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Remarks on Homogeneous Complex Manifolds Satisfying Levi Conditions

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Dedicated to the memory of Professor Aldo Andreotti on the 30th anniversary of his death.

Abstract. – *Homogeneous complex manifolds satisfying various types of Levi conditions are considered. Classical results which were of particular interest to Andreotti are recalled. Convexity and concavity properties of flag domains are discussed in some detail. A precise classification of pseudoconvex flag domains is given. It is shown that flag domains which are in a certain sense generic are pseudoconcave.*

1. – Introduction.

In the early 1960's Andreotti devoted a great deal of his attention to complex spaces satifying various types of Levi-conditions. Major works in this direction include his description of the field of meromorphic functions on a pseudoconcave complex space ([A]), showing that it is an algebraic function field of transcendence degree at most the dimension of the space, and his fundamental work with Grauert ([AGr1]) on finiteness and vanishing of cohomology on q-convex spaces. At the time the case of spaces possessing strictly plurisubharmonic exhaustions, or, under further assumptions, exhaustions having only a semipositive Levi-form, were well-understood. It was indeed quite natural to initiate a study of manifolds which can be viewed as lying between compact and Stein.

Coming from algebraic geometry Andreotti was interested in the examples of such manifolds which are obtained by removing high codimensional subvarieties from (compact) projective algebraic manifolds. If Z is compact and E is the subvariety which is removed, then the set of algebraic cycles contained in $X := Z \setminus E$, an open set in the Chow variety of X, is of basic importance. Transferring cohomology from the pseudoconcave, q-pseudoconvex space X to the level of functions on its cycle space was the topic of his basic joint works with Norguet ([AN1, AN2]). In § 3 of this note we underline another setting, that of flag domains, where cycle spaces, pseudoconcavity and q-convexity go hand-in-hand. Our research in this area (see, e.g., [FHW]) strongly relies on Barlet's methods for constructing and dealing with cycle spaces (see, e.g., [Ba]), and there is no doubt that Andreotti's viewpoints on this subject were among the inspiring factors for Barlet's early works.

Andreotti was well-acquainted with the method of discrete group quotients for constructing projective or quasi-projective varieties and was particularly interested in such quotients which arise in moduli problems. In a jewel which is perhaps only known to specialists ([AGr2]) he and Grauert introduced the notion of a *pseudoconcave discrete group action* which is the appropriate translation to the level of D of the notion that the discrete group quotient D/Γ is pseudoconcave. As an example they showed that the quotient of the Siegel upper half-plane by the modular group is pseudoconcave and as a consequence that interesting spaces of automorphic forms are finite-dimensional. Borel, who took this result to its appropriate level of generality ([Bo]), once enthusiastically recalled to us how struck he was with the beauty of this simple idea.

Of course it was clear to Andreotti that the notions of pseudoconcavity and/or mixed Levi-conditions are of basic importance, and that one should employ these as Leitfäden for discovering and analyzing interesting new classes of manifolds. He also knew that it makes sense to involve symmetry at least in the initial stages of such considerations. Thus it should come as no surprise that at the end of his Stanford course on several complex variables, which covered most of the topics mentioned above, he asked the student who was responsible for the notes to look for new classes of pseudoconcave manifolds. Typically he suggested an extremely simple starting point: complex Lie groups.

The present note is organized as follows. In § 2.1 we discuss the case of Lie groups. It turns out that one easily sees that the only such manifolds which are pseudoconcave are compact tori. However, this result is not as negative as first meets the eye, because in the process of considering candidates for pseudoconcave Lie groups one meets Levi-degenerate, pseudoconvex manifolds where first examples of interesting, number-theoretic guided foliations play a role. We continue the discussion of analogous pseudoconvexity phenomena for nilmanifolds in § 2.2 and flag domains in § 2.3. In Theorem 2.15 we show in particular that the Remmert reduction of pseudoconvex flag domain is a precisely defined homogeneous bundle over a Hermitian symmetric space.

Our work in § 3 is devoted to a study of pseudoconcave flag domains. We suspect that virtually all flag domains which are not pseudoconvex are in fact pseudoconcave, but at the present time we are only able to prove this for flag domains which are in a certain sense generic (§ 3.2). These include period domains for moduli problems, e.g., for marked K3-surfaces, which were certainly of interest to Andreotti.

2. – Pseudoconvexity.

As mentioned above, our original starting point was to determine if there are interesting pseudoconcave complex Lie groups. In the first section here we pursue this as a guideline, but in fact end up showing that Lie groups are more interesting from the point of view of pseudoconvexity. In the following paragraph we prove analogous results for homogeneous nilmanifolds. The final section is devoted to a detailed description of pseudoconvex flag domains. The results on complex Lie groups are classical, but the proofs given here underline the importance of Levi-foliations, a theme that is of recent interest and which also plays a role in our discussion of nilmanifolds. Although new, the results on nilmanifolds only require implementation of classically known information, in particular a basic fact due to Loeb ([L]) concerning the relation of geodesic convexity and Levi-pseuodoconvexity in a Lie group setting. Our characterization of pseudoconvex flag domains utilizes the notion of cycle connectivity which is the flag domain analog of the condition of *rationally connected* in algebraic geometry.

Before going further let us recall that by definition a (connected) pseudoconcave complex manifold X contains a relatively compact open set Z so that for every point $p \in \mathrm{cl}(Z)$ there is a holomorphic mapping $\psi : \Delta \to \mathrm{cl}(Z)$ of the unit disk Δ in the complex plane with $\psi(0) = p$ and $\psi(\mathrm{bd}(\Delta)) \subset Z$. The following is the first basic property of these manifolds.

PROPOSITION 2.1. – *Pseudoconcave manifolds possess only constant holomorphic functions.*

PROOF. – Let $f \in \mathcal{O}(X)$ and note that the restriction of $|f|$ to $\mathrm{cl}(Z)$ takes on its maximum at some point p. Since p is contained in a holomorphic disk $\psi(\Delta)$ whose boundary lies in Z, the maximum principle implies that f is constant on that disk and therefore takes on its maximum at a point of the open set Z. Another application of the maximum principle implies that f is constant on X. □

2.1 – *Complex Lie groups.*

If G is a connected complex Lie group with $\mathcal{O}(G) \cong \mathbb{C}$, then there is no nonconstant holomorphic homomorphism $G \to \mathrm{GL}_{\mathbb{C}}(V)$ to the general linear group of a complex vector space. This is due to the fact that $\mathrm{GL}_{\mathbb{C}}(V)$ is an open subset of $\mathrm{End}(V) \cong V \otimes V^*$ which is itself a complex vector space and is in particular holomorphically separable. Let $\mathfrak{g}$ denote the Lie algebra of G and recall that the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}_{\mathbb{C}}(\mathfrak{g})$ is exactly such a map. Since in general the kernel of this map is the center of G, we have the following first remark.

PROPOSITION 2.2. – *A connected complex Lie group G_1 with $\mathcal{O}(G_1) \cong \mathbb{C}$ is Abelian.*

In particular, if G is pseudoconcave, then it is Abelian.

2.1.1 – Background on Abelian Lie groups.

Let us investigate the case of a complex Abelian Lie group more closely. Recall that the exponential map $\exp : \mathfrak{g} \to G$ of a (connected) Abelian Lie group is a surjective homomorphism from the additive group $(\mathfrak{g}, +)$. In the case of a complex Lie group the exponential map is holomorphic and therefore such groups are of the form V/Γ, where Γ is a discrete additive subgroup of a complex vector space V.

If $\mathcal{O}(G) \cong \mathbb{C}$, then Γ must be rather large. A key object for understanding its size and position with respect to the linear complex structure of V is the real subspace $\mathrm{Span}_{\mathbb{R}}(\Gamma) =: V_\Gamma$.

