out to be the probability distribution of*

$$V_t := \sum_{\ell=1}^{\nu_t} \beta_\ell(\tau_t) v_{\ell,t} \tag{15}$$

for any $t > 0$. (For $t = 0$, V_0 is to be meant as any random variable with probability distribution μ_0.)

At this stage, it is worth emphasizing a few important facts such as:

(a) The identity

$$\sum_{\ell=1}^{\nu_t} \beta_\ell^2(\tau_t) = 1 \tag{16}$$

is valid for any $t > 0$ and τ_t in $\mathbb{G}$.

(b) For any $n \geq 2$, one has

$$\int_{[0,2\pi)^\infty} \prod_{\ell=1}^{n} \varphi_0(\beta_\ell(\gamma)\xi) u^*(d\theta) = \int_{[0,2\pi)^\infty} \prod_{\ell=1}^{n} \varphi_{0,s}(\beta_\ell(\gamma)\xi) u^*(d\theta) \quad (\xi \in \mathbb{R})$$

for every γ in $\mathbb{G}(n)$, with

$$\varphi_{0,s}(\xi) := \frac{\varphi_0(\xi) + \varphi_0(-\xi)}{2} = (\mathrm{Re}\,\varphi_0)(\xi) \quad (\xi \in \mathbb{R}). \tag{17}$$

The probability distribution of any random variable having characteristic function (17) is given by

$$\mu_{0,s}(A) = \frac{\mu_0(A) + \mu_0(-A)}{2} \tag{18}$$

for every A in $\mathscr{B}(\mathbb{R})$ and $-A := \{x \in \mathbb{R} : -x \in A\}$. Hence, designating the solution of (5) by $\varphi_s(\cdot,t)$, when the initial datum φ_0 is replaced by $\varphi_{0,s}$, one gets

$$\varphi(\xi,t) = \varphi_s(\xi,t) + \frac{\varphi_0(\xi) - \varphi_0(-\xi)}{2} e^{-t}$$

i.e.

$$\mu(A,t) = \mu_s(A,t) + \frac{\mu_0(A) - \mu_0(-A)}{2} e^{-t} \qquad (A \in \mathscr{B}(\mathbb{R}), t > 0)$$

μ_s being the probability whose Fourier-Stieltjes transform is just $\varphi_s(\cdot,t)$.

In view of (b), one is allowed to investigate the integro-differential problem (5) assuming that the initial datum is a symmetric distribution, without real loss of generality. This is useful since, in general, the symmetry assumption simplifies certain types of computations and reasoning. In particular, one gets

(c) If μ_0 is a symmetric probability distribution on $\mathbb{R}$, then V_t and

$$\sum_{\ell=1}^{\nu_t} |\beta_\ell(\tau_t)| v_{\ell,t}$$

have the same probability distribution.

Before clarifying the link Theorem 3.1 establishes between convergence of the solution of (2) or (5) and central limit problem, let us look at a possible physical interpretation of (15). Recall that the Kac kinetic equation originates from the n-particle Kac model, as n diverges to infinity, through he propagation in time of the Boltzmann factorization property. Now, fix one of these infinite particles and let ν_t be the (random) number of particles with which the fixed one collides at time t. Each McKean tree provides a description of the collisions experienced by each of the ν_t particles, represented by the leaves, before each particle contributes to the velocity V_t of the fixed particle, which, in turn, is represented by the root. For an ℓ in $\{1, \dots, \nu_t\}$ the contribution of particle ℓ is given by its initial velocity $v_{\ell,t}$ multiplied by the reducing factor β_ℓ. The factor is determined by the number of collisions (the depth of ℓ) before contributing to V_t, by the rotation angles $\theta_1, \theta_2, \dots$ and by the position of ℓ in each collision. All these circumstances are characterized by the path that, in the tree, connects leaf ℓ and the root. See also [5].

Getting down to examining connections with the central limit problem, it should be noted that the probability distribution of V_t, i.e. the solution of (5), can be written as expectation of any version of the conditional distribution of V_t given $U_t := (\nu_t, \tau_t, \theta_t)$. It is easy to check that there is a version of the conditional distribution of $(\nu_t, \tau_t, \theta_t, v_t)$, given U_t, with respect to which the random variables $v_{t,\ell}$ are i.i.d.. Pick one of these versions, say $P^*(\cdot; U_t)$, and consider its determination for $U_t = (n, \gamma, a)$ with γ in $\mathbb{G}(n)$ and a in $[0, 2\pi)^\infty$. Moreover, let $q_\ell = q_\ell(n, \gamma, a)$ be the value of $\beta_\ell(\tau_t)$ at $U_t = (n, \gamma, a)$. At this point, we can determine the probability distribution of $\sum_{\ell=1}^{n} q_\ell v_\ell$ where $v_1, v_2, \dots$ are i.i.d. random variables with common probability distribution μ_0. Notice that, substituting, in the expression of this distribution, $(\nu_t, \tau_t, \theta_t)$ for (n, γ, a), we get a version of the conditional distribution of V_t given U_t. In any case, one can write

