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Periodic Solutions of Scalar Differential Equations without Uniqueness

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Abstract. – *The note presents a simple proof of a result due to F. Obersnel and P. Omari on the existence of periodic solutions with an arbitrary period of the first order scalar differential equation, provided equation has an n-periodic solution with the minimal period $n > 1$.*

1. T. Y. Li and J. Yorke in their pioneering work [6] proved that the existence of the periodic solution of the period 3 of the map of the interval J into itself implies *chaos* i.e. the existence of large set of periodic solutions of arbitrary periods. In differential equations the problem of chaos has been also studied and there is vast literature on this subject, see e.g. [1, 3, 5, 8] and references therein.

Recently F. Obersnel and P. Omari in [7] and J. Andres, T. Fürst and K. Pastor in [2] investigated the chaos for the scalar equation

$$x' = f(t, x), \tag{1}$$

where $f(t+1, x) = f(t, x)$ for $(t, x) \in \mathbb{R}^2$.

One of results of [7], proved using the technique of upper and lower upper solutions of (1) and for details referring to the earlier work [8] of authors, is the following theorem.

THEOREM 1 (See [7, Thm 2]). – *Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is 1-periodic in t and satisfies the L^1-Carathéodory condition ($f(\cdot, x)$ is measurable for every $x \in \mathbb{R}$, $f(t, \cdot)$ is continuous for a.e. $t \in [0, 1]$, $|f(t, x)| \leq m(t)$ for a.e. $t \in [0, 1]$ with $m \in L^1(0, 1)$). If (1) has a subharmonic solution of order $n > 1$, then for every $k > 1$ there exists a subharmonic solution of (1) of order k. Moreover, the set $\mathcal{X}_k$ of all subharmonic solutions of order k has dimension at least k as the subset of $L^\infty(\mathbb{R})$.*

Recall that the solution of (1) is said to be subharmonic of order $n \in \mathbb{N}$ (n-subharmonic), provided it is periodic with the minimal period $n > 1$.

The original proof of F. Obersnel and P. Omari has been simplified in [2] under the additional assumption of the global existence of solutions to (1).

The purpose of this note is to present a still simpler direct proof of Theorem 1 exploiting the geometry of the plane $\mathbb{R}^2$ and not requiring the global existence of solutions as assumed in [2].

2. We begin with the following obvious lemma, fundamental for further considerations.

LEMMA 1. – *If f is 1-periodic in t, and u, defined for $t \in [\alpha, \beta]$, is a solution of* (1), *then the functions $u(\cdot \pm 1)$ (defined on $[\alpha \mp 1, \beta \mp 1]$) are also solutions of* (1).

PROOF OF THEOREM 1. – The first part of the theorem will be completed if it can be shown that (1) has two distinct 1-periodic solutions having a common point, i.e. that for certain p and τ the boundary value problem

$$x' = f(t,x), \quad x(\tau) = p, \quad x(\tau+1) = p, \tag{2}$$

has two distinct solutions.

In fact, if u_i, $i = 1,2$ are distinct and satisfy (2), then for any fixed $k \in \mathbb{N}$ defining a function $\tilde{z} : [\tau, \tau+k] \to \mathbb{R}$ by

$$\tilde{z}(t) = \begin{cases} u_1(t-j) & \text{for } t \in [\tau\ +j, \tau\ +j+1] \qquad j = 0,1,\ldots,l-1; \\ u_2(t-j) & \text{for } t \in [\tau\ +j, \tau\ +j+1] \qquad j = l,\ldots,k-1, \end{cases}$$

we have $\tilde{z}(\tau) = u_1(\tau) = u_2(\tau+1) = \tilde{z}(\tau+k)$ and the extension z of $\tilde{z}$ onto $\mathbb{R}$ as a k-periodic function is one of the possible k-subharmonic solutions of (1).

The proof that (2) has two different solutions will be carried out assuming firstly that y is a 2-subharmonic solution of (1).

Without the loss of generality one can suppose that y attains its minimum at $t = 0$. If $y(1) = y(0)$ then, since y is of order 2, it must be $y(t) \neq y(t+1)$ for a certain $t \in (0,1)$ which in turn implies that $y(\cdot)$ and $y(\cdot +1)$ are two distinct solutions satisfying (2) on $[0,1]$.

If $y(0) < y(1)$, then defining $w(t) = y(t+1)$ we have $w(0) = y(1)$ and $w(1) = y(2) = y(0)$, hence $(w(0) - y(0))(w(1) - y(1)) < 0$. Let $\tau \in (0,1)$ be the root of $w(t) - y(t) = 0$. By 2-periodicity of w and y, $w(\tau) = y(\tau) = w(\tau+1) = y(\tau+1)$ and, consequently, functions $u = y|_{[\tau,\tau+1]}$, $v = w|_{[\tau,\tau+1]}$ satisfy (2).

The case when $y(t)$ is an n-subharmonic ($n > 2$) solution of (1) is reduced to the previous one, by showing that (1) also admits a 2-subharmonic solution.

Let $y(0) = \min\{y(t) : t \in [0,n]\}$ and let $r > 1$, $q < n$ be integers such that $y(s) = y(0)$ for $s = 0,1,\ldots,r-1$ and $y(r) > y(0)$, $y(q) > y(0)$ and $y(s) = y(0)$, for $s = q+1, q+2, \ldots, n$.

If r does not exist, then $y(0) = \cdots = y(n)$. But, since y is subharmonic of order n, $y|_{[0,1]}(\cdot) \neq y|_{[j,j+1]}(\cdot)$ for a certain j. Hence $y|_{[0,1]}(\cdot)$ and $y|_{[j,j+1]}(\cdot - j)$ are different and satisfy (2) and the proof is complete.