Every connected (not necessicarily complex) Lie group G possesses a compact subgroup K having the property that G/K is diffeomorphic to a vector space. One shows that such groups are maximal compact subgroups and that any two are conjugate. If $G = V/\Gamma$ is Abelian, then the torus $V_\Gamma/\Gamma =:= K$ is compact and G/K is diffeomorphic to any (real) subspace of V which is complementary to V_Γ. Thus K is the unique maximal compact subgroup of G.

Observe that if U_2 is a complex subspace of V which is complementary to $U_1 := V_\Gamma + iV_\Gamma$, then G is holomorphically isomorphic to $U_1/\Gamma \times U_2$. Thus it is enough to consider the situation where Γ generates V as a complex vector space, i.e., where $\mathrm{Span}_{\mathbb{C}}(\Gamma) = V$.

Define the additive complex subgroup W_Γ of V to be the maximal complex subgroup in V_Γ, in other words $W_\Gamma = V_\Gamma \cap iV_\Gamma$, and regard the W_Γ-orbits in G as foliating the torus K. As abstract complex manifolds the leaves of this foliation are all equivalent to the orbit of the identity which is the subgroup $W_\Gamma/(W_\Gamma \cap \Gamma)$. The closure of this orbit is a subtorus K_1, and we may split K as a product $K = K_1 \times K_2$ of two subtori where K_2 is a totally real subgroup of G. Let $\mathfrak{k}_j$ be the uniquely defined subspaces of V so that $K_j = \mathfrak{k}_j/\Gamma$ and define $V_j := \mathfrak{k}_j + i\mathfrak{k}_j$, $j = 1, 2$. Assuming that Γ generates V as a complex vector space, it therefore follows that $G = V/\Gamma = V_1/(V_1 \cap \Gamma) \times V_2/(V_2 \cap \Gamma)$.

Let us summarize the above discussion in the proof of the following decomposition theorem of Remmert-Morimoto (see [K, M]).

THEOREM 2.3. – *A connected Abelian complex Lie group $G = V/\Gamma$ is the direct product $G = G_1 \times G_2 \times G_3$, where $G_3 \cong (\mathbb{C}^n, +)$, $G_2 \cong ((\mathbb{C}^*)^m, \cdot)$ and $\mathcal{O}(G) \cong \mathbb{C}$.*

PROOF. – The complex group G_3 arises (noncanonically) as the complement of the canonically defined complex subspace $U_1 = V_\Gamma + iV_\Gamma$. The factor U_1/Γ is canonically embedded in G and it contains the canonically defined complex subgroup $G_1 = V_1/(V_1 \cap \Gamma)$. The noncanonical splitting $K = K_1 \times K_2$ of the maximal compact subgroup defines the complementary complex subgroup $G_2 = V_2/(V_2 \cap \Gamma)$. Since K_2 is totally real, $G_2 \cong (\mathbb{C}^*)^m$.

Finally, recall that K_1 is foliated by the dense orbits of the complex subgroup W_Γ. If $f \in \mathcal{O}(G_1)$, then we consider its restriction to K_1 which is compact so that the restriction of $|f|$ takes on its maximum at some point $p \in K_1$. Since the orbit map $W_\Gamma \to G$, $w \to w(p)$, is holomorphic, the pullback of f to W_Γ is holomorphic and thus the maximum principle implies that this pull-back is constant. Consequently, f is constant on the (dense!) W_Γ-orbit of p and is therefore constant on the torus K_1. But $K_1 = \mathfrak{k}_1/(\mathfrak{k}_1 \cap \Gamma)$ and $V_1 = \mathfrak{k}_1 + i\mathfrak{k}_i$. Hence, it follows from the identity principle that the pullback of f to V_1 is constant and consequently f is constant on $V_1/(V_1 \cap \Gamma) =: G_1$. □

2.1.2 – Cousin groups.

Restricting to the Abelian case and regarding $\mathbb{C}^n$ and $(\mathbb{C}^*)^m$ as being well-understood, in the notation of the above decomposition theorem it is reasonable to further restrict to the case where $G = G_1$. Of course the case where G is compact has a long history, but it was first in the early 20th century that Cousin called attention to interesting complex analytic phenomena in the noncompact case (see [C]). Thus if $\mathcal{O}(G) \cong \mathbb{C}$, we refer to G as being a *Cousin group*.

Let us now turn to the matter of pseudoconcavity/pseudoconvexity of Cousin groups. Recalling the notation above, in this situation $G = V/\Gamma$ and $V = V_\Gamma + iV_\Gamma$ where $V_\Gamma = \mathrm{Span}_{\mathbb{R}}(\Gamma)$. The maximal complex subgroup $W_\Gamma = V_\Gamma \cap iV_\Gamma$ has dense orbits in the maximal compact subgroup $K = V_\Gamma/\Gamma$. Let C be a complementary subspace of V_Γ such that $V_\Gamma = W_\Gamma \oplus C$. Note that C is totally real and that V decomposes as a complex vector space as

$$V = W_\Gamma \oplus (C \oplus iC)\,.$$

Let η be an exhaustion of iC which is defined as the norm-squared function of some (positive-definite) inner-product. If we regard η as being defined on V, then its full Levi-form $i\partial\bar{\partial}\eta$ is positive-semidefinite with degeneracy W_Γ.

Now define $\rho : G \to \mathbb{R}^{\geq 0}$ by pulling back η by the linear projection to iC. Its properties are summarized as follows.

PROPOSITION 2.4. – *A noncompact Cousin group $G = V/\Gamma$ possess a plurisubharmonic exhaustion $\rho : G \to \mathbb{R}^{\geq 0}$ which is invariant by the maximal compact subgroup $K = V_\Gamma/\Gamma$. For every $p \in G$ the degeneracy of the Levi-form* *of ρ at p is the tangent space to the orbit $W_\Gamma.p$ of the maximal complex subgroup $W_\Gamma = V_\Gamma \cap iV_\Gamma$ which is dense in $K.p$.*

In particular, noncompact Cousin groups are pseudoconvex in a very strong sense and the following is therefore immediate.

PROPOSITION 2.5. – *Pseudoconcave complex Lie groups are compact complex tori.*

PROOF. – Let G be a pseudoconcave Lie group and assume that it is noncompact with a relatively compact open subset Z defining its pseudoconcavity. For $r > 0$ define $B_r := \{\rho < r\}$, where ρ is the exhaustion function defined above, and let $r_0 := \inf\{r : B_r \supset Z\}$. Thus $\mathrm{cl}(Z) \subset \mathrm{cl}(B_{r_0})$ and there exists $p \in \mathrm{bd}(Z)$ which is also contained in the level surface $M_{r_0} := \{\rho = r_0\}$. Let $\psi : \Delta \to \mathrm{cl}(Z)$ be the holomorphic disk at p which is guaranteed by the pseudoconcavity and note that $\widehat{\rho} := \psi^*(\rho)$ is plurisubharmonic on Δ with $\widehat{\rho}(0) = r_0$. Since $\widehat{\rho} < r_0$ on all boundary points of Δ, this violates the maximum principle. □

Before concluding this paragraph we should make a number of remarks on the history of this subject, particularly focused on Andreotti's involvement. First, the proof of Proposition 2.5 in ([AH]) differs somewhat from the one above: If $G = V/\Gamma$ is a noncompact Cousin group, then (in the words of Andreotti) *we can introduce a small earthquake* and move Γ to a nearby group Γ_ε so that the resulting variety $G_\varepsilon = V/\Gamma_\varepsilon$ is still pseudoconcave, but $\mathcal{O}(G) \not\cong \mathbb{C}$. It should be noted that it can be arranged that such an earthquake produces a holomorphically convex manifold so that the Levi-problem has a positive answer for a dense set of discrete groups.