$$\text{(19)} \quad P_t\{V_t \le x\} = \sum_{n \ge 1} \sum_{\gamma \in \mathbb{G}(n)} e^{-t}(1 - e^{-t})^{n-1} p_n(\gamma) \cdot \int_{[0,2\pi)^\infty} P^*\left(\sum_{\ell=1}^{n} q_\ell v_\ell \le x; U_t = (n, \gamma, a)\right) u^*(da)$$

for every x in $\mathbb{R}$. It must be noted that, with respect to P^*, $\sum_{\ell=1}^{n} q_\ell v_\ell$ turns out to be a (standard) weighted sum of independent random variables. Under suitable conditions, the asymptotic behavior (as $n \to +\infty$) of the probability laws of these sums can be successfully studied by resorting to the central limit theorem. Hence, in view of the structure of (19), it is easy to understand the role played by such a theorem in studying the convergence to equilibrium of the solution of Kac's equation.

4. – Analysis of the convergence of the solution with respect to "weak" metrics: Kolmogorov and Monge-Wasserstein.

A sequence of probability measures $\mu_n, n = 1, 2, \ldots$, defined on $(\mathbb{R}, \mathscr{B}(\mathbb{R})$, is said to *converge weakly* to the probability measure μ if

$$\lim_{n\to+\infty} \mu_n(A) = \mu(A) \tag{20}$$

holds for every μ-continuity set A, i.e. any A in $\mathscr{B}(\mathbb{R})$ with $\mu(\partial A) = 0$. This is expressed by writing $\mu_n \Rightarrow \mu$. It is well-known (see, e.g., Chapter 5 of [3]) that the following three conditions are equivalent:

(*i*) $\mu_n \Rightarrow \mu$.

(*ii*) $\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu$ for every bounded, continuous real function f on $\mathbb{R}$.

(*iii*) $F_n(x) \to F(x)$ for every continuity point x of F, with $F_n(x) := \mu_n((-\infty, x])$ and $F(x) := \mu((-\infty, x])$ for any x in $\mathbb{R}$ and $n = 1, 2, \ldots$

The importance of the weak convergence of probability measures is masterfully explained, for example, in Chapter 4 of [26]. We will explain that if F is continuous, then weak convergence of μ_n to μ is equivalent to the condition $K(\mu_n, \mu) \to 0$, as $n \to +\infty$, where K stands for the Kolmogorov distance between μ_n and μ, defined by

$$K(\mu_n, \mu) = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)|. \tag{21}$$

Now, given a pair of probability measures m_1 and m_2 on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, let $\mathscr{H}(m_1, m_2)$ denote the class of all probability measures m on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$ such that

$$m(A \times \mathbb{R}) = m_1(A), \quad m(\mathbb{R} \times B) = m_2(B) \qquad (A, B \in \mathscr{B}(\mathbb{R})).$$

If $\int_{\mathbb{R}} |x|^p m_i(dx) < +\infty$ for $i = 1, 2$ and for some $p \geq 1$, then the real number

$$\mathscr{G}_p(m_1, m_2) := \left\{ \min_{m\in\mathscr{H}(m_1,m_2)} \int_{\mathbb{R}^2} |x - y|^p m(dxdy) \right\}^{1/p} \tag{22}$$

is called *Monge-Wasserstein distance* between m_1 and m_2. The Italian statistician Gini first introduced distance (22) in [22] for statistics-theoretical purposes, when p is either 1 or 2 and probabilities m_1 and m_2 are discrete. In general, convergence with respect to $\mathscr{G}_p$ entails both weak convergence and convergence of any (pseudo-) moment of order r in $[1, p]$. Viceversa, weak convergence combined with moments convergence, up to order $p \geq 1$, implies convergence with respect to $\mathscr{G}_p$. See Corollary 7.5.3 in [32].

Let us now present a few results concerning the speed of approach to equilibrium of the solution of Kac's equation with respect to both the above-mentioned metrics. In the light of the next statement, in the present case weak convergence holds if and only if either of the other two types of convergence comes true.

