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Smooth 4-Dimensional Thickening of Singular 2-Dimensional Complex in the non Compact Case

Valentin Poénaru - Corrado Tanasi

Abstract. – *The present paper extends the theory of* 4*-dimensional thickening* [3], [4], *in the* NON-PROPER *case. Concrete uses of this extensions are to be founded in* [2].

1. – Introduction

The present paper is devoted to the 4-dimensional thickening functor which associates to a singular 2-dimensional polyhedron ($X^2 \xrightarrow{f} M^3$) as defined in [3], [4], [7] and [1] together with a desingularization called φ, a smooth 4-manifold called $\Theta^4(X^2, \varphi)$ which was already studied in our previous papers [3], [4], [7]. The novelty is that not only we consider here a not necessarily compact X^2, but also we take into account situations when exhausting f via elementary zipping moves (as defined, for instance in [3], [4], [7]) requires a *not locally finite* sequence of such.

We consider now a 3-manifold M^3 which is without boundary (it can be closed or open) and we consider also an interval, call it I, which could be the interval $[0, 1]$ or the real line R. The manifold M^3 is endowed with a smooth cell decomposition called τ^3 and I is endowed with a partition called θ (cell decomposition of I). We consider the product cell decomposition $\tau^3 \otimes \theta$ of $M^3 \times I$. We will be interested in the 2-skeleton of $\tau^3 \otimes \theta$ which we denote by X^2. This X^2 has the bona fide regular neighbourhood $N^4(X^2) \subset M^3 \times I$ which is a smooth 4-dimensional manifold with non empty boundary. The first object of the present paper is to give another different and is it turns out immensely useful description of this object $N^4(X^2)$. In order to do this we proceed as follows. To begin with from $X^2 \subset M^3 \times I$ we will extract a singular 2-dimensional polyhedron $X^2 \xrightarrow{f} M^3$ in the sense of [1], [3], [4]. We should stress here that this singular polyhedron is not canonical and not unique unlike the embedding $X^2 \subset M^3 \times I$. We will need to introduce a vector field $\vec{v}$ on M^3. This vector field is supposed to be in general position with respect to X^2 and is supposed to be such that the flow exists for all times. We might think of this flow either as $h_t(x)$ with $x \in M^3$, $t \in R$ or as a diffeomorphism $H = H(x, t) = h_t(x)$ entering the commutative diagram below

$$\begin{array}{ccc} M^3 \times I & \xrightarrow{\quad H \quad} & M^3 \times I \\ & \searrow \quad \swarrow & \\ & I. & \end{array}$$

Remember also that $M^3 \times I$ comes endowed with the natural projections

$$\begin{array}{ccc} M^3 \times I & \xrightarrow{\quad \pi_0 \quad} & I \\ \Big\downarrow {\scriptstyle \pi} & & \\ M^3 & & \end{array}$$

Notice to begin with that the first very naive idea to define a singular 2-dimensional polyhedron would be to consider $X^2 \xrightarrow{\pi} M^3$, but this would be a very bad idea because this map π is clearly degenerate and so we perturb π by $\pi \circ H$ and we define $f = \pi \circ H$ with

$$X^2 \xrightarrow{f} M^3.$$

The fact that the vector field is in general position with respect to X^2 means that this map is non degenerate and it also has our bona fide undrawable singularities, so this object is a singular 2-dimensional polyhedra. Observe that while X^2 is canonical this object depends strongly on the choice of vector fields. This singular 2-dimensional polyhedron comes with the desingularization which we call $\Re$; this desingularization is defined as follows. We should notice that from any singularity start two copies of double lines at the source, living in two distinct branches. With respect to the projection π_0 one is high and the other is low. We will take $\Re = S$ when the branch is high and $\Re = N$ when it is low. Our first Theorem is the following

THEOREM 1.1. – *There is a diffeomorphism*

$$N^4(X^2) = \Theta^4(X^2, \Re).$$

Consider a singular 2-dimensional polyhedron $X^2 \xrightarrow{f} M^3_{\text{open}}$ where M^3 is supposed now open and a desingularization φ which could be the one from above or another one. Under these conditions we consider the sequence of zippings

$$\begin{array}{ccc} X^3 & \xrightarrow[\text{zipping}]{\varphi} & Y^3 \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & M^3. & \end{array}$$

There are two possibilities when g is a PROPER map, in this case Y^2 is a bona fide a 2-dimensional polyhedron or when g is an IMPROPER map, in which case Y^2 is a wild object. We assume the zipping is, anyway, coherent.

THEOREM 1.2

I. *In the case of a coherent zipping, if we call by ψ the induced desingularization and we have*

$$\Theta^4(Y^2, \psi) \overset{\text{DIFF}}{=\!=} \Theta^4(X^2, \varphi) + \{\text{2-handles possibly } \infty^{\text{ty}} \text{ many}\}.$$

II. *In the* IMPROPER *case Y^2 is no longer a locally finite simplicial complex, nevertheless we can make sense open regular neighbourhood essentially by taking small open regular neighbourhood of small pieces and putting them together. With this and with the boundary of the regular neighbourhood deleted, the conclusion above holds again.*

2. – On 4-dimensional thickening of a singular 2-dimensional spaces.