Andreotti was interested in the fields of meromorphic functions of Cousin groups, in particular in relation to their projective algebraic equivariant compactifications (quasi-abelian varieties) and the role of Θ-functions (see [AGh]). It is should be mentioned that Θ-theory is in general not adequate for describing the meromorphic functions on a (noncompact) Cousin group. On the other hand, since the time of Cousin there have been a number of interesting developments (see [AK]).

Due to our focusing on topics of particular interest to Andreotti, we have covered only a very small part of the interesting early results involving Lie theoretic considerations in complex analysis. In closing this paragraph we would, however, like to note one further result which underlines the fact that the decomposition of Theorem 2.3 can be viewed in a much more general context (see [MM] and [M]).

THEOREM 2.6. – *Let G be a connected complex Lie group equipped with the holomorphic equivalence relation $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \mathcal{O}(G)$.*

Then the quotient $G \to G/\!\sim$ is given as a holomorphic group fibration $G \to G/C$ where the fiber C is a closed, central Cousin subgroup of G and the base is a Stein Lie group.

2.2 – *Nilmanifolds.*

Here we carry through virtually the same line of discussion for nilmanifolds as that above for Abelian groups. By definition a *complex nilmanifold* X is homogeneous under the holomorphic action of a connected complex nilpotent Lie group, i.e., $X = G/H$, where G is a connected complex nilpotent group and H is a closed complex subgroup. We may assume that the G-action on X is almost effective in the sense that there are no positive-dimensional normal subgroups of G which are contained in H. In other words, the subgroup of elements in G which fix every point of X is at most discrete.

One of the first steps toward understanding any complex homogeneous space G/H is to consider the normalizer fibration $G/H \to G/N$ where N is the normalizer in G of the connected component H^0. If we consider the action of G on $\mathfrak{g}$ by the adjoint representation and regard $\mathfrak{h}$ as a point in the Grassmannian $\mathrm{Gr}_k(\mathfrak{g})$ where $k = \dim_{\mathbb{C}} \mathfrak{h}$, then the base G/N is the orbit of that point. In particular, the base of the normalizer fibration is an orbit via a G-representation in the projective space $\mathbb{P}(\wedge^k \mathfrak{g})$.

A connected solvable Lie group G acting via a linear representation of a vector space V stabilizes a full flag $0 \subset V_1 \subset \ldots \subset V_{m-1} \subset V$. of subspaces (Lie's Flag Theorem). Thus, for example, a G-orbit in the associated projective space $\mathbb{P}(V)$ either lies in the *affine* space $\mathbb{P}(V) \setminus \mathbb{P}(V_{m-1})$ or is contained in the smaller projective space $\mathbb{P}(V_{m-1})$. Repeating this until reaching the point where the orbit in question is not contained in some $\mathbb{P}(V_\ell)$ of the flag, we have the following first remark.

Proposition 2.7. – *Orbits of a connected complex solvable group acting as a group of holomorphic transformations on a projective space $\mathbb{P}(V)$ are holomorphically separable. In particular, if $X = G/H$ is a homogeneous manifold under the holomorphic action of a complex solvable Lie group and $\mathcal{O}(X) \cong \mathbb{C}$, then, assuming that the G-action is almost effective, it follows that H is discrete.*

Thus as in the case of an Abelian group, if we are guided by investigating the possibility of a nilmanifold X being pseudoconcave, we may assume that it is of the form G/Γ where Γ is discrete. So let us now restrict our considerations to such manifolds.

If $X = G/\Gamma$ is a nilmanifold with discrete isotropy and, without loss of generality, G is simply-connected, then $\exp : \mathfrak{g} \to G$ is a biholomorphic (in fact algebraic) map. In analogy to the Abelian case, realizing Γ as a discrete subset of $\mathfrak{g}$, it spans a (real) Lie subalgebra $\mathfrak{g}_\Gamma$ such that the associated group G_Γ contains Γ with G_Γ/Γ compact (Theorem of Malcev-Matsushima).

Continuing with the analogy to the Abelian case we consider the complex Lie algebra $\widehat{\mathfrak{g}}_\Gamma := \mathfrak{g}_\Gamma + i\mathfrak{g}_\Gamma$ and the associated complex subgroup $\widehat{G}_\Gamma$. As a quotient of simply-connected complex nilpotent groups $G/\widehat{G}_\Gamma$ is biholomorphically equivalent to some $\mathbb{C}^n$. The bundle $G/\Gamma \to G/\widehat{G}_\Gamma$ is holomorphically trivial and therefore there is no loss of generality to assume that $G = \widehat{G}_\Gamma$, i.e., that the Lie algebra level Γ generates $\mathfrak{g}$.

The key subalgebra for complex analytic considerations is $\mathfrak{m} = \mathfrak{g}_\Gamma \cap i\mathfrak{g}_\Gamma$. It is an *ideal* in $\mathfrak{g}$! Of course, just as in the Abelian case, the action of the associated group M on G/Γ can be wild. However, if we replace G by $N := G/M$ and G_Γ by $N_\mathbb{R} := G_\Gamma/M$, then $N_\mathbb{R}$ is a real form of N. In this situation in the Abelian case we identified the analog of $N/N_\mathbb{R}$ with iC and lifted to N an exhaustion which is defined on iC by a positive-definite inner product. In that case straightforward computations show that the lifted exhaustion has the expected plurisubharmonicity. In the nilpotent case at hand we must apply Loeb's Theorem ([L]) which states that since the adjoint representation of $N_\mathbb{R}$ has purely imaginary spectrum (the eigenvalues are all zero!), it follows that there is a smooth exhaustion η of $N/N_\mathbb{R}$ which lifts to a strictly plursubharmonic function on N.

Let us now review the situation discussed above where $X = G/\Gamma$, $G = \widehat{G}_\Gamma$ and M is the normal closed complex subgroup of G defined by the ideal $\mathfrak{m}$. Here we have the fibration

$$X = G/\Gamma \to G/G_\Gamma = N/N_\mathbb{R}$$

and we pull back the exhaustion guaranteed by Loeb's theorem to an exhaustion ρ of X which we view as a G_Γ-invariant function on G. Since this is defined on G by lifting a strictly plurisubharmonic function from $N = G/M$, it is plurisubharmonic on G and therefore ρ is a plurisubharmonic exhaustion of X. Hence we have the following result.

PROPOSITION 2.8. – *If $G = \widehat{G}_\Gamma$, then a smooth exhaustion $\rho : X \to \mathbb{R}^{\geq 0}$ guaranteed by Loeb's theorem is plurisubharmonic with Levi-degeneracy determined by the ideal $\mathfrak{m}$. In particular, $X = G/\Gamma$ is pseudoconvex.*

Several remarks are now in order. First, we arrived at the the situation where $G = G_\Gamma$ by splitting off a factor of $\mathbb{C}^n$ from an arbitrary nilmanifold of the type G/Γ. In fact one doesn't need the assumption of discrete isotropy for such a splitting, i.e., every complex nilmanifold is a product of $\mathbb{C}^n$ and a nilmanifold of the form G/Γ with $G = \widehat{G}_\Gamma$ ([LOR]). Thus we have the following

ZUSATZ. – Every complex nilmanifold possesses a smooth plurisubharmonic exhaustion.

It should also be underlined that due to the nonabelian nature of the situation, the Levi-foliations defined by ρ should be much more interesting that those in the Abelian case.