From now on, $F(\cdot, t)$ will denote the probability distribution function associated with the solution $\mu(\cdot, t)$ of (5). γ_σ and G_σ will designate the Gaussian distribution and the corresponding distribution function, respectively, with mean zero and variance σ^2. The same symbols with $\sigma = 0$ will be used for the unit mass at zero and the corresponding distribution function.

THEOREM 4.1. – *The solution $\mu(\cdot, t)$ of the Boltzmann problem* (5)-(6) *converges weakly, as $t \to +\infty$, if and only if μ_0 has finite second moment. Moreover, if $\int_{\mathbb{R}} x^2 \mu_0(dx) = \sigma^2 < +\infty$, then $\mu_0(\cdot, t) \Rightarrow \gamma_\sigma$ as $t \to +\infty$.*

For a proof of this statement, see [20]. Since G_σ is continuous whenever $\sigma > 0$, from a classical theorem due to Pólya (see, e.g., Theorem 1.11 in [31]) one obtains that $K(\mu(\cdot, t), \gamma_\sigma) \to 0$, as $t \to +\infty$, when $0 < \sigma < +\infty$. In other words, in Theorem 4.1 weak convergence can be replaced with convergence with respect to the Kolmogorov metric. An analogous conclusion holds for Monge-Wasserstein metrics $\mathscr{G}_p$ with $1 \leq p \leq 2$. To see this, first note that, if σ is finite, then the first two moments of $\mu(\cdot, t)$ satisfy

$$\begin{cases} \int_{\mathbb{R}} x\mu(dx, t) & = & e^{-t} \int_{\mathbb{R}} x\mu_0(dx) \to 0 = \int_{\mathbb{R}} x\gamma_\sigma(dx) \\ \int_{\mathbb{R}} x^2\mu(dx, t) & = & \sigma^2 = \int_{\mathbb{R}} x^2\gamma_\sigma(dx). \end{cases} \tag{23}$$

Then, recalling the above-mentioned relations between weak convergence and $\mathscr{G}_p$-convergence, combination of (23) with Theorem 4.1 implies that $\mathscr{G}_p(\mu(\,\cdot\, .t), \gamma_\sigma) \to 0$, as $t \to +\infty$, for every p in $[1, 2]$, provided that μ_0 has finite second moment. These remarks suffice to justify the equivalence statement made immediately before Theorem 4.1.

Now, before providing bounds for the rates of convergence, we give a short account of the reasoning used in [20] to prove the necessary condition specified in the previous theorem. This way of reasoning is essentially the same as in [18] and rests on the discussion (see the end of the previous section) about the condition of conditional independence. After denoting a version of the conditional distribution of V_t, given U_t, by Λ_{v_t}, the first step of the argument consists in proving that convergence in distribution of V_t as $t \to +\infty$, implies that any increasing and diverging (to infinity) sequence of positive terms $(t_n)_{n\geq 1}$ contains a subsequence $(t_{n'})$ for which

(24) *the probability law of $\Lambda_{v_{t_{n'}}}$ weakly converges to the law of Λ*

where Λ is some (random) probability measure. It is worth noticing that, for the theory of weak convergence of probability measures, we refer both to [4] and to [16]. In the second step, via the Skorohod-Dudley representation (see, e.g., pages 70-71 of [4]), one transforms (24) into a statement about (almost sure) weak convergence of a suitably defined random distribution $\Lambda^*_{v^*_{t_{n'}}}$ towards a random probability measure Λ^*, where $\Lambda^*_{v^*_{t_{n'}}}$ has the distribution of $\Lambda_{v_{t_{n'}}}$, and Λ^* the distribution of Λ. At this stage, the general central limit theorem (see, for example, Theorem 3.3 in [31]) can be employed to deduce a necessary condition for the convergence of $\Lambda^*_{v^*_{t_{n'}}}$. Finally, one concludes by showing that this condition boils down to the existence of a bounded variance for the initial distribution μ_0. To verify that the condition of the theorem is sufficient, it is enough to check that either of the distances $K(\mu(\cdot,t),\gamma_\sigma)$, $\mathscr{G}_p(\mu(\cdot,t),\gamma_\sigma)$ is $o(1)$ as t goes to infinity. In fact, putting

$$B_m := \frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^m d\theta = \frac{1}{2\pi}\int_0^{2\pi} |\sin\theta|^m d\theta$$

and

$$\overline{m}_p := \int_{\mathbb{R}} |x|^p \mu_0(dx)$$

for any positive m and p, one can prove

THEOREM 4.2. – *If μ_0 has finite second moment σ^2 and a, p, ρ are numbers obeying*

