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The last result in [1] (Example 2.5) states the following:

Example. -- There exists a separable 1st category metric space with a Baire Wijsman (ball proximal, ball, resp.) hyperspace.

Unfortunately, the construction presented in [1] does not guarantee a key step in the proof; namely, for the u' chosen, one cannot conclude that $p(u) = p(u')$. It is the purpose of this note to fill this gap and provide a correct proof.

Recall some notation and terminology from [1]: b_d stands for the *ball topology* on the hyperspace $CL(X)$ of nonempty closed subsets of a metric space (X, d) having subbasic elements of the form $V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}$ for some open $\emptyset \neq V \subseteq X$, and of the form $(X \setminus B)^+ = \{A \in CL(X) : A \cap B = \emptyset\}$, where B is a closed ball in X. Denote by $S(x, r)$ the open ball about $x \in X$ of radius r, and by $B(X)$ the collection of finite unions of closed X-balls. The *Wijsman topology* on $CL(X)$ is the weak topology generated by the distance functionals $d(x, A) = \inf\{d(x, a) : a \in A\}$ viewed as functionals of the set argument $A \in CL(X)$. It is shown in [1], that the Wijsman hyperspace is a Baire space iff the ball topology is iff the *ball proximal* (see [1]) topology is.

Proof of the Example. – Consider ω^ω with the Baire metric

$$e(x, y) = 1/\min\{n + 1 : \ x(n) \neq y(n)\},$$

and its 1st category subset $\omega^{<\omega}$ of sequences eventually equal to zero. Then the product $X = \omega^{<\omega} \times \omega^\omega$ is a separable, 1st category space endowed with the metric $d((x_0, x_1), (y_0, y_1)) = \max\{e(x_0, y_0), e(x_1, y_1)\}$.

We claim that $(CL(X), b_d)$ is a Baire space: let $p_1 : X \to \omega^{<\omega}$ (resp. $p_2 : X \to \omega^\omega$) be the projection onto the first (resp. second) axis. Let $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \dots$ be dense open sets in $(CL(X), b_d)$, and $\mathscr{U}_0 \in b_d$. For $i \geq 1$, inductively define a nonempty finite set $F_i \subset X$, $m_i \geq i + 1$, and an increasing sequence $B_i \in B(X)$ such that $\left\{S\left(u, \dfrac{1}{m_i}\right) : u \in F_i\right\}$ is pairwise disjoint with a union missing B_i, and

$$\mathscr{U}_i = (X \setminus B_i)^+ \cap \bigcap_{u \in F_i} S\left(u, \frac{1}{m_i}\right)^- \subseteq \mathscr{G}_i \cap \mathscr{U}_{i-1},$$

moreover, for each $u \in F_i$ there is $u^\star \in F_{i+1}$ with $p_1(u) = p_1(u^\star)$ and $d(u, u^\star) < \dfrac{1}{i+1}$.

We can clearly find $\mathscr{U}_1$ and $F_1 \in \mathscr{U}_1$, defined as above, such that $\mathscr{U}_1 \subseteq \mathscr{G}_1 \cap \mathscr{U}_0$. Suppose that F_i, m_i, B_i, and thus, $\mathscr{U}_i \in b_d$ have been defined for some $i \geq 1$. Since $\mathscr{G}_{i+1}$ is dense, we can find a finite set A, a $B_{i+1} \in B(X)$ with $B_{i+1} \supseteq B_i$, and a collection $\{V_a : a \in A\}$ of pairwise disjoint open X-balls such that

$$A \in \mathscr{V} = (X \setminus B_{i+1})^+ \cap \bigcap_{a \in A} V_a^- \subset \mathscr{G}_{i+1} \cap \mathscr{U}_i.$$

Without loss of generality, assume that B_{i+1} is the union of the finite pairwise disjoint collection $\left\{S\left(b_j, \dfrac{1}{n_j}\right) : j \in J\right\}$ of clopen X-balls (remember that, since d is an ultrametric, any two d-balls either are disjoint, or one of them is included in the other).

