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Chow-Lasota Theorem for BVPs of Evolution Equations

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To the memory of Andy Lasota,
friend and master of elegant Mathematics.

Abstract. – *We extend the main result of* CHOW-LASOTA [1] *to evolution equations and show some applications of the outcome.*

1. – Introduction.

In [1], S.N. Chow and A. Lasota proved a type of Fredholm alternative for linear BVPs of nonlinear ODEs that, roughly speaking, is as follows: if the boundary operator ranges in an open set of linear operators whose members produce at most a unique solution of the corresponding BVP, then the solution of the given BVP does really exist. Later, in [2] and [3], A. Lasota proved suitable variants of this theorem for multipoint BVPs of ODEs and for Hammerstein equations respectively. The interest on these results is clear: uniqueness is usually much easier to prove than existence. Among the various consequences pointed out by the mentioned authors there is a topological explanation of the applicability of the contraction fixed point theorem.

In [13], J.R. Ward provides a partial extension of Chow-Lasota theorem to evolution equations whose semigroup is compact. His result, Corollary 3.1, applies only when the boundary operator is close to the operator corresponding to the Cauchy problem and the "difference" depends heavily on the semigroup.

In this paper, we provide a full extension of Chow-Lasota theorem to evolution equations with compact semigroups. Our argument is modelled on that of CHOW-LASOTA[1] by filling up the technical differences between ODEs and evolution equations. Some corollaries and examples illustrate the applicability of our result.

2. – Notations, terminology and preliminaries.

Our terminolgy related to semigroups follows PAZY [7]. We recall that an operator is called completely continuous when it is continuous and sends bounded sets into relatively compact sets.

We shall denote by:

- X a Banach space;
- A a generator of a compact semigroup $T(t)$ on X;
- $[a,b]$ a compact subinterval of $[0,+\infty[$;
- f a function $[a,b]\times X\to X$ such that $f(\cdot,x)$ is measurable for all x, $f(t,\cdot)$ is continuous for a.e. t and for every point $z\in[a,b]\times X$ there exist a neighborhood N_z of z and $h_z\in L^1_{\text{loc}}$ such that
$$\|f(t,x)\|\leqslant h_z(t)$$
for $(t,x)\in N_z$ and a.e. t;
- C^0 the Banach space of continuous functions $[a,b]\to X$ endowed with the sup norm $\|\cdot\|_\infty$;
- $\mathcal{L}(W,Z)$ the space of bounded linear operators $W\to Z$, W and Z normed spaces;
- L a fixed element of $\mathcal{L}(C^0,X)$.

Further assumptions on these objects are explicitly mentioned on needs.

We are interested in the solvability of the general BVP

$$\textbf{(P)}\qquad \begin{cases} u' = Au + f(t,u) \\ L(u) = y \end{cases}$$

for any given $y\in X$.

We shall use three times the following result proved between the end of p. 25 and the beginning of p. 26 of PAZY [6].

PAZY'S LEMMA. – *Let $T(t)$ be a compact semigroup on X, $h\in L^1([a,b],\mathbb{R}^+)$ and*

$$F(u)(t) := \int_a^t T(t-s)u(s)\,ds$$

for $u\in L^1([a,b],X)$. The set

$$\{F(u) : u\in L^1([a,b],X),\ \|u(\cdot)\|\leqslant h\ a.e.\}$$

has compact closure in C^0.

3. – The results.

Our main result is the following.

THEOREM. – **(P)** *has a unique mild solution for every $y\in X$ when the following conditions are satisfied:*

(i) *for each* $x \in X$ *the Cauchy problem*

$$\begin{cases} u' = Au + f(t,u) \\ u(a) = x \end{cases}$$

has a unique mild solution $\varphi(\cdot, x)$ *on* $[a,b]$*;*

(ii) *the linear BVP*

$$\begin{cases} u' = Au \\ L(u) = y \end{cases}$$

has a unique mild solution for every $y \in X$*;*

(iii) L *has a neighborhood* $\mathcal{U}$ *in* $\mathcal{L}(C^0, X)$ *such that the BVP*

$$\begin{cases} u' = Au + f(t,u) \\ S(u) = y \end{cases}$$

has at most one mild solution for every $S \in \mathcal{U}$ *and* $y \in X$.