$$0 < a < 1, \quad p > 2, \quad 0 < \rho < \frac{1-2B_p}{p},$$

then, for any $t > 0$,

$$K(\mu(\cdot,t),\gamma_\sigma) \leq 12\left\{\frac{1}{\sigma^2}\int_{\sigma x_t}^{+\infty} x^2\mu_0(dx) + e^{-t(1-2B_{2+a})} + e^{-t(1-2B_p-\rho p)}\right\}$$

where $x_t := \exp\{\rho t(1-a)\}$. Furthermore, if $\overline{m}_{2+\delta} < +\infty$ for some δ in $(0,1]$, *then*

$$K(\mu(\cdot,t),\gamma_\sigma) \leq 7\,\frac{\overline{m}_{2+\delta}}{\sigma^{2+\delta}}\exp -t(1-2B_{2+\delta}) \qquad (t>0).$$

This proposition is proved in [21] by applying an improvement of the Berry-Esseen theorem, contained in [17], to the conditional distribution of V_t, given U_t, and by using certain identities given in [19]:

$$\begin{aligned} &E_t\left(\sum_{\ell=1}^{v_t}\left(\frac{c}{2}\right)^{\delta_\ell}\middle| v_t\right) = \frac{\Gamma(c+v_t-1)}{\Gamma(c)\Gamma(v_t)} \qquad (c>0)\\ &E_t\left(\sum_{\ell=1}^{v_t}|\beta_\ell(\tau_t)|^m\right) = \exp\{-t(1-2B_m)\} \qquad (m>0, t>0). \end{aligned} \tag{25}$$

An analogous statement for the Monge-Wasserstein distance $\mathscr{G}_1$ can be obtained as above, by replacing the Feller theorem with Theorem 2.1.24 in [33] and, when $\overline{m}_{2+\delta}$ is finite for some δ in $(0,1]$, with Theorem 2.1 in [10].

THEOREM 4.3. – *If μ_0 has finite second moment σ^2, then, for any triplet (a,p,ρ) with $a>0, p>2$ and $\rho>0$ such that*

$$a+\rho<\frac{1-2B_p}{p},$$

one obtains

$$\begin{aligned}\mathscr{G}_1(\mu(\cdot,t),\gamma_\sigma) \leq \sigma\Big\{ & e^{-t(1-2B_p-p(a+\rho))}(3\sqrt{2\pi}+6(1-e^{-at}))+6e^{-at}(1+e^{-\rho t})\\ &+\frac{3\sqrt{2\pi}}{\sigma^2}\int\limits_{|x|>2\sigma e^{\rho t}} x^2\rho_0(dx)\Big\} + \overline{m}_1 e^{-t} \quad (t>0).\end{aligned}$$

Moreover, if $\overline{m}_{2+\delta}<+\infty$ for some δ in $(0,1]$ then there is a universal constant C^* such that

$$\mathscr{G}_1(\mu(\cdot,t),\gamma_\sigma) \leq C^*\frac{\overline{m}_{2+\delta}}{\sigma^{2+\delta}}e^{-t(1-2B_{2+\delta})}+\overline{m}_1 e^{-t} \quad (t>0).$$

In the same paper where the previous proposition has been formulated, i.e. [21], bounds for $\mathscr{G}_2$ have been obtained. The argument employed is rather technical and difficult to be summarized in a few words.

THEOREM 4.4. – *If $\int_{\mathbb{R}} x^2\mu_0(dx)=\sigma^2<+\infty$ then, for any (a,p,ρ,ε) satisfying $a\in(0,1), p>2, 0<\rho<\frac{1-2p}{p}, \varepsilon\in(0,1)$, there is $A=A(a,p,\rho,\varepsilon)$ for which*

$$\mathscr{G}_2(\mu(\cdot,t),\gamma_0)^2 \leq A\frac{\sigma^2}{|\ln h^*(t)|^{1/2}}+8\sigma^2 e^{-t} \quad (t>0)$$

with

$$h^*(t) := \min\left\{ e^{-3}, e^{-t(1-2B_{2+a})} + 2e^{-t(1-2B_p-\rho p)} + \frac{1}{\sigma^2} \int\limits_{\sigma e^{\rho t(1-a)}}^{+\infty} x^2 \mu_0(dx) \right\}.$$

Moreover, if $\overline{m}_{2+\delta} < +\infty$, *then there is a universal constant* C^+ *for which*

$$\mathscr{G}_2^2(\mu(\cdot,t),\gamma_0) \le C^+ \frac{\overline{m}_{2+\delta}}{\sigma^\delta} \exp\{-t(1-2B_{2+\delta})\} + 8\sigma^2 e^{-t} \quad (t>0).$$

When σ^2 is finite and $\int\limits_{\mathbb{R}} |x|^{2+\delta} \mu_0(dx) = +\infty$ for every $\delta > 0$, the upper bounds in the previous theorems cannot go to zero exponentially: they depend on μ_0 essentially only through the behaviour near $r = 0$ of the function $r \mapsto \int\limits_{|v|>1/r} |v|^2 \mu_0(dv)$.