We consider a closed 3-manifold M^3, the collar $M^3 \times [0,1]$ and a compact smooth submanifold $N^4 \subset M^3 \times [0,1)$. We assume both N^4 and $M^3 \times [0,1] - N^4$ to be connected and also that there is some $\eta > 0$ such that $M^3 \times [0, \eta] \subset N^4$. With this, after an appropriate reparametrization (= isotopy) it can always be assumed that for some base point $* \in M^3$ we have

$$(* \times [0,1]) \cap N^4 = * \times [0, \eta].$$

We will make great use of the product structure $M^3 \times [0,1]$, with its two projections, in what follows

$$\begin{array}{ccc} M^3 \times I & \xrightarrow[\text{vertical projection}]{\pi_0} & I \\ {\scriptstyle \pi}\big\downarrow \text{ horizontal projection} & & \\ M^3 & & \end{array}$$

Various triangulations (or cell-divisions) of smooth manifolds will be considered. It is understood that these triangulations (or cell-divisions) are always compatible with the underlying DIFF structure. We will denote by $\tau_{(p)}$ the p-skeleton of the triangulation (or cell-division) τ. In particular, our M^3 will be endowed with such a triangulation (or cell-division) τ and $I = [0,1]$ will be endowed with a decomposition (i.e. a triangulation) θ, namely

$$0 = t_0 < t_1 < \cdots < t_j = \eta < t_{j+1} < \cdots < t_n = 1.$$

For $M^3 \times I$, two kinds of cell-decompositions will be considered. To begin with we will have the *prismatical cell-division* (which we will sometimes loosely call just a "prismatical triangulation") $\tau \otimes \theta$, where the 0-cells are of the form {vertex of $\tau\} \times t_i$ and the k-cells ($k > 0$) are of the form

(1) $$\{(k-1) \text{ simplex of } \tau\} \times [t_i, t_{i+1}] \text{ or } \{k\text{-simplex of } \tau\} \times t_i.$$

Assume now τ is a bona fide triangulation. Then, after a total order on the set of vertices of τ has been chosen once and for all, each prism (1) can be triangulated in a standard, well-known fashion so as to change $\tau \otimes \theta$ into a bona fide *triangulation* of $M^3 \times I$, with the same 0-skeleton as $\tau \otimes \theta$ and which we will denote by $\tau \otimes \theta$. The τ's which we will consider from now on will be of a very special type. We start by considering a spitting

(2) $$M^3 = \underbrace{(M^3 - \operatorname{int} B^3)}_{\text{call this } M_0^3} \bigcup_{S^2} B^3 \text{ with } * \in B^3,$$

and a smooth submersion

(3) $$M^3 - \{*\} \xrightarrow{g} R^3.$$

The existence of such a g follows from the Smale-Hirsch theory. The submersion g induces a *flat riemannian metric* on $M^3 - \{*\}$, which is of course not complete, but this is immaterial for our purposes.

So, locally at least, $M^3 - \{*\}$ has now an *affine structure*, and all the triangulation τ (respectively cell-divisions) we consider from now on for M^3 are such that

1) M_0^3 (see (2)) is a subcomplex and $M^3 - \operatorname{int} M_0^3$ is a 3-cell of τ.

2) Any simplex (cell) of M_0^3 is affine-convex on the sense that it is mapped isorphically by g (see (3)) into an affine simplex (or affine convex-cell) of R^3. Such τ's will be called "affine".

We will choose once and for all a vector $\overrightarrow{V} \in R^3$ which is *in general position with respect to* $g_*\tau$, in the sense that if σ^2 is a 2-simplex (or 2-cell) of τ, or if σ^1, s^1 are two one-simplexes with a common vertex, then $\overrightarrow{V}$ is in general position with respect to the planes generated by $g\sigma^2$ and/or by $(g\sigma^1, gs^1)$; here "general position" is meant in the linear sense and g is the submersion (3) which has been also chosen once for all. With this, we consider a vector field ξ on M^3, which is equal to the pull back of $\overrightarrow{V}$ via the local diffeomorphism g, in the neighbourhood of M^3_0, and zero in the neighbourhood of $* \in M^3$. Conditions for the norm $\|\xi\|$ of this "*constant*" *vector field* will be imposed later on. We will denote the flow of ξ by $M^3 \xrightarrow{h_t} M^3$ and/or by

$$\begin{array}{ccc} M^3 \times I & \xrightarrow{\quad H \quad} & M^3 \times I \\ & {}_{\pi_0}\searrow \quad \swarrow_{\pi_0} & \\ & I & \end{array}$$

We denote the diffeomorphism $H(x,t) = (h_t(x), t)$. The role of the flow H will be to ***perturb*** the badly degenerate projection

$$K^2 \stackrel{\text{def}}{=} \{(\tau \otimes \theta)_{(2)} \text{ on the whole } M^3 \times I\} \xrightarrow{\pi} M^3$$

into a ***non-degenerate*** map

$$K^2 \xrightarrow{f \stackrel{\text{def}}{=} \pi\circ(H|K^2)} M^3. \tag{4}$$

It is assumed that θ is such that the various $f(K^2|t_i) \subset M^3$ are in general position.

We will denote by $\mathrm{Sing}(f) \subset K^2$ the set of points where f fails to be immersive. Because of our various general requirements, $f|(K^2 - \mathrm{Sing}(f))$ is a generic immersion, and the singularities themselves are isolated points $x = (x', t_i)$ with $x' \in \tau_{(0)}$, of the following type. At the level of τ, there are simplexes σ^2, σ^1 of dimensions two and one respectively, with $\sigma^2 \cap \sigma^1 = \{x'\}$ and which are such that the planes generated by $h_{t_i}\sigma^2$ and by $\pi H(\sigma^1 \times I) = \bigcup_{t \in I} h_t \sigma^1$ meet along a line passing through $h_{t_i}x'$. Of course, in this discussion σ^2 could also mean an affine 2-cell, when τ is a cell-decomposition. The same remark applies later on. The same x can correspond to several such lines, but by very small zippings we can blow up each x into several admissible singularities of the undrawable type, without altering the almost collapsibility of K^2. So, after such minor perturbations, (4) is an almost collapsing *singular 2-dimensional polyhedron*, in the sense of [3]. The notation (4) will be used for both interpretations, before or after the small zippings have been performed. We actually want to be very specific about the *undrawable* singularities of $K^2 \xrightarrow{f} M^3$. Every connected component of $f(\tau_{(1)} \times [0, t_i]) \cap f(\tau_{(2)} \times t_i)$ (respectively of $f(\tau_{(1)} \times [0, t_i]) \cap f((\tau_{(2)} \times t_i))$ is an arc starting at a singularity $x = (x', t_i)$ of $K^2 \xrightarrow{f} M^3$, and there are no other singularities.

