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Coactions of Hopf Algebras on Algebras in Positive Characteristic

MARILENA CRUPI - GAETANA RESTUCCIA

Abstract. – *Let K be a field of positive characteristic $p > 0$. We study the coactions of the Hopf algebra of the multiplicative group H_m with underlying algebra $H = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}})$, $n \geq 1$, $s_1 \geq \dots \geq s_n \geq 1$ on a K-algebra A.*

We give the rule for the set of additive endomorphism of A, that define a coaction of H_m on A commutative.

For $s_1 = \dots = s_n = 1$, we obtain the explicit expression of such coactions in terms of n derivations of A.

1. – Introduction.

In recent papers ([7], [8]) finite-dimensional commutative Hopf algebras H with underlying algebra

$$H = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ n \geq 1, s_1 \geq \dots \geq s_n \geq 1, \tag{1.1}$$

where $K[X_1, \dots, X_n]$ denotes the polynomial ring in n indeterminates $X_1, \dots, X_n$ with coefficients in a field K of characteristic $p > 0$, have been studied.

Such assumption on H comes from groups schemes theory, since by the structure theorem of infinitesimal, finite and connected group schemes ([12], 14.4), any finite-dimensional, commutative and local Hopf algebra over a perfect field has this form.

Thanks to this remark, we have concrete examples of Hopf algebras H. After, we consider a coaction of such Hopf algebras H on a K-algebra that can be commutative or not, finitely generated or not.

Thus, let $\delta : A \to A \otimes H$ be a coaction of H over a K-algebra A. For every $a \in A$, we can write

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{a \in \mathbb{A},\, |a| \geq 2} D_a(a) \otimes x^a \tag{1.2}$$

where the x_i are the residue classes of the X_i in H, $\mathbb{A}$ is the set of all multi-indices $a = (a_1, \dots, a_n)$, $0 \leq a_i < p^{s_i}$, $1 \leq i \leq n$, and where $D_i : A \to A$, $1 \leq i \leq n$, are derivations of A defined by

$$D_i = D_{(0,\dots,0,1,0,\dots,0)},$$

with 1 in the i-th position.

In this note we focus our attention on the Hopf algebra with underlying algebra (1.1), with comultiplication given by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i,\ \ x_i = X_i + (X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),$$

for $1 \le i \le n$, counity defined by $\varepsilon(x_i) = 0$ and antipode S given by $S(x_i) = \sum_{j=1}^{\infty} (-1)^j x_i^j$.

Such algebra will be denoted by H_m and we call it the Hopf algebra of the multiplicative group.

Our aim is to study the coactions of H_m on a K-algebra A, that is the behaviour of the additive morphisms $D_a : A \to A$ that appear in the expression of the coactions (1.2), for $a \in \mathbb{A}$.

More precisely, the plane of the paper is the following.

In Section 2 we recall some definitions and results about Hopf algebras.

In Section 3 we introduce the Hopf algebra of the multiplicative algebra group H_m. Then we study the coactions of H_m on a K-algebra A, giving the explicit expression of the additive morphisms $D_a : A \to A$ which appear in the coaction δ, for $a \in \mathbb{A}$ such that $|a| \ge 1$ (Proposition 3.2).

Such coactions can be expressed only in terms of derivations $D_1, \dots, D_n$ of A such that $[D_i, D_j] = 0$, $D_i^p = D_i$, $1 \le i,j \le n$, $s_1 = s_2 = \dots = s_n = 1$. (Theorem 3.3).

Section 4 contains some applications to the case A commutative local K-algebra. If δ is a coaction of H_m on A and $A^{coH_m} = \{a \in A \,:\, \delta(a) = a \otimes 1\}$ is the subalgebra of H_m-coinvariant elements, we study the extension $A^{coH_m} \subset A$. We examine some theoretical properties of such extension such as the flatness and the property to be an H_m-Galois extension (Proposition 4.2).