If $r > 1$ is defined, then q is also defined, and setting $u(t) = y(t)$, $v(t) = y(t-r+q)$ one has $u(r-1) = y(0)$, $v(r+1) = y(q+1) = y(0)$. Hence $(u(r-1) - v(r-1))(u(r+1) - v(r+1)) \leq 0$, which implies that $u(\tau) = v(\tau)$ for a certain $\tau \in [r-1, r+1]$. Let $\tilde{z} : [r-1, r+1] \to \mathbb{R}$ be defined by

$$\tilde{z}(t) = \begin{cases} y(t) & \text{for } t \in [r-1, \tau\ ] \\ v(t) & \text{for } t \in [\tau\ , r+1]. \end{cases}$$

By letting z to be the 2-periodic extension of $\tilde{z}$ onto $\mathbb{R}$ we get the required subharmonic solution of order 2 of (1), completing thus the proof of the first statement of Theorem 1.

The proof of the second conclusion of Theorem 1 consists in showing that the boundary value problem (2) has a continuum distinct, homeomorphic to $[0,1]$, solutions and, in a consequence, $[0,1]^k$ is embedded into $\mathcal{X}_k$.

By the argument of the first part of the proof, for a certain $\tau \in [0,1]$ the boundary value problem (2) has at least two distinct solutions u_1, u_2. Without loss of generality one can assume that $u_1(t) \le u_2(t)$ in $[\tau, \tau+1]$. Denote by $\mathcal{K}$ the set of all solutions x of (1) defined on $[\tau, \tau+1]$ such that $u_1(t) \le x(t) \le u_2(t)$ for $t \in [\tau, \tau+1]$.

Fix $d \in (\tau, \tau+1)$, satisfying $u_1(d) < u_2(d)$. Choose any closed subinterval J of $(u_1(d), u_2(d))$ and denote by u_a the maximal solution of (1) through (d, a).

Suppose that for $a \in J$ the solution u_a leaving the domain bounded by u_1 and u_2 crosses u_1 (the remaining cases are handled similarly): i.e. $u_a(s_1(a)) = u_1(s_1(a))$, $u_a(s_2(a)) = u_1(s_2(a))$, where $\tau \le s_1(a) \le d \le s_2(a) \le \tau+1$.

Set $b = (b_0, \dots, b_{k-1})$. Define the functions $\varphi : J \to \mathcal{K}$ and $\gamma : J^k \to L^\infty(\mathbb{R})$ by

$$(\varphi(a))(t) = x_a(t) = \begin{cases} u_a(t) & \text{for } t \in\ [s_1(a), s_2(a)], \\ u_1(t) & \text{otherwise}, \end{cases}$$

$$\gamma(b)(t) = x_{b_j}(t-j) \quad \text{for } t \in [\tau\ + j + ik, \tau\ + j + ik + 1],\ j = 0, \dots\ k-1,\ i \in \mathbb{Z}.$$

The function φ is continuous (see e.g. [4, Ch. II, Thm 1.4]), hence for any nondegenerate closed interval $I \subset J$ the set $\varphi(I) = \mathcal{T} \subset \mathcal{K}$ is a continuum homeomorphic to $[0,1]$.

For all $b \in J$ the solution $\gamma(b)$ is periodic and if $b \in \Pi_{s=0}^{k-1} I_s$, where $I_s = [\alpha_s, \beta_s]$ ($\alpha_s < \beta_s$), $s = 0, \dots, k-1$, are arbitrary pairwise disjoint subintervals of J, then $\gamma(b)$ is a k-subharmonic solution of (1).

Setting $\varphi(I_s) = \mathcal{T}_s$, from the above considerations it follows that γ is a homeomorphism between $\Pi_{s=0}^{k-1} \mathcal{T}_s$ and $\gamma(\Pi_{s=0}^{k-1} \mathcal{T}_s) \subset \mathcal{X}_k$, which completes the proof of second part of Theorem 1.

3. The result of Theorem 1 extends to the case of differential inclusions

$$x' \in F(t, x) \tag{3}$$

where the multivalued map F is 1-periodic in t.

Denote by $cc(\mathbb{R})$ the space of all convex, compact and nonempty subsets of $\mathbb{R}$ equipped with the Hausdorff metric d.

Recall that $F : \mathbb{R} \times \mathbb{R} \to cc(\mathbb{R})$ satisfies Carathéodory conditions in $\mathbb{R} \times \mathbb{R}$ if $F(\cdot, x)$ is measurable for every $x \in \mathbb{R}$, $F(t, \cdot)$ is upper semicontinuous for a.e. $t \in [0, 1]$, $d(0, F(t, x)) \leq m(t)$ for a.e. $t \in [0, 1]$, $m \in L^1(0, 1)$.

THEOREM 2 (see [3, Thm 5] or [7, Remark 3]). – *If* $F : \mathbb{R} \times \mathbb{R} \to cc(\mathbb{R})$ *is* 1*-periodic in* t *and satisfies Carathéodory conditions, then all conclusion of Theorem* 1 *applied to solutions of* (3) *hold true, provided* (3) *has an* n*-subharmonic solution with* $n > 1$.

PROOF OF THEOREM 2. – Since Lemma 1 applies to solutions of (3) and for any $(d, a) \in \mathbb{R} \times \mathbb{R}$ the maximal solution of the initial problem (3), $u(d) = a$ exists and continuously depends on a, the argument of the previous proof is applicable to the case of differential inclusions.

REMARK. – Note that in Theorems 1 and 2 no assumption concerning the existence of solutions of (1) or (3) for all t is required.

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