Now recall that we originally were investigating the possibilty of complex homogeneous spaces being pseudoconcave but ended up with a general pseudoconvexity result. Thus with exactly the same proof as that for Proposition 2.5 we have the following remark.

PROPOSITION 2.9. – *Pseudoconcave nilmanifolds are compact.*

Finally, e.g. in the case of discrete isotropy, X is Stein if and only if G_Γ is totally real ([GH]). Furthermore, in analogy to Theorem 2.6, as in the case of Lie groups a general complex homogeneous manifold G/H has a canonically defined holomorphic reduction $X = G/H \to G/I = X/\sim$. In the nilpotent case the fiber possesses only the constant holomorphic functions and the base is Stein ([GH]). This is far from being true in the general situation.

2.3 – *Flag domains.*

2.3.1 – Background.

Recall that the radical R of a connected Lie group G is defined to be the maximal connected solvable normal subgroup of G. If R is trivial, i.e., consists only of the identity, then G is said to be *semisimple*. A fundamental difference between solvable and semisimple groups is that most semisimple groups possess intrinsic algebraic structure whereas solvable groups do not. In general a Lie group G is a product $R \cdot S$ of its radical and a maximal semisimple subgroup S. In fact, S is unique up to conjugation. The intersection $R \cap S$ is a discrete central subgroup of G and if, for example, G is simply-connected, then this is a semidirect product $G = R \ltimes S$.

Above we commented on certain aspects of the solvable case, i.e., where the complex Lie group G agrees with its radical. If G is semisimple, H is a complex closed subgroup and $X = G/H$, then the assumption of existence of meromorphic or plurisubharmonic functions on X or even that X is Kähler is very restrictive. In most cases this forces H to be an algebraic subgroup of G ([Be, BeO]). For example it is known that X is Stein if and only if H is reductive. In the other extreme of Levi conditions, even under the further condition that H is algebraic there is no known characterization of $X = G/H$ being pseudoconcave.

The situation changes dramatically if G is allowed to be a real semisimple group. In that setting the first basic examples arise as *flag domains*. Here we describe the flag domains which possess plurisubharmonic exhaustions and in the following section we discuss flag domains with Levi conditions in the opposite direction, e.g., pseudoconcavity. Let us begin with a sketch of some background information. The first basic results on flag domains can be found in ([W]). A systematic treatment, which in particular gives the details of the results needed here, is presented in ([FHW]).

Let us begin with a real Lie group G_0 and consider an action $G_0 \times X \to X$ by holomorphic transformations on a complex manifold. If this action is transitive, then we refer to X as being G_0-homogeneous. In that case we may as usual identify X with G_0/H_0 where H_0 is the isotropy group at a base point. However, unlike the case where $X = G/H$ is the homogeneous space under the holomorphic action of a complex Lie group, the complex structure of X is not transparently encoded in the Lie group structure.

At the level of vector fields the situation is slightly better, because the complexified Lie algebra, $\mathfrak{g} := \mathfrak{g}_0 + i\mathfrak{g}_0$, is represented as an algebra of holomorphic $(1,0)$-vector fields on X. In other words, the complexified Lie group G acts locally and holomorphically on X. To put this in perspective consider the example of the standard $G_0 = \mathrm{SU}(1,1)$-action on $\mathbb{P}_1(\mathbb{C})$ and let X be one of its two open orbits (both are disks!). Here, as in the general case, the complexification $G = \mathrm{SL}_2(\mathbb{C})$ acts locally on X and in addition has the advantage of acting globally on $\mathbb{P}_1$. One regards the holomorphic G-manifold $\mathbb{P}_1$ as the *globalization* of the local G-manifold X.

There is a beautiful theory of globalization of local actions due to Palais which was adapted to our complex analytic setting by Heinzner and Iannuzzi (see [HI]). However, even when X is G_0-homogeneous it is difficult to know whether or not it is embedded in a G-globalization. On the other hand, as reflected by the example of the unit disk in $\mathbb{P}_1$, the case where a globalization is implicitly given is already quite interesting. The case of flag domains is one such situation.

In order to discuss flag domains we restrict to the case where G_0 is semisimple. Due to standard splitting theorems it is usually enough to assume, as we do here, that it is even simple. For our purposes it is also enough to consider the situation where it is embedded in its complexification G. We let $G \times Z \to Z$ be a holomorphic G-action on a complex manifold and consider the induced G_0-action. A case of fundamental interest, e.g., for studying the representation theory of G_0, is that where Z is assumed to be a compact, G-homogeneous projective manifold. Choosing a base point we write $Z = G/Q$.

Whereas much is known about *flag manifolds* $Z = G/Q$ of the above form, restricting to the G_0-action adds significant complications which lead to new phenomena which are not yet understood. The following first key step is, however, proved by classical combinatorial arguments (see [W]).

PROPOSITION 2.10. – *The real form G_0 has only finitely many orbits on the flag manifold Z.*

In particular, G_0 has open orbits in Z. We refer to such as a *flag domain* and, if there is no confusion, will always denote it by D. One purpose of this paper is to give evidence for the following (perhaps naive) conjecture.

Flag domains are either pseudoconvex or pseudoconcave.

By *pseudoconvex* we mean that there exists an exhaustion $\rho : D \to \mathbb{R}^{\geq 0}$ which is plurisubharmonic outside of a compact set. *Pseudoconcavity* is understood in the usual sense of Andreotti (see § 1). Below we give a detailed description of the pseudoconvex flag domains. After doing so, we devote the remainder of the paper to describing a large class of pseudoconcave flag domains and to giving some indication of the validity of the conjecture.

2.3.2 – Background on cycle spaces.

Our discussion of pseudoconvex flag domains D makes strong use of the *cycles* in D which are defined by the actions of G_0 and G. Here we begin by introducing minimal background on this subject, referring the reader to ([FHW]) for detailed proofs.

Let K_0 be a maximal compact subgroup of G_0. Any two such are G_0-conjugate and as a result for our purposes the choice is not relevant. A basic fact, which is just the tip of the iceberg of Matsuki duality, is that there is a unique K_0-orbit in D which is a complex submanifold. Let us refer to it as the *base cycle* C_0, regarded as either a submanifold or a point in the cycle space of D. In the sense of dimension C_0 is *the* minimal K_0-orbit in D. If K denotes the complexification of K_0 which is realized as a subgroup of G, then C_0 is also a K-orbit. It can be characterized as the only K-orbit of a point in D which is contained in D.

Here not much information is needed about the cycle spaces at hand. However, let us introduce some convenient notation which will also be of use in the next section. For this let $q := \dim_{\mathbb{C}} C_0$ and let $\mathcal{C}_q(D)$ be the space of q-dimensional cycles in D. Recall that such a cycle is a linear combination $C = n_1 X_1 + \ldots + n_k X_k$ where the X_j are irreducible q-dimensional subvarieties and the coefficients n_j are positive integers. In a natural way $\mathcal{C}_q(D)$ is a complex space which can be regarded as an open subset of the cycle space $\mathcal{C}_q(Z)$. Our view of these cycle spaces is that of ([Ba]). The reader is also referred to Chapter 8 of [FHW] for a minimal presentation.

In our particular case $\mathcal{C}_q(Z)$ is smooth at C_0 (see Part IV of [FHW]) and thus it makes sense to speak of *the* irreducible component of $\mathcal{C}_q(D)$ at C_0. We simplify the notation by replacing $\mathcal{C}_q(D)$ by this irreducible component. Since the algebraic group G is acting algebraically on Z, it acts algebraically on the associated cycle spaces $\mathcal{C}_q(Z)$. The group-theoretical cycle space $\mathcal{M}_D$ which, for example, is of basic interest in representation theory is defined as the connected component of the intersection of the orbit $G.C_0$ with $\mathcal{C}_q(D)$. It is in fact a closed submanifold of $\mathcal{C}_q(D)$ ([HoH]).