Such a singularity has a *high branch* $P_x(h)$ and a *low branch* $P_x(\ell)$ with $P_x(h) \subset \tau_{(1)} \times [t_i, 1], P_x(\ell) \subset \tau_{(2)} \times t_i$ (respectively $P_x(h) \subset \tau_{(2)} \times t_i, P_x(\ell) \subset \tau_{(1)} \times [0, t_i]$). We will work from now with the basic desingularization $\mathcal{R}$ for $K^2 \xrightarrow{f} \Sigma^3$ which always specifies the high branches

$$\mathcal{R}(P_x(h)) = S, \mathcal{R}(P_x(\ell)) = N. \tag{5}$$

Here, the letter "$\mathcal{R}$" stands for "rectangular", and it is supposed to remind us that our desingularization (5) takes into account the product structure of $M^3 \times I$ (and the orientation of I). On the other hand, it is assumed here that the singularity appearing in (5) is actually of the undrawable type, obtained from the blow-up of the pristine rough singularity x.

If we consider now $N_{(2)} \stackrel{\text{def}}{=} (\tau \otimes \theta)_{(2)} | N^4$, then

$$N_{(2)} \xrightarrow{f} M^3$$

(i.e. the restriction of (4) to $N_{(2)}$) is again a singular 2-dimensional polyhedron which will be endowed with the desingularization $\mathcal{R}$ (5).

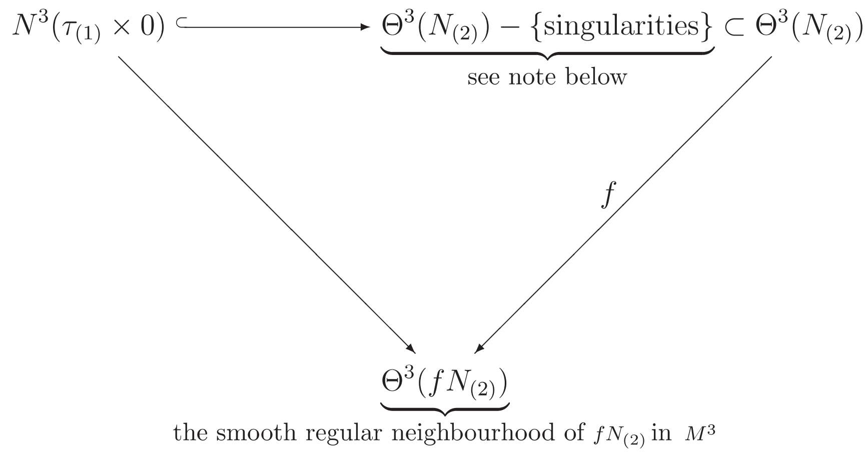
The embedding

$$(N_{(2)}, \pi_0^{-1}(0) = \tau_{(1)} \times 0) \hookrightarrow (M^3 \times [0,1], M^3 \times 0) \tag{6}$$

where τ is a triangulation of M^3 and $\tau_{(1)}$ its 1-skeleton, induces a pair of nested regular neighborhoods

$$N^3(\tau_{(1)} \times 0) \subset N^3(N_{(2)}), \text{ with } N^3 \subset \partial N^4. \tag{7}$$

But then, there is also another "natural" 3-dimensional regular neighborhood of $\tau_{(1)}$ to be considered. If we start with our singular 2-dimensional polyhedron (6) we can change it into a ***singular*** 3-manifold $\Theta^3(N_{(2)})$, and then, the $\frac{1}{2}$-collar $\tau_{(1)} \times [0, t_0) \subset N_{(2)} - \{\text{singularities}\}$, produces for us a second, canonical 3-dimensional regular neighborhood



NOTE. This can be identified with a piece of the desingularization $\check{\Theta}^3(N_{(2)}, \varphi)$.

This canonical $N^3(\tau_{(1)} \times 0)$ embeds naturally into the various $\partial\Theta^4(\cdots, \mathcal{R})$ which we will consider.

LEMMA 2.1. – *We have a diffeomorphism of pairs*

$$(N^4(N_{(2)}), N^3(\tau_{(1)} \times 0)) = (\Theta^4(N_{(2)}, \mathcal{R}), N^3(\tau_{(1)} \times 0)). \tag{8}$$

In order to simplify the exposition, we will assume that $N_{(2)} = (\tau \otimes \theta)_{(2)}|M^3 \times I$ and we will denote this by Z^2.

The case of a more general $N_{(2)}$, with N^4 subcomplex of $(\tau \oplus \theta)_{(2)}|M^3 \times I$ can be treated similarly. With all this, the two terms in (8) are two recipes for thickening Z^2 in dimension four, the first one coming from the natural embedding $Z^2 \hookrightarrow M^3 \times I$ and the second one depending on the vector field ξ which defines generic perturbation f of π. We have to show that they coincide.

We start by introducing some notations. The regular neighborhood of $Z^2 \times t_i \subset \Sigma \times t_i$ is denoted $N^3(i)$ (it is supposed to be invariant under $i \Longrightarrow i+1$) and the regular neighborhood of the 1-skeleton of $Z^2_{(1)} \times t_i$, with a Σ to be defined below, is denoted $n^3(i) \subset \operatorname{int} N^3(i)$. We denote by $N^4(i)$ the regular neighborhood of $Z^2 \times t_i \subset \Sigma^3 \times I$ and (where the Σ to be defined), if we write $\partial N^4(i) = 2N^3(i)$, we have two obvious embeddings j_i(inf,sup) of $n^3(i)$ into $\partial N^4(i)$, with disjoined images.