If σ^2 is not finite, then $\mu(\cdot,t)$ converges to the null measure vaguely. More precisely, the next theorem says that if the initial energy is infinite, then the total mass of the limiting distribution splits into two equal masses (of value $1/2$ each) which adhere to $-\infty$ and $+\infty$, respectively.

THEOREM 4.5. – *Set* $\tau_1 := (-\infty, -R], \tau_2 := [R, +\infty)$ *and*

$$L_t := \exp\left\{ t\left(1 - \frac{8}{3\pi}\right)\right\}.$$

Assume $\int\limits_{\mathbb{R}} x^2 \mu_0(dx) = +\infty$ *and let* η *be a fixed element in* $(0,1)$. *Then, there is a time* t_η *such that, for every* $t \ge t_\eta, \eta \vee (1-\eta^2) < m_0(L_t) < 1$ *is valid and, for* $1 \le i \ne j \le 2$,

$$\frac{1}{2} - A(t) + B_{i,j}(t) \le \mu(\tau_i, t) \le \frac{1}{2} + B_{i,j}(t)$$

holds for every $R \ge 2\{(m_0(L_t) - \eta)(2-\sqrt{2})\}^{-1}$, *with*

$$A(t) := \frac{R}{m_2(L_t)^{1/2}[m_0(L_t) - \eta]^{1/2}} + \frac{1}{2} e^{-t/4}$$

and

$$B_{i,j}(t) := \frac{1}{2} e^{-t} \{\mu_0(\tau_i) - \mu_0(\tau_j)\}$$

$m_k(L) := \int\limits_{[-L,L]} v^k \mu_0(dv), k = 0, 1, \ldots, L > 0.$

For the proof, see [8].

5. – Convergence of the solution in the strong sense.

Given two probability measures m_1, m_2 on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, the *total variation* (or variational) *distance* between them is defined by

$$d_{TV}(m_1, m_2) = \sup\{|m_1(A) - m_2(A)| : \ A \in \mathscr{B}(\mathbb{R})\}.$$

If m_1 and m_2 are dominated by the Lebesgue measure, then

$$d_{TV}(m_1, m_2) = \frac{1}{2}\int_{\mathbb{R}} |f_1(x) - f_2(x)|dx = \frac{1}{2}\|f_1 - f_2\|_1$$

f_i being any probability density of m_i with respect to that measure ($i = 1, 2$).

The literature on convergence to equilibrium of the solution of the Kac equation focused chiefly on the study of the behaviour of $\|f(\cdot, t) - g_\sigma\|_1$, as $t \to +\infty$, where $f(\cdot, t)$ represents the solution of (2) with initial density f_0. With a view to describing the main contributions to the subject, it is worth recalling that they have been predominantly influenced by the McKean conjecture (4) already mentioned in Section 2. In [29] he proved that

$$(4e)^{-1}\|f(\cdot, t) - g_\sigma\|_1^2 \le c_{12}t^{3/2}e^{\frac{2}{9}(8/(3\pi)-1)t} \qquad (\text{as } t \to +\infty) \tag{26}$$

holds true with a constant $c_{12} = c_{12}(f_0)$ depending upon f_0 alone, under the conditions: $\sigma^2 = 1, \int_{\mathbb{R}} |v|^3 f_0(v)dv < +\infty, (H[f] > -\infty)$ and $I[f] < +\infty$. Here, H and I stand for the entropy and the Linnik functional, respectively. For the sake of completeness, we recall that the *entropy* of a probability density function f, on $\mathbb{R}$, is defined by

$$H[f] = -\int_{\{f>0\}} f(x)\ln(f(x))dx.$$

As to the functional $I[\,\cdot\,]$, Linnik was the first to notice its importance in developing an information-theoretic proof of the central limit theorem. See [27]. In the beginning he defined such a functional as

$$I[f] = \int_{\mathbb{R}} \frac{(f'(x))^2}{f(x)}dx$$

when the probability density f is a strictly positive element of $\mathbb{C}^1(\mathbb{R})$. Afterwards, McKean extended I to the set D of all probability densities with finite variance, according to the rule

$$I[f] = \lim_{\delta \to 0^+} I]f * g_\sigma]$$

where $*$ indicates convolution.