2.3.3 – Cycle connectivity.

We say that two points $x, y \in D$ are *connected by cycles* if there are cycles $C_1, \ldots, C_m \in \mathcal{M}_D$ so that the union $C_1 \cup \ldots \cup C_m$ is connected with $x \in C_1$ and $y \in C_m$. The relation defined by $x \sim y$ if and only if x and y are connected by cycles is an equivalence relation. Since $\mathcal{M}_D$ is G_0-invariant, it is by definition G_0-invariant. Thus, if we choose a base point z_0 in D and identify D with G_0/H_0 where H_0 is the G_0-isotropy at z_0, then the quotient $D \to D/\sim$ defined by the equivalence relation is given by a homogeneous fibration $G_0/H_0 \to G_0/I_0$ where I_0 is the stabilizer of the equivalence class $[z_0]$.

By definition the equivalence classs $[z_0] = I_0/H_0 =: F$ is a closed real submanifold of D. Note that if Ω is a relatively compact open neighborhood of z_0 in F, then there is an open neighborhood U of the identity of the isotropy group G_{z_0} which maps U into F. Since G_{z_0} has only finitely many orbits in Z and since G_{z_0} is complex, this imples that F contains an open dense subset which is a complex submanifold of D. But I_0 acts transitively and holomorphically on F and therefore F is a complex submanifold of D. The stabilizer in $\mathfrak{g}$ of F, i.e., the stabilizer of F under the local G-action, is a complex Lie subalgebra $\widehat{\mathfrak{q}}$ which contains the algebra $\mathfrak{q}$ of the G-isotropy subgroup at z_0. Consequently, there exists a globally defined complex subgroup $\widehat{Q}$ so that the fiber F at the base point of $D \to D/\sim$ is an open I_0-orbit in the (compact) fiber of $G/Q \to G/\widehat{Q}$ at the base point.

Proposition 2.11. – *The cycle connectivity reduction $D \to D/\sim = \widehat{D}$ of a flag domain is given by the restriction of a canonically defined G-equivariant map $Z = G/Q \to G/\widehat{Q} = \widehat{Z}$. It is a holomorphic map onto a G_0-flag domain $\widehat{D}$ in $\widehat{Z}$. In particular, the fibers of $D \to D/\sim$ are themselves connected complex manifolds.*

Proof. – Except for one point the proof is given above: We must show that the intersection of the fibers of the G-equivariant map $Z \to \widehat{Z}$ with D are connected. But this follows immediately from the fact that $\widehat{D}$ is simply-connected (see [W] or [FHW]). □

Since the base cycle C_0 is a K-orbit and in particular $k(z_0) \sim z_0$, we know that K stabilizes $[z_0]$. In other words, $K \subset \widehat{Q}$ and it follows that the base cycle $\widehat{C}_0$ in $\widehat{D}$ is just a single point. Since it is known that this can only happen when $\widehat{D}$ is a G_0-Hermitian symmetric space of noncompact type embedded in its compact dual $\widehat{Z}$ ([W]). Let us note this for future reference.

Proposition 2.12. – *Either $D = G_0/H_0$ is cycle connected, i.e., any two points are connected by a chain of cycles in $\mathcal{M}_D$ or the cycle connectivity equivalence reduction $D \to \widehat{D}$ is such that $\widehat{D} = G_0/K_0$ is a Hermitian symmetric space embedded in its compact dual $\widehat{Z}$ and the neutral fiber $K_0/H_0 = C_0$ is the base cycle itself.*

PROOF. – We know that K_0 fixes the base point in $\widehat{D}$ and by general theory the isotropy group of a G_0-symmetric space is exactly a maximal compact subgroup of G_0. $\square$

As a consequence we see that if D is not cycle connected, then any two cycles either agree or are disjoint. In other words, in that case the fibers of the reduction $D \to D/\sim$ are cycles and the cycle space $\mathcal{M}_D$ is the Hermitian symmetric space $\widehat{D}$.

2.3.4 – Pseudoconvex flag domains.

Let us say that a complex manifold X is *pseudoconvex* if it possesses a continuous proper exhaustion function $\rho : X \to \mathbb{R}^{\geq 0}$ which is plurisubharmonic on the complement $X \setminus S$ of a compact set S. It should be underlined that, even if ρ is smooth, we are only assuming the semi-positivity of its Levi-form.

Given the preparation in the previous paragraph, it is now a simple matter to characterize pseudoconvex flag domains. For this the following is the main remark.

LEMMA 2.13. – *Cycle connected flag domains possess only constant plurisubharmonic functions.*

PROOF. – Let D be a pseudoconvex flag domain and consider a plurisubharmonic function ρ on D. Given two points $x, y \in D$, connect them with a chain $C_1, \ldots, C_m$ of cycles. Since $\rho|C_i$ is constant for every i, it is immediate that $\rho(x) = \rho(y)$. $\square$

PROPOSITION 2.14. – *Cycle connected flag domains are not pseudoconvex.*

PROOF. – Given a cycle connected flag domain D, assume to the contrary that it is pseudoconvex. Let $\rho : D \to \mathbb{R}^{\geq 0}$ be an exhaustion which is plurisubharmonic on $D \setminus S$. Define $r_0 = \min(\rho|S)$ and define $\widehat{\rho}$ to be the maximum of ρ and the constant function $r_0 + 1$. Then, contrary to the above Lemma, $\widehat{\rho}$ is a nonconstant plurisubharmonic function on D. $\square$

It follows that pseudoconvex flag domains have cycle reduction $\pi : D \to \widehat{D}$ to a Hermitian symmetric space $\widehat{D}$. The unique cycle through a given point $z \in D$ is the π-fiber $\pi^{-1}(\pi(p))$ through that point. Since $\widehat{D}$ is a contractible Stein manifold, the bundle $\pi : D \to \widehat{D}$ is trivial and D can be (noncanonically) realized as the product $C_0 \times \widehat{D}$. In summary we have the following characterization of pseudoconvex flag domains.

THEOREM 2.15. – *For a flag domain D the following are equivalent.*

(1) *D is pseudoconvex*
(2) *D is holomorphically convex with Remmert reduction $D \to \widehat{D}$ to a Hermitian symmetric space.*
(3) *D is not cycle connected with cycle reduction agreeing with its Remmert reduction.*
(4) *D possesses a nonconstant plurisubharmonic function.*

It should be underlined that domains fulfilling any one of the above conditions are of the form $D = G_0/L_0$ where L_0 is a *compact* subgroup of the group G_0 which is of Hermitian type. As a result one can also describe such domains via root-theoretic data (see [W, FHW]).

3. – Pseudoconcave flag domains.

Above we began our study of flag domains from the point of view of Levi-geometry by showing that pseudoconvex flag domains are of a very special nature (Theorem 2.3.4). As a Leitfaden for further investigations we conjecture that if a flag domain is not pseudoconvex, then it is pseudoconcave. Here we begin with a brief exposition of constructions of two natural exhaustions of flag domains whose Levi-curvature is at least in principle computable. Then, using the exhaustion constructed using cycle geometry, we describe a rather large class of flag domains which are pseudoconcave. We underline that further information concerning properties of these exhaustions in a general setting would certainly be of interest.