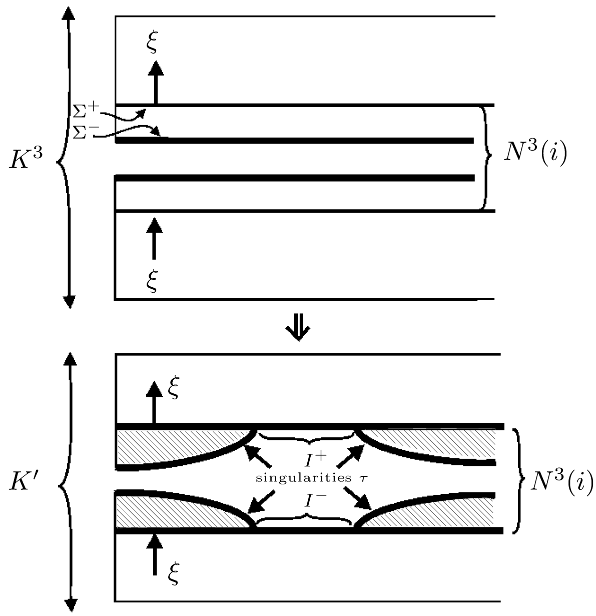


Fig. 1. – The hatched regions correspond to $\Sigma \times [t_{i-1}, t_i]$ and $\Sigma \times [t_i, t_{i+1}]$.

For $N^4(Z^2)$, we have the following obvious and very symmetrical description

$$(9)\qquad N^4(Z^2) = \cdots \cup (n^3(i) \times [t_{i-1}, t_i]) \bigcup_{j_i(\mathrm{inf})} N^4(i) \bigcup_{j_i(\mathrm{sup})} (n^3(i) \times [t_i, t_{i+1}]) \cup \cdots$$

The choice of ξ breaks the symmetry of the description of $n^3(i)$. More precisely, we have now a surface Σ which has $Z^2_{(1)} \times t_i$, as a spine and a product structure

$$(10)\qquad n^3(i) = \Sigma \times [0,1] \subset \overset{\circ}{N}{}^3(i) \subset \Sigma^3 \times t_i$$

with the $[0,1]$-factor corresponding to the vector field ξ. Here is the explicit ***construction*** of Σ. For each $p \in \Sigma^3 - (\,*\,)$, we consider a small 2-dimensional affine variety $E(p) \in p$ orthogonal to ξ, and for a small neighborhood $p \in U(p) \subset Z^2_{(1)}$ we take the orthogonal projection $U(p) \overset{\pi(p)}{\longrightarrow} E(p)$. For each $p \in Z^2_{(1)}$, we have a "local chart", which is the pair

$$C_p = \{\text{the germ of } E(p) \text{ along } \pi(p) \cup (p), U(p) \approx \pi(p)U(p)\}.$$

Whenever $U(p) \cap U(q) \neq \emptyset$, we have a diffeomorphism

$$(11)\qquad C_p|(U(p) \cap U(q)) \overset{g(p,q)}{\longrightarrow} C_p|(U(p) \cap U(q)),$$

defined by following the trajectories of ξ, and for which the cocycle condition is trivially verified. So we can put things together and get $(\Sigma, Z^2_{(1)})$, defined for the time being as a purely abstract object. We get an embedding $\Sigma \subset N^3$ which extends $Z^2_{(1)} \subset N^3$ by isotoping, to begin with $(E(p), \pi(p)U(p)) \subset N^3$, via the flow, lines of ξ, until $\pi(p)U(p)$ coincides with $U(p) \subset Z^2_{(1)}$. Further isotopies of the $E(p)$'s, along the flow lines and not budging any longer the 1-skeleton, can realize the gluing (11). We have now a singular 3-dimensional object

$$(12)\qquad K^3 \overset{\mathrm{def}}{=} \cdots \bigcup_{\Sigma} (\Sigma \times [t_{i-1}, t_i]) \bigcup_{\Sigma} N^3(i) \bigcup (\Sigma \times [t_i, t_{i-1}]) \bigcup \cdots \subset \Sigma^3 \times I$$

and $N^4(Z^2) = N^4(K^3)$; the right hand side of this equality is a second less symmetrical description of $N^4(Z^2)$ than (9). In (12) we have two copies of Σ which we will denote $\Sigma^+ \subset \overset{\circ}{N}{}^3(i)$, respectively $\Sigma^- \subset \overset{\circ}{N}{}^3(i)$, via which $\Sigma \times [t_{i-1}, t_i]$. Respectively $\Sigma \times [t_i, t_{i+1}]$ are glued to $N^3(i)$. Each of the $\Sigma^\pm$ can be independently isotoped, even allowing the two copies to cross each other (COHERENT" O(4)-move" see [1]) without changing $N^4(K^3)$. We can even bring parts of $\Sigma^\pm$ to $\partial N^3(i)$ via isotopy, without changing $N^4(K^3)$ ("O(2)-moves"). In particular, for the basic Σ, we will consider a collection of 2-by-2 disjoined squares $I_j \subset \Sigma$ resting with exactly two opposite sides on $\partial\Sigma$, such that each connected component of $\Sigma - \bigcup_j I_j$ is a 2-cell (a polygon with an even number of sides). It is understood that from the standpoint of the 2-dimensional Z^2, the I_i's correspond to little arcs in Z^2 which are far from the singularities of $Z^2 \longrightarrow \Sigma^3$. The vector field $\pm\,\xi$ gives a recipe for pushing the $I_j^\pm$ into $\partial N^3(i)$; this recipe is suggested in figure 1. This operation changes K^3 into an object which we will denote by K'; this is a 3-manifold except for singularities τ which are polygons with an even number of sides. These are actually a generalization of our usual undrawable singularities σ, in their 3-dimensional version. In order to explain this, we will go back to our usual undrawable σ's and present them in an equivalent way, where they look very much like the new τ's.