Moreover, if (A, m) contains the field K of characteristic $p > 0$, H a finite-dimensional commutative Hopf algebra over the field K satisfying (1.1) and δ a coaction of H on A, we prove the existence of a p-basis of A^{coH} over K (Proposition 4.3).

2. – Preliminaries on Hopf algebras.

Let K be a field of characteristic $p > 0$. Let H be a Hopf algebra over the field K with comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$, multiplication $\mu : H \otimes H \to H$ and antipode $S : H \to H$.

Let A be an algebra and let

$$\delta : A \to A \otimes H$$

be an algebra map and a right H-comodule structure on A. (A, δ) is called a right H-comodule algebra, and

$$A^{coH} = \{a \in A \,:\, \delta(a) = a \otimes 1\},$$

is called the subalgebra of H-coinvariant elements.

Let K be a field of characteristic $p > 0$. Let $s_1, \ldots, s_n$ be positive integers and let $\mathbb{A}$ be the set of all multi-indices

$$a = (a_1, \ldots, a_n), \qquad 0 \leq a_i < p^{s_i}, \qquad 1 \leq i \leq n.$$

For $a = (a_1, \ldots, a_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we define

$$a + \beta = (a_1 + \beta_1, \ldots, a_n + \beta_n)$$

and

$$\binom{a}{\beta} = \binom{a_1}{\beta_1} \cdots \binom{a_n}{\beta_n}.$$

Consider the Hopf algebras which "live" on the coalgebras

$$C_\sigma = \left(\sum_{a \in \mathbb{A}} K e_a, \varDelta, \varepsilon \right), \ \ \sigma = (s_1, \ldots, s_n) \in \mathbb{A},$$

where

$$\varDelta(e_a) = \sum_{\beta+\gamma=a,\ \beta,\gamma \in \mathbb{A}} e_\beta \otimes e_\gamma, \quad \varepsilon(e_a) = \delta_{a,0},$$

with $\delta_{a,\beta} = 1$ if $a = \beta$ and $\delta_{a,\beta} = 0$ if $a \neq \beta$.

EXAMPLE 2.1. – Consider

$$H_\sigma = (C_\sigma, \mu : C_\sigma \otimes C_\sigma \to C_\sigma,\ S : C_\sigma \to C_\sigma,\ \eta : k \to C_\sigma),$$

$\sigma = (s_1, \ldots, s_n) \in \mathbb{N}^n$, with

$$\mu(e_a \otimes e_\beta) = \begin{cases} \binom{a+\beta}{a} e_{a+\beta}, & \text{if } a_i + \beta_i < p^{s_i} \\ 0, & \text{otherwise;} \end{cases}$$

antipode S determined by the equalities:

$$\sum_{a+\beta=\gamma} e_a S(e_\beta) = \delta_{\gamma,0}$$

and

$$\eta(t) = te_0,\ 0 \in \mathbb{N}^n, \qquad \text{for every } t \in \mathbb{N}.$$

If an Hopf algebra "lives" on the coalgebra C_σ, then the dual Hopf algebra H^* "lives" on the algebra $H_\sigma = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}})$ (i.e. H^* as an algebra is equal to H_σ), because $C_\sigma^* = H_\sigma$, where $K[X_1, \dots, X_n]$ is the polynomial ring in n indeterminates $X_1, \dots, X_n$.

Moreover there is a one to one correspondance between the actions D of H^* on A that preserves the ring invariants $A^H = A^D = \{a \in A : D(a) = a \otimes 1\}$. Then it turns out that is better to deal with coactions of H^* than with actions of H.

The importance of such algebras is underlined by a result due to Oort and Mumford ([6], Cor. 5.2) that assures that all cocommutative Hopf algebras can be described by the above algebras H_σ, by using the formal groups.

Now assume that H is a commutative Hopf algebra with underlying algebra

$$H = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ n \geq 1, s_1 \geq \dots \geq s_n \geq 1. \tag{2.1}$$

Let $\mathbb{A}$ be the set of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, such that $0 \leq \alpha_i < p^{s_i}$ and $1 \leq i \leq n$.