Since the rate of exponential decay in (26) is rather different from the one conjectured by McKean (see (4)), in [9] Carlen, Gabetta and Toscani tackled the McKean conjecture and obtained an estimate that can be considered to be arbitrarily close to the desired rate, i.e.

$$\|f(\cdot,t)-g_1\|_1 \leq C_\varepsilon \exp\left\{-\frac{1}{4}(1-\varepsilon)t\right\} \tag{27}$$

where ε is an arbitrary strictly positive number and C_ε a constant which, in general, depends both on f_0 and, unfortunately, on ε in such a way that C_ε goes to infinity as ε goes to zero. Moreover, they obtained (27) assuming rather strong hypotheses of three different kinds on the initial density f_0: finiteness of all absolute moments; Sobolev regularity in the sense that f_0 must belong to $H_m(\mathbb{R})$ for any integer m; finiteness of the Linnik functional at f_0. A further noteworthy progress is made in [6], where it is shown that the above second group are unnecessary in order to get (27).

The first actual validation of the McKean conjecture (4) has been obtained recently by resorting to suitable developments of the probabilistic viewpoint we explained in Section 3. A noteworthy feature of the approach is that the proof rests on a set of assumptions which are definitely weaker than those considered so far.

THEOREM 5.1. – *Assume that the initial probability density function, f_0, of Kac's equation* (2) *has finite fourth moment. Moreover, suppose*

$$\varphi_0(\xi) := \int_{\mathbb{R}} e^{i\xi x} f_0(x)dx = o(|\xi|^{-p}) \qquad (|\xi| \to +\infty) \tag{28}$$

for some strictly positive p. Then there is a constant C depending only on the behaviour of f_0 for which

$$\|f(\cdot,t)-g_\sigma\|_1 \leq Ce^{-t/4} \qquad (t \geq 0)$$

where $\sigma^2 = \int_{\mathbb{R}} x^2 f_0(x)dx$.

A complete proof of this proposition can be found in [15]. Here we confine ourselves to providing brief descriptions of the main steps. First, to pave the way for application of classical central limit arguments, we deal with i.i.d. real-valued random variables $X_1, X_2, \ldots, X_n$ on some probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, with common non-degenerate distribution $\tilde{\mu}_0$. It is assumed that $\tilde{\mu}_0$ is symmetric and has moment of fourth power. We denote the k-th moment and the absolute k-th moment of $\tilde{\mu}_0$ by m_k and $\overline{m}_k$, respectively. Moreover, we define $\tilde{\varphi}_0$ to be the Fourier-Stieltjes trasform of $\tilde{\mu}_0$, consider real constants $c_1, \ldots, c_n$ such that $\sum_{j=1}^{n} c_j^2 = 1$, and form the sum V_n of $Y_1, \ldots, Y_n$ where $Y_j = c_j X_j/\sqrt{m_2}$ for

$j = 1, \dots, n$. We prove that there are universal constants c_1 and c_2 such that the following inequalities

$$\left(\int_{-A}^{A} |\tilde{\varphi}_n(\xi) - e^{-\xi^2/2}|^2 d\xi \right)^{1/2} \leq c_1 \Gamma_n^4 \tag{29}$$

$$\left(\int_{-A}^{A} \left| \frac{d}{d\xi} \left\{ \tilde{\varphi}_n(\xi) - e^{-\xi^2/2} \right\} \right|^2 d\xi \right)^{1/2} \leq c_2 \Gamma_n^4 \tag{30}$$

hold true for $A = a/\Gamma_n$ whenever a belongs to $(0, 1/2]$ and $\Gamma_n^4 := \left(m_4 \sum_{j=1}^{n} c_j^4 \right) / m_2^2$. Secondly, it should be noted that Theorem 5.1 is proved once it is verified that $\int_{\mathbb{R}} |f_s(v,t) - g_\sigma)v_| dv \leq C_* e^{-t/4}$ holds for some constant C_* and for any density $f_s(\cdot, t)$ of the symmetrized probability μ_s defined in Section 3. Thus, without real loss of generality we can assume that f_0 and, consequently, $f(\cdot, t)$ are even functions. At this stage, one can start working at some version, say F^*, of the conditional probability distribution function for V_t given U_t, to obtain

$$E_t \left[\left\| \frac{d}{dV} F^*(\sigma v) - g_1(v) \right\|_1 ; U \right] \leq 2P_t(U) \leq 2(\overline{n} + 2^{\overline{n}} \cdot \overline{n}!) e^{-t/4}. \tag{31}$$