So, here comes ***an asymmetric model for the (usual) undrawable singularities*** σ. Let X be the union of the two rectangular boxes A^3, B^3 from figure 2 along the long rectangle $[a_2, b_2, c_2, d_2, d_3, b_3, a_3]$. It is understood here that the part $\sigma = [b_2, b_3, c_3, c_2]$ goes towards the interior of A^3, except for its two opposite sides $[b_1, b_3], [c_1, c_3]$. We leave it to the reader to check that this X is the same thing as our usual description of an undrawable singularity.

The genus one surface which is supposed to split X from a larger singular 3-manifold is

$$\{\partial A^3 - \{[a_1, b_1, b_4, a_4] \cup [c_1, d_1, d_4, c_4]\}\} \cup \{\partial B^3 - \{([a_2, d_2, d_3, a_3] \times 0] \\ \cup ([a_2, d_2] \times [0,1]) \cup ([a_3, d_3] \times [0,1])\}\}.$$

In this new model, if we pass to a desingularization φ with $\varphi(A^3) = S$, then we have to blow up $[b_2, c_2, c_3, b_3] \subset A^3$ into two copies $[b_2, c_2, c_3, b_3]$ (***sup***) and $[b_2, c_2, c_3, b_3]$(***inf***) having $[b_2, c_2], [b_3, c_3]$ in common; one leaves B^3 glued to the blown up A^3 along $[a_3, b_3, b_2, a_2] \cup [b_2, c_2, c_3, b_3]$(***inf***)$\cup [c_2, d_2, d_3, c_3]$. If $\varphi(A^3) = N$, then we just unglue B^3 and A^3 along the hatched areas, but

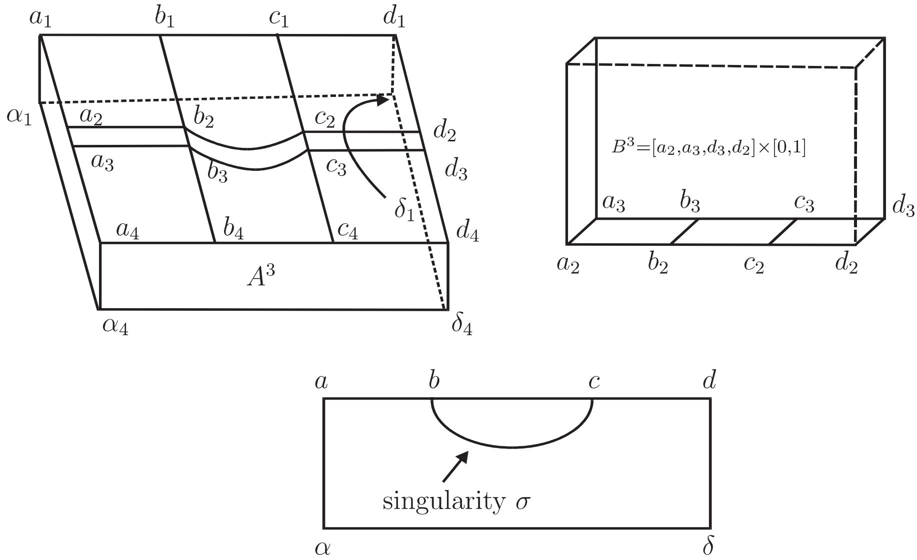


Fig. 2. – Usual, undrawable singularity σ(= a square).

leaving $[b_2, b_3], [c_2, c_3]$ put. This is our asymmetric model. But the point here is that this way of describing the undrawable σ's (which are little squares) can be immediately generalized to singularities τ which are polygons with $2q$ sides, for $q > 2$. For the sake of the argument, we will choose $q = 4$. Now we consider the union $Y = C^3 \cup D^3$ (see figure 3) the two pieces being glued along the large octagon with sides $[a_1, a_2], [a_2, a_2, \delta_1, d_1]$ (which for typographical convenience appears as a broken line), $[d_1, d_2], [d_2, \delta_2, \gamma_1, c_1], [c_1, c_2]$, $[c_2, \gamma_2, \beta_1, b_1], [b_1, b_2]$, $[b_2, \beta_2, a_1, a_1]$

It is understood that the little hatched octagon

$$\tau = [a_2, \delta_1, \delta_2, \gamma_1, \gamma_2, \beta_1, \beta_2, a_1]$$

is pushed towards the interior of C^3, except for its four opposing sides $[a_1, a_2]$, $[\beta_1, \beta_2]$, $[\gamma_1, \gamma_2]$, $[\delta_1, \delta_2]$. We can define singular 3-dimensional manifolds W^3 with singularities τ, and when our local model is part of such a W^3, then it is split off from the rest, by the surface of genus three

$$\{\partial C^3 - \{\text{the four half-disks } (a', a_1, a_2, a''), (b', \beta_1, \beta_2, b''), (c', \gamma_1, \gamma_2, c''), (d', \delta_1, \delta_2, d'')\}\}$$

$$\cup \Big\{\partial D^3 - \big\{(\text{the large octogon}) \times 0) \cup ([a_2, d_1] + [d_2, c_1] + [c_2, b_1] + [b_2, a_1] \times [0, 1])\big)\big\}\Big\}.$$

We again have desingularization φ and if

$$\varphi(C^3) = S = \varphi^*(D^3)$$

then we blow up the hatched octagon $\tau \subset C^3$ into $\tau(\text{inf}) \cup \tau(\text{sup})$ the two copies