For all i let x_i be the residue class of X_i in H. Then the elements

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ \alpha \in \mathbb{A},$$

are a K-basis of H.

Note that $\varepsilon(x_i) = 0$ for all i, where $\varepsilon : H \to K$ is the counit of H, since H is local with maximal ideal $(x_1, \dots, x_n)$.

Let A be an algebra, and $\delta : A \to A \otimes H$, a right H-comodule algebra structure on A.

We will always write

$$\delta(a) = \sum_{\alpha \in \mathbb{A}} D_\alpha(a) \otimes x^\alpha, \quad \forall a \in \mathbb{A}.$$

Then for all $\alpha \in \mathbb{A}$ and $a, b \in A$,

$$D_\alpha(ab) = \sum_{\beta + \gamma = \alpha,\ \beta, \gamma \in \mathbb{A}} D_\beta(a) D_\gamma(b),$$

and $D_{(0,\dots,0)} = id$.

For all i, let $\delta_i = (\delta_{ij})_{1 \leq j \leq n} \in \mathbb{A}$, where $\delta_{ij} = 1$, if $j = 1$, and $\delta_{ij} = 0$ otherwise.

We define

$$D_i = D_{\delta_i} = D_{(0,\dots,0,1,0,\dots,0)},$$

with 1 in the i-th position for $i = 1, \dots, n$.

Note that the linear maps $D_i : A \to A$ are derivations of the algebra A, and for all $a \in A$, we have

$$\delta(a) = a \otimes 1 + \sum_{1 \le i \le n} D_i(a) \otimes x_i + \sum_{a \in \mathbb{A},\, |a| \ge 2} D_a(a) \otimes x^a, \tag{2.2}$$

where $|a| = |(a_1, \dots, a_n)| = a_1 + \cdots + a_n$.

In particular

$$[D_i, D_j] \in \sum_{1 \le t \le n} KD_t, \qquad D_i^p \in \sum_{1 \le t \le n} KD_t, \qquad 1 \le i,j \le n$$

and in the case when $s_1 = s_2 = \cdots = s_n = 1$ ([7])

$$A^{coH} = A^{\{D_1, \dots, D_n\}} = \{a \in A \,:\, D_i(a) = 0,\ 1 \le i \le n\}.$$

EXAMPLE 2.2. – The commutative Hopf algebra H_a with underlying algebra

$$H_a = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ n \ge 1, s_1 \ge \cdots \ge s_n \ge 1$$

and comultiplication given by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \qquad x_i = X_i + (X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}), \qquad 1 \le i \le n,$$

is called the Hopf algebra of the additive group, where n and $s_1, \dots, s_n$ are fixed.

If $s_1 = \cdots = s_n = 1$ ([7]), $D_a = \frac{D_1^{a_1}}{a_1!} \cdots \frac{D_n^{a_n}}{a_n!}$, $a = (a_1, \dots, a_n)$, $0 \le a_i < p$, $1 \le i \le n$, where $D_1, \dots, D_n \in Der(A, A)$ with $[D_i, D_j] = 0, D_i^p = 0, 1 \le i,j \le n$.

3. – The Hopf algebra of the multiplicative group.

Let

$$H_m = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ n \ge 1,\ s_1 \ge \cdots \ge s_n \ge 1,$$

be the K-algebra with comultiplication given by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i,\ \ x_i = X_i + (X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ \ 1 \le i \le n,$$

counity defined by $\varepsilon(x_i) = 0$ and antipode S given by $S(x_i) = \sum_{j=1}^{\infty} (-1)^j x_i^j$.

H_m will be called *the Hopf algebra of the multiplicative group*.

LEMMA 3.1. – *Let*

$$H_m = K[X_1, \dots, X_n]/(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}),\ \ n \ge 1$$

be the Hopf algebra of the multiplicative group with $s_1 \geq s_2 \geq \cdots \geq s_n = 1$.
Then

$$\Delta(x^\gamma) = \sum_{a_1=0}^{\gamma_1} \cdots \sum_{a_n=0}^{\gamma_n} \sum_{\beta_1=\gamma_1-a_1}^{\gamma_1} \cdots \sum_{\beta_n=\gamma_n-a_n}^{\gamma_n} \binom{\gamma}{a} \binom{a}{a+\beta-\gamma} x^a \otimes x^\beta,$$

for every $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{A}$ *such that* $|\gamma| \geq 1$.