Here $E_t[\cdot; S]$ denotes integral – with respect to P_t – over the measurable set S, while $U := \{\nu_t \leq \overline{n}\} \cup \left\{ \prod_{\ell=1}^{\nu_t} \nu_t \beta_\ell(\tau_t) = 0 \right\} \cup \left\{ \sum_{\ell=1}^{\nu_t} \beta_\ell^4(\tau_t) \geq \overline{\delta} \right\}$ with $\overline{\delta} := (2^{\overline{n}} \overline{n}!)^{-1}$, $\overline{n}$ being equal to the least integer not less than $9/(2a)$ and a determined along with λ in such a way that $|\varphi_0(\xi)| \leq [\lambda^2/(\lambda^2 + \xi^2)]^a$ for every ξ. Existence of a and λ follows from (28). Moreover, by resorting to a result due to Beurling (see [2]) we can write

$$\begin{aligned} E_t \left[\left\| \frac{d}{dV} F^*(\sigma v) - g_1(v) \right\|_1 ; U^c \right] &\leq E_t \left[\left\{ \int_{\mathbb{R}} |\Delta|^2 + \int_{\mathbb{R}} |\Delta'|^2 \right\}^{1/2} ; U^c \right] \\ &\leq E_t \left[\left(\int_{|\xi| \leq A} |\Delta|^2 \right)^{1/2} ; U^c \right] + E_t \left[\left(\int_{|\xi| > A} |\Delta|^2 \right)^{1/2} ; U^c \right] \\ &\quad + E_t \left[\left(\int_{|\xi| \leq A} |\Delta'|^2 \right)^{1/2} ; U^c \right] + E_t \left[\left(\int_{|\xi| > A} |\Delta'|^2 \right)^{1/2} ; U^c \right] \end{aligned}$$

with $A := \sigma^4 \left\{ 2m_4 \left(\sum_{\ell=1}^{\nu_t} \beta_\ell^4(\tau_t) \right)^{1/4} \right\}^{-1}$ and $\Delta := \varphi^*(\xi/\sigma) - e^{-\xi^2/2}$, $\Delta' := d\Delta/d\xi$. Then, recalling (29), (30) and (25) we state there are constants c_3 and c_4 for which

$$E_t \left[\left(\int_{|\xi| \leq A} |\Delta|^2 d\xi \right)^{1/2} \right] \leq c_3 \frac{m_4}{\sigma^4} e^{-t/4} \tag{32}$$

and

$$E_t\left[\left(\int_{|\xi|\le A} |\Delta'|^2 d\xi\right)^{1/2}\right] \le c_4 \frac{m_4}{\sigma^4} e^{-t/4}. \tag{33}$$

Finally, we apply the Minkowski inequality to get

$$\left(\int_{|\xi|>A} |\Delta|^2\right)^{1/2} \le \left(\int_{|\xi|>A} |\varphi^*(\xi/\sigma)|^2\right)^{1/2} + \left(\int_{|\xi|>A} e^{-\xi^2}\right)^{1/2}$$

and

$$\left(\int_{|\xi|>A} |\Delta'|^2\right)^{1/2} \le \left(\int_{|\xi|>A} \left|\frac{d}{d\xi}\varphi^*(\xi/\sigma)\right|^2\right)^{1/2} + \left(\int_{|\xi|>A} \xi^2 e^{-\xi^2}\right)^{1/2}.$$

Now, from elementary inequalities for the error function, we find constants c_5 and c_6 so that

$$E_t\left(\int_{|\xi|\ge A} e^{-\xi^2} d\xi\right)^{1/2} \le c_5\left(\frac{2m_4}{\sigma^4}\right)^{a/2} e^{-t/4} \tag{34}$$

$$E_t\left(\int_{|\xi|\ge A} \xi^2 e^{-\xi^2} d\xi\right)^{1/2} \le c_6\left(\frac{2m_4}{\sigma^4}\right)^{a/2} e^{-t/4} \tag{35}$$

and from Berry-Esseen-like arguments there is c_7 such that

$$E_t\left[\left(\int_{|\xi|\ge A} |\varphi^*(\xi/\sigma)|^2\right)^{1/2} + \left(\int_{|\xi|\ge A} \left|\frac{d}{d\xi}\varphi^*(\xi/\sigma)\right|^2\right)^{1/2}\right] \le c_8 \frac{M^4}{\sigma^4} e^{-t^4}. \tag{36}$$