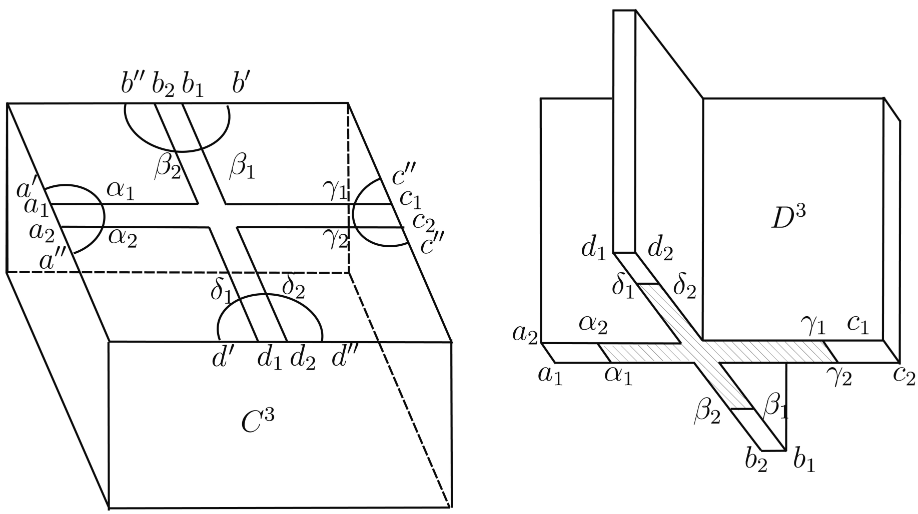


Fig. 3. – New singularity τ (the case of an octagon)
$D^3 = \{\text{the large octagon } [a_1, a_2, d_1, d_2, c_1, c_2, b_1, b_2]\} \times [0, 1]$.

staying glued along $[a_1, \beta_2] + [\beta_1, \gamma_2] + [\gamma_1, \delta_2] + [\delta_1, a_2]$. We leave D^3 glued to C^3 along a large octagon obtained from the usual one $\tau \Longrightarrow \tau(\text{inf})$.

If $\varphi(C^3) = N = \varphi^*(D^3)$ then we unglue D^3 from C^3 along τ leaving the sides $[a_1, a_2] + [\beta_1, \beta_2] + [\gamma_1, \gamma_2] + [\delta_1, \delta_2]$ untouched. All our little theory of 4-dimensional thickening $\Theta^4(V^3, \varphi)$ where V^3 has only singularities σ, can be extended to a theory of 4-dimensional thickening $\Theta^4(W^3, \varphi)$ where the more general τ's are involved.

REMARKS.

I) We would have a hard time working with singularities like the original Σ. It is essential here that a complete set of opposing sides should be on the boundary:

II) If we go to dimension two, the τ's blow up into usual singularities (see the example below).

For the $W^3 = K'$ (see figure 1) we choose the desingularization $\mathcal{R}$ which is defined by

$$\mathcal{R}(N^3(i)) = S = \mathcal{R}^*(\Sigma \times [t_{i-1}, t_i]) \text{ for singularities } \tau \subset \Sigma^-$$

respectively

$$\mathcal{R}(N^3(i)) = N = \mathcal{R}^*(\Sigma \times [t_i, t_{i+1}]) \text{ for singularities } \tau \subset \Sigma^+.$$

LEMMA 2.2.

1) *We have a diffeomorphism*

$$N^4(Z^2) = \Theta^4(K', \mathcal{R}), \tag{13}$$

where $N^4(Z^2)$ *is like in* (9) *and where the right hand side is built in the context of the theory with new singularities* τ (*see figure* 3).

2) *There is a passage*

$$(K', \mathcal{R}) \Longrightarrow (\Theta^3(Z^2), \mathcal{R}) \tag{14}$$

(where $\Theta^3(Z^2)$*) is the singular* 3*-dimensional manifold which corresponds to* $Z^2 \xrightarrow{f} \Sigma^3$*) consisting of acyclic* $O(i)^{\pm 1}$ *moves and of* COHERENT $O(4)$*'s. As a consequence of this we have a diffeomorphism*

$$\Theta^4(K', \mathcal{R}) = \Theta^4(Z^2, \mathcal{R}). \tag{15}$$

In (14), (15), the left hand sides belong to the new, generalized theory, while the right hand sides belong to the usual one.

PROOF. – The diffeomorphism (13) is easy, but it only establishes the connection of 4-dimensional regular neighborhoods in $\Sigma^3 \times I$ with our new bizarre

Θ^4. It remains to show the connection of this Θ^4 with our usual one. We will only sketch the description of (14). The general idea is to start with a singularity τ (figure 1) and then push the hatched area completely into the boundary, using the vector field $\pm\xi$ and thereby creating a whole bunch of usual singularities σ. The easiest way to understand the process is to look at a concrete example. Consider a localized piece X^2 of $Z^2 \times t_i$ looking like in figure 4. We assume that the corresponding piece of $Z^2_{(1)}$ is $Y^1 = [0,a] \cup [0,b] \cup [0,c] \cup [0,d]$ and that the constant vector field ξ points towards the $(x > 0, y > 0, z > 0)$-sector. To make things really precise, let us also say that it is orthogonal to the plane (a, b, c).

Locally, $N^3(i)$ is a very thin tubular neighbourhood of $X^2 \subset R^3$. Inside this $N^3(i)$, the $\Sigma^\pm$ are essentially two copies of a very thin 2-dimensional regular neighbourhood of Y^1, defined with the help of ξ.

We will describe what happens to these $\Sigma^\pm$ after one has performed (14). As far as Σ^- is concerned, it is simply a regular neighborhood of Y^1 living completely on the $(y < 0, z < 0)$-side; it is completely non-singular.