Before we give the proof, note that assumption (ii) is not needed when $\dim(X) < \infty$ [(ii) is used here to by-pass the fact that L is not necessarily completely continuous when $\dim(X) = \infty$].

PROOF. – To prove the theorem it suffices to show that the map $F(x) := L\big(\varphi(\cdot, x)\big)$ is surjective, i.e. $F(X) = X$. To achieve this goal we show that the set $F(X)$ is open and closed.

Proof that $F(X)$ *is open*. The map F is injective by (iii) and is continuous because $x \rightsquigarrow \varphi(\cdot, x)$ is a continuous map $X \to C^0$ by virtue of Lemma 1 in VIDOSSICH [10]. The map $G(x) := L\big(T(\,\cdot - a)x\big)$ is a linear bijection by (ii) and is continuous by Lemma 1 in VIDOSSICH [10], hence it is a homeomorphism by the Banach open mapping theorem. Thus $F(X)$ is open if and only if $G^{-1}\big(F(X)\big)$ is. From the standard representation of mild solutions to evolution equations we get

$$F(x) = L\big(T(\,\cdot - a)x\big) + L\Bigg(\int_a^{\cdot - a} T(\,\cdot - s) f\big(s, \varphi(s,x)\big)\,ds\Bigg)$$

and so

$$G^{-1}\big(F(x)\big) = x + G^{-1}\Bigg(L\Bigg(\int_a^{\cdot - a} T(\,\cdot - s) f\big(s, \varphi(s,x)\big)\,ds\Bigg)\Bigg).$$

This means that $G^{-1} \circ F$ is a perturbation of the identity which is locally completely continuous by virtue of Pazy's Lemma. Then $G^{-1}\big(F(X)\big)$ is open by a well-known application of the Leray-Schauder topological degree saying that injective completely continuous perturbations of the identity are open mappings, cf. e.g. Theorem 4.3.12 of LLOYD [4], a property that here is used locally.

Proof that $F(X)$ is closed. We have to show that if $F(x_n) \to y$, then there is x such that $y = F(x)$. Assume that $F(x_n) \to y$. To state our aim, it suffices to show that $(x_n)_n$ is a convergent sequence because F is continuous. Assume that $(x_n)_n$ is not convergent and argue for a contradiction. Then there exist $\varepsilon > 0$, $n_k \uparrow \infty$ and $j_k \geq 1$ such that

$$\|\varphi(\cdot, x_{n_k}) - \varphi(\cdot, x_{n_k+j_k})\|_\infty \geq \varepsilon$$

for all k. One of the corollaries to the Hahn-Banach theorem ensures the existence of a linear functional $h_k : C^0 \to \mathbb{R}$ such that

$$h_k\big(\varphi(\cdot, x_{n_k}) - \varphi(\cdot, x_{n_k+j_k})\big) = 1 \text{ and } \|h_k\| = 1/\|\varphi(\cdot, x_{n_k}) - \varphi(\cdot, x_{n_k+j_k})\|_\infty \leqslant \frac{1}{\varepsilon}.$$

Define $L_k \in \mathcal{L}(C^0, X)$ by

$$L_k(u) := -h_k(u) \cdot \big(F(x_{n_k}) - F(x_{n_k+j_k})\big).$$

We have

$$\|L_k\| \leqslant \|h_k\| \cdot \|F(x_{n_k}) - F(x_{n_k+j_k})\| \leqslant \frac{\|F(x_{n_k}) - F(x_{n_k+j_k})\|}{\varepsilon},$$

hence

$$\lim_k \|L_k\| = 0.$$

It follows that $L + L_k \in \mathcal{U}$ for k sufficiently large. Fix one of these k's and set

$$y_k := (L + L_k)\big(\varphi(\cdot, x_{n_k})\big).$$

In view of

$$\begin{aligned}
&(L + L_k)\big(\varphi(\cdot, x_{n_k+j_k})\big) - (L + L_k)\big(\varphi(\cdot, x_{n_k})\big) \\
&\quad = F(x_{n_k+j_k}) - F(x_{n_k}) - h_k\big(\varphi(\cdot, x_{n_k+j_k}) - \varphi(\cdot, x_{n_k})\big) \cdot \big(F(x_{n_k}) - F(x_{n_k+j_k})\big) \\
&\quad = 0,
\end{aligned}$$