PROOF. – Since the comultiplication Δ is a K-algebra morphism, if $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{A}$, then

$$\Delta(x^\gamma) = \Delta(x_1^{\gamma_1} \cdots x_n^{\gamma_n}) = \Delta(x_1)^{\gamma_1} \cdots \Delta(x_n)^{\gamma_n}. \tag{3.1}$$

For every $i = 1, \ldots, n$, we have:

$$\begin{aligned}\Delta(x_i)^{\gamma_i} &= (1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i)^{\gamma_i} = [(1 \otimes x_i + x_i \otimes 1) + x_i \otimes x_i]^{\gamma_i} \\ &= \sum_{k=0}^{\gamma_i} \binom{\gamma_i}{k} (1 \otimes x_i + x_i \otimes 1)^k (x_i \otimes x_i)^{\gamma_i - k} \\ &= \sum_{k=0}^{\gamma_i} \sum_{h=0}^{k} \binom{\gamma_i}{k} \binom{k}{h} (1 \otimes x_i)^h (x_i \otimes 1)^{k-h} (x_i \otimes x_i)^{\gamma_i - k} \\ &= \sum_{k=0}^{\gamma_i} \sum_{h=0}^{k} \binom{\gamma_i}{k} \binom{k}{h} x_i^{\gamma_i - k} \otimes x_i^{\gamma_i + h - k}.\end{aligned}$$

Set $\gamma_i - h = a_i$ and $\gamma_i + h - k = \beta_i$, then

$$\Delta(x_i)^{\gamma_i} = \sum_{a_i=0}^{\gamma_i} \sum_{\beta_i=\gamma_i-a_i}^{\gamma_i} \binom{\gamma_i}{a_i} \binom{a_i}{a_i+\beta_i-\gamma_i} x_i^{a_i} \otimes x_i^{\beta_i}.$$

Hence, from (3.1),

$$\Delta(x^\gamma) = \sum_{a_1=0}^{\gamma_1} \cdots \sum_{a_n=0}^{\gamma_n} \sum_{\beta_1=\gamma_1-a_1}^{\gamma_1} \cdots \sum_{\beta_n=\gamma_n-a_n}^{\gamma_n} \binom{\gamma}{a} \binom{a}{a+\beta-\gamma} x^a \otimes x^\beta,$$

for every $\gamma \in \mathbb{A}$ such that $|\gamma| \geq 1$. □

PROPOSITION 3.2. – *Let*

$$H_m = K[X_1, \ldots, X_n]/(X_1^{p^{s_1}}, \ldots, X_n^{p^{s_n}}), \quad n \geq 1$$

be the Hopf algebra of the multiplicative group with $s_1 \geq s_2 \geq \cdots \geq s_n = 1$ *and let* A *be a commutative* K*-algebra. Let* $\delta : A \to A \otimes H_m$ *be a coaction of* H_m *on* A *satisfying* (2.2).

Then

$$D_\alpha D_\beta = \sum_{\gamma_1=\max\{\alpha_1,\beta_1\}}^{\alpha_1+\beta_1} \cdots \sum_{\gamma_n=\max\{\alpha_n,\beta_n\}}^{\alpha_n+\beta_n} \binom{\gamma}{\alpha}\binom{\alpha}{\alpha+\beta-\gamma} D_\gamma.$$