The Σ^+ is a union of pieces which we will describe now. It will be understood that those pieces which are contained in $[a, c, g, f] \cup [a, i, h, c] \subset X^2$ are always on the $(y > 0, z > 0)$-side. When they live on $[o, b, e, d] \subset X^2$, they can be either on the $x > 0$ side ***or*** on the $x < 0$ side. A singularity σ is created exactly where there is a change of sign for x. Schematically speaking, we will have

$$\Sigma^+ = [a_1, a, a_2, b_1, \underbrace{b, b_2, \beta, a, d_2, d}_{\text{this is with } x>0}, d_1, a_1, a] \bigcup_{[\beta,a]}$$

$$\bigcup_{[\beta,a]} [c_1, c, c_2, \gamma, \delta, c_1] \bigcup_{[\gamma,\delta]} \underbrace{[\delta, a, \beta, \gamma]}_{\text{this is with } x<0} .$$

The site $[a, \beta]$ corresponds to a usual undrawable singularity σ. Any other local model can be treated similarly and the passage from local to global should be clear.

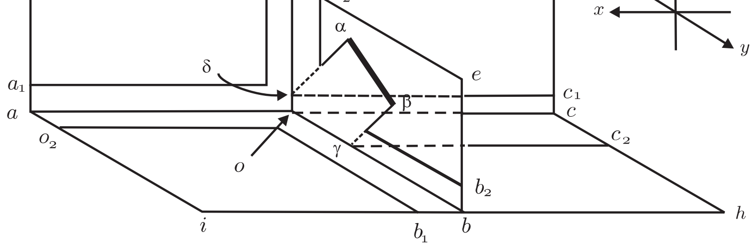


Fig. 4. – We see here a localized piece X^2 of $Z^2 \times t_i \subset \Sigma^3 \times t_i$.

So, this finishes the proof of the absolute diffeomorphism

$$N^4(Z^2) = \Theta^4(Z^2, \mathcal{R})$$

and we make now the following claims, the proofs of which are left to the reader.

i) The proof above extends to more general subcomplexes X^2 of $(\tau \oplus \theta)_{(2)}|M^3 \times I$, like for instance the $N_{(2)}$ from the lemma 2.3 in [5].

ii) The diffeomorphism $N^4(X^2) = \Theta^4(X^2, \mathcal{R})$ has a number of nice features, some of which we list now. Assume that there is a 1-dimensional subcomplex $X^1 \subset X^2$ such that X^1 is for from the singularities $X^2 \to M^3$ and such that $X^1 \subset X^2$ has ***collar*** $X^1 \times [0,1] \subset X^2$ with $X^1 \times 0 = X^1, X^1 \times [0,1)$ open and far from the singularities. We assume, also, that for each connected component $X^1_i \subset X^1$ we have $X^1_i \subset \pi_0^{-1}(t_i)$ with the corresponding piece of collar either also in $\pi_0^{-1}(t_i)$ ***or*** compatible with π. [In our real life case from [5], we can take $X^1 = \tau_{(1)} \times 0 + \Gamma$.] Under the conditions both N^4 and Θ^4 come with a canonical 3-dimensional regular neighbourhood of X^1 embedded in the boundary and the claim is that the two regular neighborhoods are the same and that the diffeomorphism $N^4 = \Theta^4$ respects them.

3. – IMPROPER zipping and its thickenings.

In this section we will discuss the proof of the case II of theorem 1.2. So, we will consider a singular 2-dimensional polyhedron $X^2 \xrightarrow{f} M^3$ with $\Psi(f) = \Phi(f)$, meaning that all the double points $M_2(f) \subset X^2$ can be extended via zippings, starting from the singularities. We also give ourselves a desingularization φ.

We want to discuss now the IMPROPER case and the simplest way of characterizing it is to say that we will consider the situation when the subset $M_2(f) \subset X^2$ is ***not*** a ***closed*** subset. We will only discuss here the following paradigmatic case, where the issues involved appear already in full clarity. So, we will look at the following local model. Inside M^3 we have a coordinate chart $U = R^3 = (x, y, z)$, inside which live $\infty + 1$ planes, namely $W = (z = 0)$ and the $V_n = (x = x_n)$, where $x_1 < x_2 < \cdots$ with $\lim x_n = x_\infty$. Our local model for $X^2 \xrightarrow{f} M^3$ is here $f^{-1}U = W + \sum_1^\infty V_n \subset X^2$ with $f|\big(W + \sum_1^\infty V_n\big)$ being the obvious map.The fact that $M_2(f)$ is not closed manifests itself here by the limit line $x = x_\infty$ NOT being a double line. Notice that the situation is here, nevertheless, the "the next best", short of being PROPER: a tight transversal $\Lambda \subset X^2$ to the double point set is such that the accumulation set

$$\lim\,(M_2(f) \cap \Lambda)$$

is of FINITE cardinality. According to our previous paper [6] this is, essentially, the generic situation.

Now, we will pick up a sequence of positive numbers converging very fast to zero $\varepsilon > \varepsilon_1 > \varepsilon_2 > \cdots$ and with this, on the road to $\Theta^4(Y^2, \psi)$ from theorem 1.2 which we will take here simply to be $\Theta^4(fX^2)$, an object still to be made sense out of we will start by replacing the $fW \cup \sum_1^\infty V_n \subset fX^2$, with the following 3-dimensional non-compact 3-manifold with boundary

$$(16) \quad M \stackrel{\text{def}}{=} \left[W \times (-\varepsilon \leq z \leq \varepsilon) - \{x = x_\infty\} \times \{z = \pm\varepsilon\}\right] \cup \sum_1^\infty V_n \times (x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n).$$

In such a formula, notations like "$W \times (-\varepsilon \leq z \leq \varepsilon)$" should be read "$W$ thickened into $-\varepsilon \leq z \leq \varepsilon$". Next, we will go 4-dimensional and, start by replacing R^3 with $R^4 = (x, y, z, t)$. Our local model should live now inside $R^4 = (x, y, z, t)$, and we will try to locate it there conveniently for the geometric realization of the zipping. We will show how we would like to achieve this for a generic section $y =$ constant.