$\varphi(\cdot, x_{n_k})$ and $\varphi(\cdot, x_{n_k+j_k})$ are solutions to

$$\begin{cases} u' = Au + f(t, u) \\ (L + L_k)(u) = y_k \end{cases}.$$

As $\varphi(\cdot, x_{n_k}) \neq \varphi(\cdot, x_{n_k+j_k})$, we have contradicted (iii), and we are done. □

The next result uses the concept of semi-inner product introduced by LUMER-PHILLIPS [5]. When $\dim(X) < \infty$ and $A = 0$, Corollary 1 provides a correct version of the corollary on page 287 of PICCININI-STAMPACCHIA-VIDOSSICH [8] whose statement is incorrect because equation (5.9) of [8] is false when $M < 0$ [and consequently some of the examples in § 5.1 of [8] make no sense].

COROLLARY 1. – *Under the following assumptions*

- *there exist a semi-inner product* $[\cdot,\cdot]$ *on* X *and a constant* $M \in \mathbb{R}$ *such that*

$$[f(t,x) - f(t,y), x-y] \leqslant M\|x-y\|^2$$

for a.e. t *and all* x, y*;*
- $T(t)$ *is a contraction semigroup;*
- f *is continuous and bounded on bounded sets;*

(P) *has a unique mild solution for every* $y \in X$ *and every* $L \in \mathcal{L}(C^0, X)$ *such that*

$$\|L - L_C\| < \min\{1, \mathrm{e}^{-M(b-a)}\}$$

where $L_C \in \mathcal{L}(C^0, X)$ *is the operator corresponding to the Cauchy problem, i.e.* $L_C(u) := u(a)$*.*

PROOF. – In view of Theorem 1 in VIDOSSICH [9] and Theorem 2 in VIDOSSICH[10], the hypotheses guarantee that every Cauchy problem for our evolution equation has a unique mild solution defined in $[a,b]$. That $x \rightsquigarrow L\big(T(\,\cdot - a)x\big)$ is a linear bijection, i.e. the validity of (ii) in Theorem, is a consequence of the fact that a perturbation of the identity map by a contraction produces a homeomorphism and actually $L\big(T(\,\cdot - a)\,\cdot\,\big)$ is indeed a perturbation of the identity map by virtue of the following remarks:

$$L\big(T(\,\cdot - a)\,\cdot\,\big) = L_C\big(T(\,\cdot - a)\,\cdot\,\big) + L\big(T(\,\cdot - a)\,\cdot\,\big) - L_C\big(T(\,\cdot - a)\,\cdot\,\big),$$

$L_C\big(T(\,\cdot - a)\,\cdot\,\big)$ is the identity map and

$$\begin{aligned}\|L\big(T(\,\cdot - a)\,\cdot\,\big) - L_C\big(T(\,\cdot - a)\,\cdot\,\big)\| &= \sup_{\|x\|=1} \|L\big(T(\,\cdot - a)x\big) - L_C\big(T(\,\cdot - a)x\big)\| \\ &\leqslant \|L - L_C\|\,\|T(\,\cdot\,)\|\,\|x\| \\ &< 1\end{aligned}$$

as $\|T(\,\cdot\,)\| \leqslant 1$.