PROOF. – From the definition of a coaction

$$(1\otimes \Delta)\delta = (\delta\otimes 1)\delta.$$

Since, for every $a\in A$,

$$\delta(a) = \sum_{\alpha\in\mathbb{A}} D_\alpha(a)\otimes x^\alpha,$$

it follows that

$$\begin{aligned}(\delta\otimes 1)\delta(a) &= \sum_\alpha \delta(D_\alpha(a))\otimes x^\alpha \\ &= \sum_\alpha \left[\sum_\beta D_\beta D_\alpha(a)\otimes x^\beta\right]\otimes x^\alpha \qquad (3.2)\\ &= \sum_\alpha\sum_\beta D_\beta D_\alpha(a)\otimes x^\beta\otimes x^\alpha.\end{aligned}$$

Moreover, for every $a\in A$, from Lemma 3.1,

$$\begin{aligned}(1\otimes\Delta)\delta(a) &= \sum_\alpha D_\alpha(a)\otimes \Delta(x^\alpha)\\ &= \sum_\alpha D_\alpha(a)\otimes\left[\sum_{\gamma=0}^{\alpha}\sum_{\tau=\alpha-\gamma}^{\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{\gamma+\tau-\alpha}x^\gamma\otimes x^\tau\right]\\ &= \sum_\alpha\sum_\gamma\left[\sum_{\tau=\alpha-\gamma}^{\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{\gamma+\tau-\alpha}D_\alpha(a)\right]\otimes x^\gamma\otimes x^\tau \qquad (3.3)\\ &= \sum_\tau\sum_\gamma\left[\sum_{\tau=\alpha-\gamma}^{\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{\gamma+\tau-\alpha}D_\alpha(a)\right]\otimes x^\gamma\otimes x^\tau.\end{aligned}$$

Comparing (3.2) and (3.3), we have

$$\begin{aligned}D_\gamma D_\tau(a) &= \sum_{\tau=\alpha-\gamma}^{\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{\gamma+\tau-\alpha}D_\alpha(a)\\ &= \sum_{\tau_1=\alpha_1-\gamma_1}^{\gamma_1}\cdots\sum_{\tau_n=\alpha_n-\gamma_n}^{\gamma_n}\binom{\alpha}{\gamma}\binom{\gamma}{\gamma+\tau-\alpha}D_\alpha(a)\\ &= \sum_{\alpha_1=\max\{\gamma_1,\tau_1\}}^{\tau_1+\gamma_1}\cdots\sum_{\alpha_n=\max\{\gamma_n,\tau_n\}}^{\tau_n+\gamma_n}\binom{\alpha}{\gamma}\binom{\gamma}{\gamma+\tau-\alpha}D_\alpha(a),\end{aligned}$$

i.e.

$$D_a D_\beta = \sum_{\gamma_1=\max\{a_1,\beta_1\}}^{a_1+\beta_1} \cdots \sum_{\gamma_n=\max\{a_n,\beta_n\}}^{a_n+\beta_n} \binom{\gamma}{a}\binom{a}{a+\beta-\gamma} D_\gamma.$$

□

Under certain conditions the coactions of H_m on a K -algebra A can be expressed in terms of derivations $D_1, \ldots, D_n \in Der(A, A)$. Precisely, we have:

THEOREM 3.3. – *Let*

$$H_m = K[X_1, \ldots, X_n]/(X_1^{p^{s_1}}, \ldots, X_n^{p^{s_n}}), \;\; n \geq 1$$

be the Hopf algebra of the multiplicative group with $s_1 = s_2 = \cdots = s_n = 1$ *and let* A *be a commutative* K*-algebra.*

Then the coactions of H_m *on* A *are given by derivations* $D_1, \ldots, D_n \in Der(A, A)$ *such that*

$$D_i D_j = D_j D_i, \qquad D_i^p = D_i, \qquad 1 \leq i, j \leq n,$$

and

$$D_a = \frac{\prod_{j_1=0}^{a_1-1}(D_1 - j_1)}{a_1!}\frac{\prod_{j_2=0}^{a_2-1}(D_2 - j_2)}{a_2!}\cdots\frac{\prod_{j_n=0}^{a_n-1}(D_n - j_n)}{a_n!} = \frac{\prod_{t=1}^{n}\left(\prod_{j_t=0}^{a_t-1}(D_t - j_t)\right)}{a!},$$

$a = (a_1, \ldots, a_n)$, $0 \leq a_i < p$, $1 \leq i \leq n$ *and* $a! = a_1!\, a_2! \cdots a_n!$.