3.1 – *Exhaustions.*

Here we discuss two natural methods for constructing K_0-invariant exhaustions of flag domains. From the point of view of Levi-geomtry both have their advantages and disadvantages. The first was introduced by Schmid for a flag domain D which is a G_0-orbit in $Z = G/B$ where B is a Borel subgroup of G ([S]). This was generalized to *measureable* flag domains in ([SW], see also § 4.6 in [FHW]). This type of exhaustion has the advantage that it is smooth and clearly q-convex in the sense of ([AGr1]). However, determining the concavity properties requires root calculations which vary from case to case and which could be rather subtle.

Exhaustions of a second type were recently constructed in ([HW]). These are canonically related to a given irreducible G-representation and the Levi-geometry of $\mathcal{M}_D$. They have the disadvantage of only being continuous, but they are

q-convex in a very strong sense and, as shown in the sequel, their concavity properties (which are related to cycle geometry) are more transparent than those of the exhaustions of the first type.

3.1.1 – Schmid-Wolf exhaustions.

As above D denotes a flag domain which is a G_0-orbit in a flag manifold $Z = G/Q$. The first observation relevant for the construction of the Schmid-Wolf exhaustion is the fact that the anticanonical bundle $K^{-1} \to Z$ is very ample. Assuming that we have chosen G to be simply-connected, this is a G-bundle $G \times_\chi \mathbb{C}$ where $\chi : Q \to \mathbb{C}^*$ is an explicitly computable character. Recall that if h is a Hermitian bundle metric (unitary structure) on a line bundle $L \to X$ on a complex manifold with associated norm-squared function $\| \cdot \|^2$, then the Chern form $c_1^h(L)$ is the negative of the Levi-form $\frac{i}{2}\partial\bar{\partial}\log(\| \cdot \|^2)$ of the exhaustion $\log(\| \cdot \|^2)$ of the bundle space.

In the case at hand, having fixed a Cartan involution θ on $\mathfrak{g}_0$ which defines the Lie algebra $\mathfrak{k}_0$ of the maximal compact subgroup K_0, we extend θ to a holomorphic involution of $\mathfrak{g}$ and define the antiholomorphic involution $\sigma := \tau\theta$, where τ is the antiholomorphic involution which defines the real form $\mathfrak{g}_0$ on $\mathfrak{g}$. The Lie group G_u corresponding to $\mathfrak{g}_u := \mathrm{Fix}(\sigma)$ is the maximal compact subgroup of G which is canonically associated to the real form G_0 with the choice of maximal compact subgroup K_0.

If $L \to Z$ is any nontrivial G-bundle on $Z = G/Q$, then G has exactly two orbits in the bundle space L, the 0-section, which corresponds to the fixed point of Q in the neutral fiber, and its complement. In this complement all G_u-orbits are real hypersurfaces and G_u acts transitively on the 0-section as well. Define $V_u := G_u \cap Q$ so that $Z = G/Q = G_u/V_u$. Writing L as a G_u-bundle, $L = G_u \times_\chi \mathbb{C}$, we note that, since the restriction of χ to V_u is nontrivial and there is a unique S^1-invariant unitary structure on $\mathbb{C}$ normalized at $1 \in \mathbb{C}$, there is an essentially unique G_u-invariant unitary structure on L. The level surfaces of the associated norm-squared function are exactly the G_u-orbits in L. If, as in the case $L = K^{-1}$, the bundle L is ample, then Chern form is positive-definite or, equivalently, from the point of view of the 0-section the G_u-hypersurface orbits are strongly pseudoconcave.

Let us now consider the restriction of the anticanonical bundle K^{-1} to a flag domain D. It is a G_0-homogeneous bundle $G_0 \times_{\widehat{\chi}} \mathbb{C}$. Here $\widehat{\chi}$ is a $\mathbb{C}^*$-valued character from the G_0-isotropy $V_0 = G_0 \cap Q$. One is of course interested in the situation where $\widehat{\chi}$ is S^1-valued so that, as in the case of the G_u-bundle on Z, the anticanonical bundle on D would possess a G_0-invariant unitary structure. The condition for this is called *measurable* ([W], see also § 4.5 in [FHW]).

There are a number of equivalent conditions for D to be measurable ([W]). Here are those of a less technical nature:

(1) D possesses a G_0-invariant pseudokählerian metric.
(2) D posseses a G_0-invariant volume form.
(3) The isotropy group V_0 is reductive in the sense that its complexification V is a complex reductive subgroup of G.
(4) The isotropy group V_0 is the centralizer of a compact subtorus $T_0 \subset G_u \cap V_0$ so that D is realized as a coadjoint orbit. The symplectic form induced from this realization is the invariant form defined by the pseudokählerian metric.

One can show that if one flag domain in Z is measurable, then all others flag domains in Z are also measurable. Thus measurable is a property of the G_0-action on the flag manfold Z. For example, flag manifolds $Z = G/B$ are measurable for any real form. Furthermore, every flag manifold Z is measurable if G_0 is of Hermitian type. On the other hand it is seldom the case that a flag manifold Z is measurable for $G_0 = \mathrm{SL}_n(\mathbb{R})$.

Now if Z is measurable and D is a flag domain in Z, then we have two Hermitian norm-squared functions on its anticanonical bundle, the restriction $\|\cdot\|_u^2$ of the G_u-invariant norm on the full anticanonical bundle of Z and the G_0-invariant function $\|\cdot\|_0^2$ coming from the coadjoint symplectic form or from the the pseudokählerian metric. The characters which define these norms are actually defined on the same torus T_0 which splits off of both isotropy groups and on that torus they are the same. Thus the ratio

$$R = \frac{\|\cdot\|_0^2}{\|\cdot\|_u^2}$$

is a well-defined function on the base D and one can show that $\rho := \log(R)$ is an exhaustion function of D. We refer to this as the Schmid-Wolf exhaustion of D (see [S, SW]). Note that since h_u is G_u-invariant and h_0 is G_0-invariant, ρ is invariant with respect to the maximal compact subgroup $K_0 = G_u \cap C_0$.

The Levi-form of ρ is the difference $c_1^{h_u} - c_1^{h_0}$. A direct calculation with roots shows that $c_1^{h_0}$ is of signature $(q, n-q)$ where q is the dimension of the cycle C_0 and $n = \dim(D)$. Since $c_1^{h_u} > 0$, the exhaustion ρ is q-complete in the sense of Andreotti and Grauert, i.e., at every point of D the Levi-form of ρ has at least $n-q$ positive. Let us note this result.

THEOREM 3.1. – *The Schmid-Wolf exhaustion of a measurable flag domain D is q-complete in the sense of Andreotti and Grauert.*

The Schmid-Wolf exhaustions have the advantage that one can directly apply the Andreotti-Grauert vanishing theorem for higher cohomology groups. One disadvantage is that without further root-theoretic computation one does not know the degree of concavity. Furthermore, one only knows the existence of these exhaustions on measurable domains.

3.1.2 – Exhaustions via Schubert slices.

Here we explain the construction of ([HW]) which uses cycle space geometry to produce an exhaustion $\rho_D : D \to \mathbb{R}^{\geq 0}$ of any given flag domain. It has the disadvantage of only being continuous, but it is q-pseodoconvex in a strong sense. Its concavity properties are closely related to the cycle geometry of D.

The construction of ρ_D requires basic information concerning *Schubert slices*. We sketch this here and refer the reader to § 9 of ([FHW]) for details. In order to define a Schubert slice we must recall that G_0 possesses an Iwasawa-decomposition $G_0 = K_0A_0N_0$. Here K_0 is a maximal compact subgroup as above, A_0 is an Abelian subgroup noncompact type, N_0 is a certain nilpotent subgroup defined by root-theory and which is normalized by A_0. Writing K, A and N for the complexifications of these subgroups which are subgroups of G, we note the fundamental fact that the set KAN is a *proper* Zariski open subset of G.