At this point in order to apply the above Theorem we need only to show that the open ball $\mathcal{U}$ in $\mathcal{L}(C^0, X)$ with center L_C and radius $\min\{1, \mathrm{e}^{-M(b-a)}\}$ satisfies condition (iii) in its statement. To this aim, we assume the existence of an $L \in \mathcal{U}$ such that **(P)** has two different solutions u, v for some y and we argue for a contradiction. Set $w := u - v$. The mentioned uniqueness of solutions to Cauchy problems for our evolution equation ensures that $w(a) \neq 0$. Moreover, following the patterns of the proof of Theorem 1 in VIDOSSICH[9] we can see that

$$\|w(t)\| \leqslant \|w(a)\| \cdot \mathrm{e}^{M(t-a)} \leqslant \|w(a)\| \cdot \max\{1, \mathrm{e}^{M(b-a)}\}. \tag{1}$$

Then we have the following contradiction

$$
\begin{aligned}
0 &= \|L(w)\| \\
&\quad [\text{as } L(u) = y = L(v)] \\
&= \|L_C(w) + (L - L_C)(w)\| \\
&\geqslant \|L_C(w)\| - \|L - L_C\| \, \|w\|_\infty \\
&\geqslant \|w(a)\| - \|L - L_C\| \, \|w(a)\| \, \max\{1, \mathrm{e}^{M(b-a)}\} \\
&\quad [\text{by } (1)] \\
&> \|w(a)\| - \min\{1, \mathrm{e}^{-M(b-a)}\} \, \|w(a)\| \, \max\{1, \mathrm{e}^{M(b-a)}\} \\
&\quad [\text{by the hypotheses and } w(a) \neq 0] \\
&= 0
\end{aligned}
$$

and so we are done. □

The next corollary generalizes partially Theorem 2.1 in WARD [14] and its proof provides a simplification of the proof as well as a generalization of Theorem 2 in VIDOSSICH [11].

COROLLARY 2. – *If A, f, L fulfil the assumptions of the above Theorem and f is Lipschitz in u, then the BVP*

$$
\begin{cases} u' = Au + f(t,u) + g(t,u) \\ L(u) = N(u) \end{cases}
$$

has at least one mild solution for every continuous mapping $N : C^0 \to X$ with relatively compact range and every mapping $g : [a,b] \times X \to X$ such that $g(\cdot, x)$ is measurable for all x, $g(t, \cdot)$ is continuous for a.e. t, $\|g(t,x)\| \leqslant h(t)$ for all x and a.e. t with $h \in L^1$ and moreover the Cauchy problem

$$
\begin{cases} u' = Au + f(t,u) + g(t,u) \\ u(a) = x \end{cases}
$$

has a unique mild solution $\sigma(\cdot, x)$ on $[a,b]$ for every x.

PROOF. – We have to show that

$$
L\big(\sigma(\cdot, x)\big) = N\big(\sigma(\cdot, x)\big) \tag{2}
$$

for at least one x. To this aim, we argue as follows.

Let $\varphi(\cdot, x)$ be the unique mild solution on $[a,b]$ of the Cauchy problem

$$
\begin{cases} u' = Au + f(t,u) \\ u(a) = x \end{cases}.
$$

In view of the Lipschitz continuity of f in u and the "L^1-boundedness" of g, an elementary application of the Gronwall lemma implies

$$\sup_{t,x} \|\varphi(t,x) - \sigma(t,x)\| < +\infty. \tag{3}$$

In view of the representation of mild solutions to evolution equations, (2) can be rewritten as follows:

$$L\big(T(\,\cdot - a)x\big) + L\Big(\int_a^{\cdot - a} T(\,\cdot - s) f\big(s, \sigma(s,x)\big)\,ds\Big) + L\Big(\int_a^{\cdot - a} T(\,\cdot - s)\, g\big(s, \sigma(s,x)\big)\,ds\Big) = N\big(\sigma(\cdot, x)\big)$$

hence also as:

$$\begin{aligned} L\big(T(\,\cdot - a)x\big) &+ L\Big(\int_a^{\cdot - a} T(\,\cdot - s) f\big(s, \varphi(s,x)\big)\,ds\Big) \\ &= L\Big(\int_a^{\cdot - a} T(\,\cdot - s) f\big(s, \varphi(s,x)\big)\,ds\Big) - L\Big(\int_a^{\cdot - a} T(\,\cdot - s) f\big(s, \sigma(s,x)\big)\,ds\Big) \\ &\quad - L\Big(\int_a^{\cdot - a} T(\,\cdot - s)\, g\big(s, \sigma(s,x)\big)\,ds\Big) + N\big(\sigma(\cdot, x)\big). \end{aligned}$$