Pick $u \in F_i$, and $a \in S\left(u, \dfrac{1}{m_i}\right) \setminus B_{i+1}$. If $u \notin B_{i+1}$, choose $u^\star = u$. If $u \in S\left(b_{j_0}, \dfrac{1}{n_{j_0}}\right)$ for some $j_0 \in J$, and $n_{j_0} \leq m_i$, then $a \in S\left(b_{j_0}, \dfrac{1}{n_{j_0}}\right) \subseteq B_{i+1}$, which is impossible, so $n_{j_0} > m_i$. Choose some $k \in \omega \setminus \{p_2(b_j)(m_i) : j \in J\}$, and notice that such a k is also different from $p_2(u)(m_i)$, as $u \in S\left(b_{j_0}, \dfrac{1}{n_{j_0}}\right)$ and $n_{j_0} > m_i$ imply that $p_2(u)(m_i) = p_2(b_{j_0})(m_i)$. Let $u_2 \in \omega^\omega$ be such that

$$u_2(s) = \begin{cases} p_2(u)(s), & \text{if } s \neq m_i, \\ k, & \text{if } s = m_i. \end{cases}$$

Then for $u^\star = (p_1(u), u_2)$ we have $p_1(u^\star) = p_1(u)$, and $d(u, u^\star) = \dfrac{1}{m_i + 1} < \dfrac{1}{i+1}$. Moreover, $u^\star \notin B_{i+1}$: indeed, take any $j \in J$, and assume first that $n_j \leq m_i$. Then $j \neq j_0$ (as $n_{j_0} > m_i$), therefore from $u \notin S\left(b_j, \dfrac{1}{n_j}\right)$ $\left(\text{recall that } \left\{S\left(b_j, \dfrac{1}{n_j}\right) : j \in J\right\}\right.$ is pairwise disjoint and $\left.u \in S\left(b_{j_0}, \dfrac{1}{n_{j_0}}\right)\right)$ we deduce that for some $s < n_j \leq m_i$

$$\text{either } p_1(u^\star)(s) = p_1(u)(s) \neq p_1(b_j)(s) \text{ or } p_2(u^\star)(s) = p_2(u)(s) \neq p_2(b_j)(s),$$

hence in both cases, $d(u^\star, b_j) \geq \dfrac{1}{s+1} \geq \dfrac{1}{n_j}$, i.e. $u^\star \notin S\left(b_j, \dfrac{1}{n_j}\right)$. If, on the other side, $n_j > m_i$, then from $p_2(u^\star)(m_i) = k \neq p_2(b_j)(m_i)$ we deduce that

$$d(u^\star, b_j) \geq e(p_2(u^\star), p_2(b_j)) \geq \frac{1}{m_i+1} \geq \frac{1}{n_j},$$

hence again, $u^\star \notin S\left(b_j, \frac{1}{n_j}\right)$.

Define $F_{i+1} = A \cup \{u^\star : u \in F_i\}$, and choose $m_{i+1} \geq i+2$ so that

$$\mathscr{U}_{i+1} = (X \setminus B_{i+1})^+ \cap \bigcap_{u \in F_{i+1}} S\left(u, \frac{1}{m_{i+1}}\right)^- \subseteq \mathscr{V}.$$

Now, the sequence $u, u^\star, u^{\star\star}, \ldots$ is a Cauchy sequence in $\{p_1(u)\} \times \omega^\omega$; hence, it converges to some $u^\infty \in S\left(u, \dfrac{1}{m_i}\right)$. Because the B_i's are disjoint from the $S\left(u, \dfrac{1}{m_i}\right)$'s, the set $\left\{u^\infty : u \in \bigcup_{n\geq 1} F_n\right\}$ misses the clopen B_i for each $i \geq 1$. Then

$$\emptyset \neq \overline{\{u^\infty : u \in \bigcup_{n\geq 1} F_n\}} \in \bigcap_{n\geq 1} \mathscr{U}_n \subseteq \mathscr{U}_0 \cap \bigcap_{n \geq 1} \mathscr{G}_n;$$

thus, $(CL(X), b_d)$ is a Baire space. □

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