PROOF. – From Proposition 3.2, we have that

$$D_a D_\beta = \sum_{\gamma_1=\max\{a_1,\beta_1\}}^{a_1+\beta_1} \cdots \sum_{\gamma_n=\max\{a_n,\beta_n\}}^{a_n+\beta_n} \binom{\gamma}{a}\binom{a}{a+\beta-\gamma} D_\gamma, \tag{3.4}$$

where $\binom{\gamma}{a} = \binom{\gamma_1}{a_1}\binom{\gamma_2}{a_2}\cdots\binom{\gamma_n}{a_n}$, for $a = (a_1, \ldots, a_n), \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{A}$.

We claim that

$$(D_i - a_i)D_a = (a_i + 1)D_{a+\delta_i}, \tag{3.5}$$

where $D_i = D_{\delta_i} = D_{(0,\ldots,1,\ldots,0)}$ with 1 in the i-th position, for $i = 1, \ldots, n$.

In fact

$$\begin{aligned} D_i D_a &= \sum_{\gamma_1=a_1}^{a_1} \cdots \sum_{\gamma_{i-1}=a_{i-1}}^{a_{i-1}} \sum_{\gamma_i=a_i}^{a_i+1} \sum_{\gamma_{i+1}=a_{i+1}}^{a_{i+1}} \cdots \sum_{\gamma_n=a_n}^{a_n} \binom{\gamma}{\delta_i}\binom{\delta_i}{\delta_i+a-\gamma} D_\gamma \\ &= a_i D_a + (a_i+1)D_{a+\delta_i} \end{aligned}$$

and equality (3.5) follows.

Now we proceed by induction on $|a|$.

Let $|a| = 1$.

We have to prove that

$$i!D_i = \prod_{j_i=0}^{i-1} (D_i - j_i). \tag{3.6}$$

In this case $i! = 1$, $i = 1$ and so (3.6) is true.

Now let $|a| > 1$ and suppose

$$a_1!a_2! \cdots a_n!D_a = \prod_{j_1=0}^{a_1-1} (D_1 - j_1) \cdots \prod_{j_n=0}^{a_n-1} (D_n - j_n) = \prod_{t=1}^{n} \left(\prod_{j_t=0}^{a_t-1} (D_t - j_t) \right).$$

We want to prove that

$$\begin{aligned} &a_1!a_2! \cdots a_{i-1}!(a_i+1)!a_{i+1}! \cdots a_n!D_{a+\delta_i} \\ &= \prod_{j_1=0}^{a_1-1} (D_1 - j_1) \cdots \prod_{j_{i-1}=0}^{a_{i-1}-1} (D_{j-1} - j_{i-1}) \prod_{j_i=0}^{a_i} (D_i - j_i) \prod_{j_{i+1}=0}^{a_{i+1}-1} (D_{j+1} - j_{i+1}) \cdots \prod_{j_n=0}^{a_n-1} (D_n - j_n) \\ &= (D_i - a_i) \prod_{t=1}^{n} \left(\prod_{j_t=0}^{a_t-1} (D_t - j_t) \right). \end{aligned}$$

By the inductive hypotheses, and from (3.5),

$$\begin{aligned} a_1!a_2! \cdots a_{i-1}!(a_i+1)!a_{i+1}! \cdots a_n!D_{a+\delta_i} &= a_1! \cdots a_n!(a_i+1)D_{a+\delta_i} \\ &= a_1! \cdots a_n!(D_i - a_i)D_a \\ &= (D_i - a_i)a_1! \cdots a_n!D_a \\ &= (D_i - a_i) \prod_{t=1}^{n} \left(\prod_{j_t=0}^{a_t-1} (D_t - j_t) \right). \end{aligned}$$

Finally, we obtain

$$a!D_a = \prod_{j_1=0}^{a_1-1} (D_1 - j_1) \cdots \prod_{j_n=0}^{a_n-1} (D_n - j_n),$$

where $a! = a_1!\, a_2! \cdots a_n!$, for $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$.