Applying the inverse of $x \rightsquigarrow L\big(T(\,\cdot - a)x\big)$ to both sides of this formula we get an equality of the type

$$I + F = G$$

where I is the identity mapping of X and $G : X \to X$ is continuous and has relatively compact range by virtue of Pazy's Lemma [due to (3), the Lipschitz continuity of f in u and the "L^1-boundedness" of g]. Moreover, the mapping $I + F$ is a homeomorphism as it follows from the above Theorem and its proof. Then the equation

$$x = (I + F)^{-1}\big(G(x)\big)$$

is solvable by Schauder fixed point theorem, hence we are done. $\square$

EXAMPLE 1. – Consider the following BVP

$$\begin{cases} u_t = u_{xx} + F(t,x,u) + G(t,x,u) & \text{on }]0,1[\, \times \,]c,d[\\ u(t,x) = 0 & \text{on }]0,1[\, \times \{c,d\} \\ u(0,x) - a\,u(1,x) = \displaystyle\int_0^1\!\!\int_c^d H\big(t,x,y,u(t,y)\big)\,dt\,dy & \text{for every } x \end{cases}$$

where $0<|a|<1$, $H:[0,1]\times[c,d]^2\times\mathbb{R}\to\mathbb{R}$ is uniformly continuous and $F,G:[0,1]\times[c,d]\times\mathbb{R}\to\mathbb{R}$ are continuous and satisfy $F(t,c,0)=G(t,c,0)=0=F(t,d,0)=G(t,d,0)$. If F is decreasing in u, if G is locally Lipschitz in u and G,H are uniformly bounded, then the given BVP has at least one mild solution.

PROOF. – Let $X:=C^0([c,d])$ and $C^0:=C^0([0,1],X)$. We introduce on X a semi-inner product $[\cdot,\cdot]$ in the following way: for every $u\in X$, we fix a point x_u where $|u(\,\cdot\,)|$ takes its maximum and set

$$[u,v]:=u(x_v)\cdot v(x_v).$$

Let $A:=u_{xx}$ with $\mathrm{dom}(A):=\{u\in X:\ u\in C^2(]c,d[),\ u(c)=0=u(d)\}$. Obviously

$$[Au,u]\leqslant 0$$

whenever $u\in\mathrm{dom}(A)$, so that A generates a contraction semigroup on X by Theorem 2.1 of LUMER-PHILLIPS [5]. It is well-known that this semigroup is compact. Let $L_C,\,L,\,N:C^0\to X$ and $f,\,g:[0,1]\times X\to X$ be defined by

$$L_C(u):=u(0),\quad L(u):=u(0)-a\,u(1),\ \ N(u)(x):=\int_0^1\!\!\int_c^d H\big(t,x,y,u(t)(y)\big)\,dt\,dy,$$

$$f(t,u):=F\big(t,\cdot,u(\,\cdot\,)\big),\qquad g(t,u):=G\big(t,\cdot,u(\,\cdot\,)\big).$$

Obviously $0<|a|<1$ implies $\|L_C-L\|<1$. Moreover, the decreasing property of $F(t,x,\cdot)$ implies

$$\big[f(t,u)-f(t,v),u-v\big]\leqslant 0\qquad (t,u,v)$$

while the continuity of F implies that f is bounded on bounded sets of X. Consequently the BVP

$$\begin{cases} u'=Au+f(t,u)\\ L(u)=y\end{cases}$$

fulfils the assumptions of Theorem by virtue of Corollary 1. The boundedness and the uniform continuity of H imply that N is a continuous operator with relatively compact range by virtue of the Ascoli theorem. Thus the conclusion follows from Corollary 2. $\square$

When G is p-periodic in t and $H\equiv 0$, the following example provides the existence of p-periodic mild solutions.