In order to complete the proof it suffices to combine inequalities from (31) to (36). □

After touching on the proof of the theorem, it is worth comparing its assumptions with previous work. To start with, our moment assumption shows that the finiteness of all moments is actually redundant. Also the finiteness of $I[f_0]$ is not needed since, in view of Lemma 2.3 in [9], one can write $\int_{\mathbb{R}} i^{i\xi x} f_0(x)dx \le |\xi|^{-1}\sqrt{I[f_0]}$. Hence, the tail assumption on φ_0, i.e. (28), turns out to be weaker than finiteness of $I[f_0]$. It should also be noted that assumptions in Theorem 5.1 are substantially independent. Indeed, for instance, initial characteristic functions like

$$\varphi_0(\xi) = \sum_{n\geq 1} a_n \left(\frac{1}{1+\xi^2}\right)^{1/n} \qquad \left(a_n > 0 \text{ for every } n, \sum_{n\geq 1} a_n = 1\right)$$

possess the moment property but do not meet the tail condition. Conversely, the Fourier transform of

$$f_0(x) = \frac{c_m}{1+|x|^{m+1}} \qquad \left(x \in \mathbb{R}; m \geq 1, \frac{1}{c_m} = \int_{\mathbb{R}} \frac{1}{1+|x|^{m+1}}\,dx\right)$$

has "good" tails but f_0 does not possess m-th moment.

One could wonder whether the assumptions made in Theorem 5.1 may be weakened in some significant manner preserving, at the same time, the validity of the rate $-1/4$. In Subsection 2.2 of [15] one can find an example of symmetric initial density like

$$f_0(x) = \frac{\beta}{2|x|^{1+\beta}} 1_{\{|x|\geq 1\}} \qquad (x \in \mathbb{R}; 3 < \beta < 4)$$

which yields a solution $f(\cdot, t)$ for (2) satisfying

$$\|f(\cdot,t) - g_\sigma\|_1 \geq C \exp\{-(1-2B_\beta)\,t\} \qquad (t \geq 0).$$

Now, since $(1-2B_\beta) < 1/4$, one can say that $m_4 = +\infty$ can in general imply that $\|f(\cdot,t) - g_\sigma\|_1$ goes to zero exponentially, but with a rate which is slower than the one provided in Theorem 5.1 under the assumption of finiteness of the fourth moment. For the sake of completeness, it should be noted that assumptions about finiteness of the entropy or of the Linnik functional could not compensate a possible lack of finiteness for moments of a certain order.

As to the tail hypothesis (28), note that it is connected with (it is actually slightly stronger than) statements like: there is some integer N such that the N-fold convolution of f_0 is bounded. In fact, the latter statement often recurs, with a role of sufficient condition, in local limit theorems within the classical Lindeberg-Lévy framework. See, for example, Chapter 7 of [30].

6. – Concluding remarks.

In the previous sections we studied a significant example of investigation into the limiting behavior of solutions of Boltzmann-like problems (the Kac equation, specifically) by resorting to methods in the domain of the central limit theorem of probability theory. We also explained how to use them in order to obtain sharp bounds for the error of convergence, under hypotheses which are definitely weaker than those considered so far in the state-of-the-art literature. This approach, in line with the McKean stance on the convergence to equilibrium in the Kac model, proves quite innovative with respect to results attained in the last few decades, when the authors have prevalently followed strategies of an analytical nature. See, e.g., the recent survey in [36]. In a forthcoming paper, a new problem will be tackled, i.e.: can the upper bound stated in Theorem 5.1 be improved? It will be proved that the answer is in the affirmative only if the fourth cumulant of $\mu_{0,s}$ $\left(\int_{\mathbb{R}} x^4 d\mu_{0_s} - 3\left(\int_{\mathbb{R}} x^2 d\mu_{0,s}\right)^2\right)$ is zero, which, in any case, is a rather peculiar condition.

To conclude, it should be recalled that the Kac model provides the pattern for the analysis of certain more physically realistic kinetic models. Many of the essential features of these more realistic models are, in any case, preserved in the Kac simplified setting. In particular, the specific probabilistic methods utilized in the previous sections can be applied to the study of the asymptotic behaviur of solutions of equations of Maxwellian molecules with constant collision kernels supported by compact subsets. Additionally, these very same methods can have applicability to the approach to equilibrium of solutions of certain inelastic variants of the Kac model in connection with the study of the behaviour of granular materials and of the redistribution of wealth in simple market economies. For these two topics, cf. [35], [28], [1].

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