Now if $C_0 := K_0.z_0$ is a base cycle, then every orbit of A_0N_0 in D must have nonempty intersection with C_0. The following is basic for our discussion (see § 7.3 in [FHW] for details).

THEOREM 3.2. – *The set I of points $z \in C_0$ which are such that the orbit $A_0N_0.z$ is of minimal dimension under all A_0N_0-orbits in D is finite. For every $z \in I$ the orbit $\Sigma := A_0N_0.z$ has the following properties:*

(1) *Σ is closed in D and open in $AN.z$ which is a Schubert cell in Z.*
(2) *The intersection of Σ with every cycle $C \in \mathcal{M}_D$ consists of exactly one point and C is transversal to Σ at that point.*

For obvious reasons we refer to the orbits Σ as Schubert slices. We should note that the Schubert cells in the above statement are meant to be the orbits of Borel groups B which contain an Iwasawa-factor AN. These are very special Borel groups, being those whose fixed point is in the (unique) closed G_0-orbit in Z.

If $r_\Sigma : \Sigma \to \mathbb{R}^{\geq 0}$ is a strictly plurisubharmonic function on Σ, then we define a plurisubharmonic function $\rho_\Sigma : \mathcal{M}_D \to \mathbb{R}^{\geq 0}$ by $\rho_\Sigma(C) := r_\Sigma(\sigma_C)$, where σ_C is the unique point of intersection of C and Σ. After checking that ρ_Σ is a plurisubharmonic function on $\mathcal{M}_D$ one might hope that if Σ is Stein and r_Σ is an exhaustion, then ρ_Σ might be a plurisubharmonic exhaustion of $\mathcal{M}_D$. Simple examples, e.g., the one interesting flag domain defined by the $\mathrm{SU}(2,1)$-action on the 3-dimensional manifold of full flags in $\mathbb{C}^3$, show that in general Σ is not Stein. Furthermore, even if r_Σ is an exhaustion, ρ_Σ may not be an exhaustion.

The difficulties mentioned above can be remedied by simultaneously considering a number of Schubert slices. To do this we start with strictly plurisubharmonic functions r_Σ which arise in a natural way, in this case associated to an irreducible representation of G. For this we recall that if $L \to Z$ is a G-line bundle, then the G-representation on $\Gamma(Z,L)$ is irreducible. Conversely, every irreducible holomorphic representation of G occurs in this way.

Now recall that a given $\Sigma = A_0N_0.z$ is open in the Schubert cell $\mathcal{O}_S := B.z \cong \mathbb{C}^{n-q}$ which closes up to the Schubert variety S. Given an of an ample bundle $L \to Z$, we let V be the space of sections of $L|S$ which are defined as restrictions of sections of L on Z. Let $s \in V$ be a B-eigenvector which is not identically zero. It follows that s vanishes exactly on $S \setminus \mathcal{O}_S$ (see, e.g., [FHW], § 7.4C) and if we equip L with the canonically defined G_u-invariant norm-squared function $\|\cdot\|^2$, then the restriction of $r_\Sigma := s^*(\log(\|\cdot\|^2)$ is a strictly plurisubharmonic exhaustion of the Schubert cell $\mathcal{O}_S$. The associated function ρ_Σ on the cycle space is plurisubharmonic, but normally not an exhaustion. Thus we define $\rho_{\mathcal{M}_D}$ to be the supremum of the ρ_Σ as Σ ranges over *all* possible Schubert slices for a fixed Iwasawa component A_0N_0 and over all Iwasawa decompositions. Since this is a compact family of Schubert slices, it can be shown that $\rho_{\mathcal{M}_D}$ is a continuous plurisubharmonic function. Using our analysis of the boundary behavior of the Schubert slices (see, e.g., § 9.2 in [FHW]), one proves the following first result.

PROPOSITION 3.3. – *The function $\rho_{\mathcal{M}_D} = \sup_\Sigma(\rho_\Sigma)$ associated to an irreducible representation of G on the space of sections of an ample bundle on Z is a continuous plurisubharmonic K_0-invariant exhaustion of the cycle space $\mathcal{M}_D$.*

The procedure for transferring $\rho_{\mathcal{M}_D}$ back to the domain D is quite natural. For this we let $\mathfrak{X} := \{(z,C) \in D \times \mathcal{M}_D : z \in C\}$ and denote by $\mu : \mathfrak{X} \to D$ and $\nu : \mathfrak{X} \to \mathcal{M}_D$ the canonical projections. Note that the fiber $\mu^{-1}(p) = F_p$ can be identified with the set of cycles in D which contain the point p. Now define $\rho_{\mathfrak{X}} := \rho_{\mathcal{M}_D} \circ \nu$ and let

$$\rho_D(p) := \inf_{F_p}(\rho_{\mathfrak{X}}) .$$

The following can be proved by tracing through the construction of ρ_D ([HW]).

PROPOSITION 3.4. – *The function $\rho_D : D \to \mathbb{R}^{\geq 0}$ is a continuous K_0-invariant exhaustion of D which is q-pseudoconvex in the following sense: For every $r<0$ and every z in the boundary of the sublevel set $\{\rho_D < r\}$ there exists a neighborhood $U = U(z)$ and a smooth function h on U such that $h(z) = r$, $h \leq \rho_D|U$ and the Levi-form $L(h)$ restricted to the complex tangent space of $\{h = r\}$ at z has an $(n-q)$-dimensional positive eigenspace.*

It would be useful if either ρ_D could be smoothed to an exhaustion which is q-pseudoconvex in the sense of Andreotti-Grauert or if the finiteness/vanishing theorems of Andreotti-Grauert could be proved under the assumption of a continuous exhaustion with the above pseudoconvexity property.

3.1.3 – Flag domains are q-pseudoflat.

In ([NH]) we introduced the notion of q-pseudoflatness as a weakening of both q-Leviflatness and q-pseudoconcavity (See [HSt] for elementary complex analytic properties of such manifolds.). By definition a q-pseudoflat (connected) complex manifold X is required to contain a relatively compact open set Z such that every point p of its closure $\mathrm{cl}(Z)$ is contained in a q-dimensional (locally defined) analytic set A_p which itself is contained in $\mathrm{cl}(Z)$. Examples on the pseudoconvex side which possess exhaustions by plurisubharmonic functions whose level sets are foliated by q-dimensional leaves are given by the Lie groups in (2.1), the nilmanifolds in (2.2) and the flag domains in (2.3.4). In the flag domain case the number q is the dimension of the base cycle C_0. An optimal dichotomy might be that a flag domain is either q-Leviflat as in Theorem 2.15 or q-pseudoconcave. We have stated a weakened version of this conjecture in (2.3.1) and prove the pseudoconcavity (without any particular degree q) of certain flag domains in (3.2). Here we note the following general result on q-pseudoflatness. Its proof follows by direct inspection of the definition of an exhaustion defined by the Schubert-slice method.