EXAMPLE 2. – Consider the following BVP

$$\begin{cases} u_t = \Delta u + B(t)u + G(t,x,u) & \text{on }]0,p[\, \times \Omega \\ u(t,x) = 0 & \text{on }]0,p[\, \times \partial\Omega \\ u(0,x) - u(p,x) = \displaystyle\int_0^p\int_\Omega H\big(t,x,y,u(t,y)\big)\,dt\,dy & \text{for every } x \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $G : [0,+\infty[\, \times \overline{\Omega} \times \mathbb{R}^k \to \mathbb{R}^k$ and $H : [0,+\infty[\, \times \overline{\Omega}^2 \times \mathbb{R}^k \to \mathbb{R}^k$ are Lipschitz and uniformly bounded, $G(t,x,0) = 0$ whenever $x \in \partial\Omega$ and $B(t)$ is a symmetric $k \times k$-matrix for each $t \in \mathbb{R}$. If

- $B(\,\cdot\,)$ is continuous and p-periodic;
- $\big(B(t)y\big|y\big) < \lambda_0\,\|y\|^2$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^k$;

where λ_0 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega,\mathbb{R}^k)$, then the given BVP has at least one mild solution.

PROOF. – Let $X := L^2(\Omega,\mathbb{R}^k)$, $C^0 := C^0([0,p],X)$ and $A := \Delta$ with $\mathrm{dom}(A) := \{u \in H_0^1(\Omega,\mathbb{R}^k) :\ \Delta(u) \in X\}$. Define L, $N : C^0 \to X$ and f, $g : [0,p] \times X \to X$ by

$$L(u) := u(0) - u(p), \qquad N(u)(x) := \int_0^p\int_\Omega H\big(t,x,y,u(t)(y)\big)\,dt\,dy,$$

$$f(t,u) := B(t)u, \qquad g(t,u) := G\big(t,\cdot,u(\,\cdot\,)\big).$$

Let $\Phi(t,x)$ be the value at t of the unique mild solution to the linear Cauchy problem

$$\begin{cases} z' = Az + f(t,z) = Az + B(t)z \\ z(0) = x \end{cases}.$$

The linear periodic BVP

$$\begin{cases} z' = Az + f(t,z) = Az + B(t)z \\ z(0) = z(p) \end{cases}$$

has a solution if and only if there exists $x \in X$ such that $L\big(\Phi(\cdot,x)\big) = 0$. As the maximum eigenvalue of $B(t)$ depends continuously on t by the continuity of $B(\,\cdot\,)$ and as $B(\,\cdot\,)$ is p-periodic, there is $0 < \lambda < \lambda_0$ such that

$$\big(B(t)y\big|y\big) \leqslant \lambda\,\|y\|^2 \qquad (0 \leqslant t \leqslant p,\ y \in \mathbb{R}^k).$$

Therefore the Corollary to Theorem 5 in VIDOSSICH [12] guarantees that the above linear periodic BVP has only the trivial solution. In view of

$$L\big(\Phi(\cdot,x)\big) = x - T(p)x - \int_0^p T(p-s)\,B(s)\,\Phi(s,x)\,ds$$

and the fact that

$$x \rightsquigarrow T(p)x + \int_0^p T(p-s)\,B(s)\,\Phi(s,x)\,ds$$

is a compact linear operator by virtue of Pazy's lemma, $x \rightsquigarrow L\big(\Phi(\cdot,x)\big)$ is a linear homeomorphism $X \to X$ by the Fredholm alternative. Since the set of linear homeomorphisms $X \to X$ is open in $\mathcal{L}(X,X)$ and since the map $\mathcal{L}(C^0,X) \to \mathcal{L}(X,X)$ assigning to S the linear operator $x \rightsquigarrow S\big(\Phi(\cdot,x)\big)$ is continuous, there exists a neighborhood $\mathcal{U}$ of L in $\mathcal{L}(C^0,X)$ such that

$$\begin{cases} z' = Az + f(t,z) = Az + B(t)z \\ S(z) = y \end{cases}$$

has a unique mild solution for every $S \in \mathcal{U}$ and $y \in X$. Moreover, $N : C^0 \to C^0(\overline{\Omega}, \mathbb{R}^k) \hookrightarrow X$ is a continuous operator with relatively compact range by the Ascoli theorem in view of the boundedness and uniform continuity of H. Thus we are now in a position to apply Corollary 2 to get the desired conclusion.□

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