In characteristic p

$$D_a = \frac{\prod_{j_1=0}^{a_1-1} (D_1 - j_1)}{a_1!} \frac{\prod_{j_2=0}^{a_2-1} (D_2 - j_2)}{a_2!} \cdots \frac{\prod_{j_n=0}^{a_n-1} (D_n - j_n)}{a_n!},$$

$a = (a_1, \ldots, a_n)$, $0 \le a_i < p$, $1 \le i \le n$.

Moreover it is easy to verify that the derivations D_i satisfy the following conditions:

$$[D_i, D_j] = 0, \; D_i^p = D_i, \; 1 \leq i,j \leq n.$$

□

REMARK 3.4. – The relations stated in Theorem 3.3 are known for $n = 1$ and $H_m = H_{s_1}(F_m)$, where $H_{s_1} = k[X]/(X^{p^{s_1}})$ and $F_m = X + Y + XY$ is the multiplicative formal group ([1], [9]), $s_1 \geq 1$.

4. – Some applications.

In this section we study the extension of rings $A^{\mathrm{co}\,H} \subset A$, by using techniques and results of ([2], [7], [10]).

Let (A, δ) be a right H-comodule algebra and let $R = A^{\mathrm{co}\,H}$.

Recall that the extension $R \subset A$ is called an *H-Galois extension* or A is H-Galois if the Galois map

$$\beta : A \otimes_R A \to A \otimes H, \; a \otimes b \mapsto a\delta(b),$$

is bijective.

The following result, due to H.-J. Schneider ([10]), gives a characterization of an H-Galois extension with the property of being faithfully flat. Recall that an H-Galois extension $A \subset B$ is faithfully flat if A is faithfully flat over B as a left (or equivalently right) module over B.

THEOREM 4.1. – *Let H be an Hopf algebra with a bijective antipode and A a right H-comodule algebra with $B = A^{\mathrm{co}\,H}$. Then the following are equivalent:*

1) a) *$A \subset B$ is a right H-Galois extension and*
   b) *A is faithfully flat left (or right B-module);*

2) a) *the Galois map β is surjective and*
   b) *A is an injective H-comodule.*

PROPOSITION 4.2. – *Let H_m be the Hopf algebra of the multiplicative group and A a right H-comodule algebra with structure map $\delta : A \to A \otimes H_m$. Define the derivations $D_1, \ldots, D_n$ by (2.2) and let $R = A^{coH_m}$.*

Assume $s_1 = s_2 = \cdots = s_n = 1$ and A is a commutative local algebra with maximal ideal m_A and $R + m_A = A$.

Then the following are equivalent:

(1) *$R \subset A$ is a faithfully flat H_m-Galois extension.*

(2) *There are $y_1, \ldots, y_n \in A$ with $\delta(y_i) = y_i \otimes 1 + 1 \otimes x_i + y_i \otimes x_i$, for all $1 \leq i \leq n$.*

(3) *There exist* $y_1, \ldots, y_n \in m_A$ *such that*

$$D_i(y_j) = (1 + y_j)\delta_{ij},$$

for $i,j = 1, \ldots, n$.

PROOF. – (1) $\Rightarrow$ (2): By (1), A is an injective H_m-comodule (Theorem 4.1). Hence the right H_m-colinear map $k \to A$, $1 \to 1$ can be extended to an H_m-colinear map

$$\gamma : H_m \to A.$$

Then for all i, since γ is H_m-colinear and $\gamma(1) = 1$

$$\delta(\gamma(x_i)) = (\gamma \otimes id)\Delta(x_i)$$
$$= (\gamma \otimes id)(x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i) = \gamma(x_i) \otimes 1 + 1 \otimes x_i + \gamma(x_i) \otimes x_i.$$

Hence the assert follows by putting

$$y_i := \gamma(x_i),$$

for all $1 \leq i \leq n$.