PROPOSITION 3.5. – *Let $\rho_D : D \to \mathbb{R}^{\geq 0}$ be an exhaustion of a flag domain D which is defined by the Schubert-slice method and $D_r = \{\rho_D < r\}$ be a sublevel set. Then every $p \in \mathrm{bd}(D_r)$ is contained in a cycle $C \in \mathcal{M}_D$ which itself is contained in* $\mathrm{cl}(D_r)$. *In particular, D is q-pseudoflat.*

PROOF. – If $\rho_D(p) = r$, then by definition there exists a cycle $C_p \in \mathcal{M}_D$ with $p \in C_p$ such that

$$\rho_{\mathfrak{X}}(p, C) = r = \min\{\rho_{\mathfrak{X}}(p, C) : C \in F_p\}\,.$$

Now consider another point $\widehat{p} \in C$ and note that, since $C \in F_{\hat{p}}$ and

$$\rho_{\mathfrak{X}}(p, C) = \rho_{\mathfrak{X}}(\widehat{p}, C) = \rho_{\mathcal{M}_D}(C)\,,$$

it follows that $\rho_D(\widehat{p}) \leq \rho_D(p) = r$, i.e., $C \subset \mathrm{cl}(D_r)$. □

3.2 – *Pseudoconcavity via cycles.*

Here we prove that D is pseudoconcave if it is cycle connected in a certain strong sense which we refer to as *generically 1-connected*. To define this notion first note that for p an arbitrary point in D and C an arbitrary cycle in $G.C_0$ the set of cycles in $G.C_0$ which contain p is just the orbit $G_p.C$ of the G-isotropy group at C. We therefore say that D is generically 1-connected if C has nonempty intersection with the open G_p-orbit in Z. One checks that this notion does not depend on the choice of p or C.

Throughout this paragraph we assume that D is generically 1-connected. Under this assumption we will show that D is pseudoconcave in the sense of Andreotti, i.e., that D contains a relatively compact open subset $\mathrm{int}(\mathcal{K})$ such that for every point of its closure $\mathcal{K}$ there is a 1-dimensional holomorphic disk Δ with p at its center such that $\mathrm{bd}(\Delta)$ is contained in $\mathrm{int}(\mathcal{K})$. In fact the construction is such that every $p \in \mathcal{K}$ is contained in a cycle C in $\mathcal{M}_D$ which is itself contained in $\mathcal{K}$. This cycle has the further property that $C \cap \mathrm{int}(\mathcal{K}) \neq \emptyset$. Hence, in a certain sense one may regard D as being q-pseudoconcave. At the present time, however, we are not able to replace C with a q-dimensional polydisk.

The compact set $\mathcal{K}$ is constructed as follows. For p_0 an arbitrary point in C_0 let U be a relatively compact open neighborhood of the identity in the isotropy subgroup G_{p_0}. Choose U to be sufficiently small so that

$$\mathcal{K}_{p_0} := \{u(p); p \in C_0,\ u \in \mathrm{cl}(U)\}$$

is contained in D and let

$$\mathcal{K} := \bigcup_{k \in K_0} k.K_{p_0}\,.$$

PROPOSITION 3.6. – *The K_0-invariant set $\mathcal{K}$ is a compact subset of D which is the closure of its interior* $\mathrm{int}(\mathcal{K})$*. The base cycle C_0 is contained in* $\mathrm{int}(\mathcal{K})$ *and every point of $\mathcal{K}$ is contained in a cycle C which is contained in $\mathcal{K}$ and which has nonempty intersection with C_0.*

PROOF. – Since $\mathcal{K} = \{ku(p); k \in K_0, u \in \mathrm{cl}(U), p \in C_0\}$ and K_0, $\mathrm{cl}(U)$ and C_0 are compact, it is immediate that $\mathcal{K}$ is compact. If $z = ku(p) \in \mathcal{K}$, then we let $\{p_n\}$ be a sequence in C_0 which is contained in in the open orbit of G_p and which converges to p. It follows that $z_n := ku(p_n)$ is in the interior of $\mathcal{K}$ and $z_n \to z$. Thus $\mathcal{K}$ is the closure of its interior $\mathrm{int}(\mathcal{K})$. By definition every point of the intersection of C_0 with the open G_{p_0}-orbit is in $\mathrm{int}(\mathcal{K})$. Thus, since K_0 acts transitively on C_0, it follows that $C_0 \subset \mathrm{int}(\mathcal{K})$. Finally, every point $z \in \mathcal{K}$ is of the form $z = kuk^{-1}k(p_1)$, where $p_1 \in C_0$. Thus $z \in kuk^{-1}(C_0) := C \subset \mathcal{K}$. Since kuk^{-1} fixes $k(p_0)$, it follows that $C \cap C_0 \neq \emptyset$. □

In order to replace the *supporting cycles* with q-dimensional polydisks, the construction of $\mathcal{K}$ may have to be refined. However, without further refinements we are able to construct 1-dimensional supporting disks at each boundary point of $\mathcal{K}$.

THEOREM 3.7. – *Generically 1-connected flag domains are pseudoconcave.*

PROOF. – Let $z \in \mathrm{bd}(\mathcal{K})$, choose $C = gC_0g^{-1} \in \mathcal{M}_D$ to be contained in $\mathcal{K}$ with $z \in C$ and $C \cap C_0 \neq \emptyset$, and let z_0 be in this intersection. Choose a 1-parameter unipotent subgroup of gKg^{-1} whose orbit of z_1 has nonempty intersection with the open orbit of the isotropy subgroup of gKg^{-1} at z_1 and define Y to be the closure of this orbit. After an injective normalization, Y is just a copy of $\mathbb{P}_1$. Replacing Y by $h(Y)$ where h is in the gKg^{-1}-isotropy group at z_1, i.e., by conjugating the 1-parameter subgroup by h, we may assume that Y contains points y_0 which are arbitrarily near z_0. Choose y_0 in $\mathrm{int}(\mathcal{K})$ and define Δ to be the complement in Y of the closure of a disk about y_0 which is likewise in $\mathrm{int}(\mathcal{K})$. $\square$

Our feeling is that most flag domains *are* 1-connected and that the domains which are not 1-connected can be classified by elementary root computations. The argument in the proof of the following remark gives some indication of this.

PROPOSITION 3.8. – *Every flag domain of* $\mathrm{SL}_n(\mathbb{R})$ *is 1-connected.*

PROOF. – Let $z_0 \in C_0$ be the base point and assume that it cannot be connected to some point $z \in D$ by a cycle. It follows that C_0 has empty intersection with the open Q-orbit. In particular, it has empty intersection with the open orbit of every Borel subgroup B contained in Q. Thus C_0 is contained in an irreducible B-invariant complex hypersurface H in Z. Given such a hypersurface there is a maximal parabolic subgroup $\widehat{Q}$ containing Q such that H is the preimage of the unique B-invariant hypersurface $\widehat{H}$ in $\widehat{Z} := G/\widehat{Q}$ by the projection $G/Q \to G/\widehat{Q}$.

We may assume that the unipotent radical $U := R_u(\widehat{Q})$ is contained in B and note that it is Abelian and acts freely and transitively on the open B-orbit in $\widehat{Z}$. Thus it acts with 1-dimensional ineffectivity on $\widehat{H}$. By construction the base cycle $\widehat{C}_0$ is contained in $\widehat{H}$. Thus the stabilizer of $\widehat{C}_0$ in G acts on $\widehat{C}_0$ with nontrivial ineffectivity. On the other hand $K = \mathrm{SO}_n(\mathbb{C})$ is a simple Lie group and consequently this stabilizer contains K as a proper subgroup.

As a result the domain $\widehat{D} = G_0.\widehat{z}_0$ is of Hermitian holomorphic type (see § 5 in [FHW]) and in particular G_0 is of Hermitian type, a contradiction. $\square$

The reader will note that, except for the fact that we use the simplicity of K, the discussion in the above proof is completely general. However, even in the case where K is not simple, we would expect that it still would be possible to reduce to the Hermitian holomorphic case. A concrete example of this is the action of $G_0 = \mathrm{SO}(3, 19)$ on the 20-dimensional quadric $Z = G/Q$ which is also 1-connected. The flag domain D of positive lines in Z is the moduli space of marked K3-surfaces, an example of interest to Andreotti. It also should be mentioned that if stronger conditions are imposed, then fine classification results can be proved. For example, for the case of strong pseudocavity see ([HS1, HS2]).

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