For reasons to become soon clear, we will replace the normal section $y =$ const corresponding to W and which should be

$$N_y = \left[-\infty < x < \infty, y = \text{const}, \ -\varepsilon \leq z \leq \varepsilon, \ 0 \leq t \leq 1\right] - (x = x_\infty, \ y = \text{const}, z = \pm\varepsilon, \ 0 \leq t \leq 1),$$

by the smaller $N_y - \sum_1^\infty$ DITCH $(n)_y$, which is defined as follows. The DITCH $(n)_y$ is a thin column of height $-\varepsilon \leq z \leq \varepsilon$ and of (x, t)-width $4\varepsilon_n$, which is concentrated around the arc

$$(x = x_n, \ y = \text{const}, \ -\varepsilon \leq z \leq \varepsilon, \ t = 1).$$

This thin indentation inside N_y is such that, with our fixed $y =$const being understood here, we should have

$$(17) \qquad \lim_{n\to\infty} \text{DITCH } (n)_y = (x = x_\infty, \ -\varepsilon \leq z \leq \varepsilon, \ t = 1).$$

Notice that, in the RHS of (17) it is exactly the $z = \pm\varepsilon$ which corresponds to punctures eliminated in (16).

Continuing to work here with a fixed, generic y, out of the normal y-slice corresponding to V_n, namely $(x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n, \ -\infty < z < \infty, \ 0 \leq t \leq 1)$, we will keep only a much thinner, isotopically equivalent version, namely the following

$$(18) \qquad (x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n, \ -\infty < z < \infty, \ 0 \leq t \leq 1).$$

The (18) has the virtue that it can fit now inside the corresponding DITCH$(n)_y$, without touching at all the $N_y - \{\text{DITCHES}\}$. See here the figure 5.

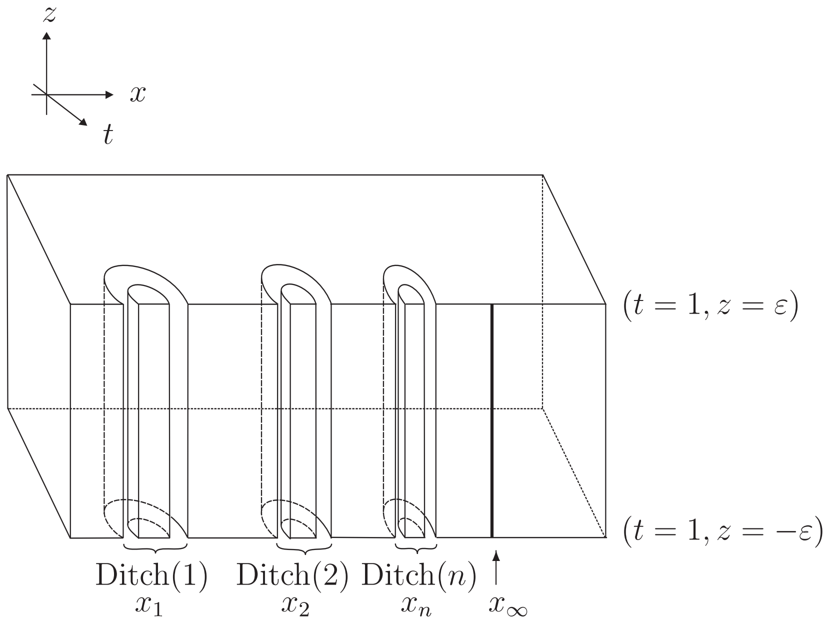


Fig. 5.

What has been carefully described here, when all y's are being taken into account, is a very precise way of separating the $\infty + 1$ branches of (the thickened) (16), at the level of R^4, taking full advantage of the additional dimensions (i.e. the factor $[0 \leq t \leq 1]$ in our specific case). With some work, this kind of thing can be done consistently for the whole global fX^2. The net result is an isotopically equivalent new model for $\Theta^4(X^2)$, which invites us to try the following naive approach for the geometric realization of the zipping. Imitating the successive folding maps of the actual zipping, fill up all the empty room left inside the ditches, by using only Whitehead dilatations and additions of handles of index $\lambda > 1$, until one has reconstructed completely the $\Theta^4(fX^2)$. Formally there is no obstruction here and then also what at a single y =const may look like a handle of index one, becomes "only index ≥ 2", once the full global zipping is taken into account. But there ***is*** actually a big problem with this naive approach, *via* which one can certainly reconstruct $\Theta^4(fX^2)$ as a set, but with the ***wrong topology***, as it turns out.

The precise problem is the following. Because of (17) above, when we try to imitate the zipping

$$X^2 \Longrightarrow fX^2$$

which we suppose here COHERENT, via an infinite sequence of additions, i.e. inclusion maps instead of quotient space maps, in four dimensions, then we get for the set $\Theta^4(fX^2)$ a topology which is NOT metrizable. This is a topic which will be discussed in full detail in the paper [2].

There is an obvious cure for this disease, namely to delete the line

$$(x = x_\infty,\ -\varepsilon \leq z \leq \varepsilon,\ t = 1).$$

for all values y, from the prospective $\Theta^4(fX^2)$. What theorem 1.2 says, is that there is here a better, more global cure, superseding the preceding one, namely throw away completely the $\partial\Theta^4(fX^2)$.

But then, this should ***not*** be a brutal step, it should be integrated in the infinite succession of additions (= inclusion maps), which mimic the zipping process: intercalate steps which are part of an infinite Whitehead dilatation sending the $\partial\Theta^4$ to infinity, to the normal ones. The reader should not find it hard to fill in the details here. Anyway there issues are more fully developed in [2].

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