(2) $\Rightarrow$ (3): Since $A = R + m_A$, for every $j = 1, \ldots, n$, $y_j = z_j + \tilde{y}_j$, $z_j \in R$ and $\tilde{y}_j \in m_A$.

Hence $D_i(y_j) = D_i(\tilde{y}_j)$ and we can assume $y_1, \ldots, y_n \in m_A$.

By hypothesis (2) and from Theorem 3.3, we obtain $D_i(y_j) = (1 + y_j)\delta_{ij}$, $1 \leq i,j \leq n$.

(3) $\Rightarrow$ (1): It is easy to verify that $det(D_i(y_j))_{1 \leq i,j \leq n}$ is invertible and so $R \subset A$ is an H_m-Galois extension ([7], Theorem 4.1).

On the other hand, from ([7], Theorem 4.3), $y^a = y_1^{a_1} \cdots y_n^{a_n}$ is an R-basis of A, hence A is a free R-module and consequently a faithfully flat R-module. □

Now we consider the next application.

Let A be a commutative ring of characteristic $p > 0$ and A^p denotes the subring $\{x^p : x \in A\}$ of A. Let B be a subring of A. A subset Γ of A is said to be p-independent over B, if the monomials $x_1^{e_1} \cdots x_n^{e_n}$, where $x_1, \ldots, x_n$ are distinct elements of Γ and $0 \leq e_i < p$, are linearly independent over $A^p[B]$. Γ is called a p-basis of A over B if it is p-independent over B and $A^p[B, \Gamma] = A$.

Then, if B is a subring of a ring A, a subset $x_1, \ldots, x_n$ of A is a p-basis of A over $A^p[B]$ if and only if A is a free $A^p[B]$-module with basis $x_1^{e_1} \cdots x_n^{e_n}$, $0 \leq e_i < p$.

For more details on p-bases see [3].

We state the following:

PROPOSITION 4.3. – *Let* (A, m) *be a commutative local algebra containing a field* K *of characteristic* $p > 0$ *and of dimension* $d > 0$. *Let* H *be an Hopf algebra* *over K satisfying* (2.1) *with* $s_1 = s_2 = \cdots = s_n = 1$, *and* δ *a coaction of* H *on* A *with derivations* $D_1, \ldots, D_n$ *defined by* (2.2).

Suppose

i) *there exist* $y_1, \ldots, y_n \in \mathrm{m}$ *such that the matrix* $\left(D_i(y_j)\right)_{1 \le i,j \le n}$ *is invertible;*

ii) $A \otimes K^{p^{-1}}$ *is reduced;*

iii) A/m *is a separable extension of* K.

If A *has a finite* p*-basis over* k, *then* A^{coH} *has a finite* p*-basis over* K.

PROOF. – From ([7], Lemma 3.4), there exist $\partial_1, \ldots, \partial_n \in Der(A, A)$ such that

$$[\partial_i, \partial_j] = 0, \qquad \partial_i^p = 0, \qquad \partial_i(y_j) = \delta_{ij}, \qquad 1 \le i,j \le n,$$

and $R = A^{coH} = A^{D_1,\ldots,D_n} = A^{\partial_1,\ldots,\partial_n}$.

Since A has a p-basis over K, A is a regular local ring of dimension d ([11], Lemma 1) and R is a regular local ring of dimension d ([7], Theorem 5.1).

Moreover A is a noetherian ring that is a finite $K[A^p]$-module. Hence the subring $K[A^p] \subset A$ is noetherian too and consequently R is a finite $K[A^p]$-module.

On the other hand A has a finite p-basis over R ([7], Lemma 3.4), A is a finite R-module, $K[A^p]$ is a finite $K[R^p]$-module and R is a finite $K[R^p]$-module.

It follows that, in order to prove the statement, it is sufficient to show that the universal module of differentials $\Omega_K(R)$ is a free R-module ([3], § 38. Proposition).

We obtain the assert by following the arguments in ([2], Theorem 2). □

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