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Groups with Normality Conditions for Non-Periodic Subgroups(*)

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Abstract. – *The structure of (non-periodic) groups in which all non-periodic subgroups have a prescribed property is investigated. Among other choices, we consider properties generalizing normality, like subnormality, permutability and pronormality. Moreover, non-periodic groups whose proper non-periodic subgroups belong to a given group class are studied.*

1. – Introduction.

Let G be a non-periodic group. If G is generated by its elements of infinite order, the behaviour of such elements has a strong influence on the structure of G. For instance, in this situation, if the elements of infinite order pairwise commute, the group G is obviously abelian, and the same conclusion holds if all infinite cyclic subgroups of G are normal. The situation is much more complicated if the elements of infinite order generate a proper subgroup: in the infinite dihedral group D_∞ the elements of infinite order (together with the identity) form an infinite cyclic subgroup of index 2, so that in particular the elements of infinite order of D_∞ pairwise commute and generate cyclic normal subgroups, but the centre of D_∞ is trivial. On the other hand, it is easy to show that if all non-periodic subgroups of an arbitrary group G are normal, then G is either periodic or abelian.

The aim of this paper is the investigation of the structure of (non-periodic) groups in which every non-periodic subgroup has a given subgroup theoretical property, with special interest for properties generalizing normality. In particular, Section 2 deals with groups in which all non-periodic subgroups are either subnormal or permutable or pronormal, while in Section 3 properties close to normality up to a finite section are considered.

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is said to be *minimal non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$; the structure of

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minimal non-$\mathfrak{X}$ groups has been investigated for several relevant choices of the group class $\mathfrak{X}$, also in the case of infinite groups (of course assuming suitable conditions generalizing solubility). In the final Section 4, we study non-periodic groups whose proper non-periodic subgroups belong to a given group class; among other choices, we describe the situation for the classes of abelian, nilpotent, minimax, polycyclic and supersoluble groups.

Most of our notation is standard and can be found in [16].

2. – Subnormality, permutability, pronormality.

Recall that the *finite residual* of a group G is the intersection of all (normal) subgroups of finite index of G, and that G is called *residually finite* if its finite residual is trivial. It is well known that all polycyclic-by-finite groups are residually finite.

A classical theorem of J. E. Roseblade (see [16] Part 2, Theorem 7.42) states that a group G is nilpotent if and only if all subgroups of G are subnormal and there exists a bound for their defects. Our first result shows that in the case of non-periodic groups it is enough to check the behaviour of the non-periodic subgroups.

THEOREM 2.1. – *Let G be a non-periodic group in which all non-periodic subgroups are subnormal with defect at most k, for some positive integer k. Then every subgroup of G is subnormal with defect at most k. In particular, G is nilpotent and its class is bounded by a function of k.*

PROOF. – Assume for a contradiction that the statement is false, and choose a counterexample G containing an infinite cyclic subgroup $\langle a\rangle$ with smallest subnormal defect m. As the normal closure $\langle a\rangle^G$ of $\langle a\rangle$ inherits the hypotheses on G, it follows from our minimal choice that $\langle a\rangle^G$ is nilpotent (this fact being obvious if $m = 1$, i.e. if $\langle a\rangle$ is normal in G). Moreover, $G/\langle a\rangle^G$ is a Dedekind group, and so in particular G is soluble. Let X be any finite subgroup of G, and let E be a finitely generated non-periodic subgroup of G such that

$$X \leq E \leq X\langle a\rangle^G.$$

Since $X\langle a\rangle^G$ is nilpotent-by-finite, the subgroup E is residually finite, and hence it contains a normal subgroup J of finite index such that $X \cap J = \{1\}$. Let $(J_i)_{i\in I}$ be a collection of E-invariant subgroups of finite index of J such that

$$\bigcap_{i\in I} J_i = \{1\}.$$

Clearly, each J_i is non-periodic, so that XJ_i is subnormal in G with defect at most

k, and so

$$X = \bigcap_{i \in I} XJ_i$$

is likewise subnormal in G with defect at most k. Therefore all finitely generated subgroups of G are subnormal with defect at most k. It follows that each subgroup of G is subnormal with defect at most k and hence G is nilpotent with bounded class by Roseblade's theorem. This contradiction proves the theorem. □

The structure of groups in which all subgroups are subnormal has been investigated by several authors. Although a famous example by H. Heineken and I. J. Mohamed proves that infinite groups with such property may have trivial centre, a relevant theorem by W. Möhres [11] states that any group in which all subgroups are subnormal is soluble. We leave here as an open question whether there exists a non-periodic group G such that all non-periodic subgroups of G are subnormal, but G contains some periodic non-subnormal subgroups. However, it is possible to prove at least that any non-periodic group whose non-periodic subgroups are subnormal is soluble.

Recall that the *Baer radical* of a group G is the subgroup generated by all cyclic subnormal subgroups of G, and G is a *Baer group* if it coincides with its Baer radical. It is well known that Baer groups are locally nilpotent.

LEMMA 2.2. – *Let G be a non-periodic group in which all non-periodic subgroups are subnormal. Then G is a Baer group.*

PROOF. – Clearly, the Baer radical B of G contains all elements of infinite order of G. Let x be any element of finite order of G, and consider an element of infinite order a of G. Then the finitely generated subgroup $\langle a, x \rangle$ is contained in $\langle B, x \rangle$, and hence it is nilpotent-by-finite. Moreover, since every subgroup of finite index of $\langle a, x \rangle$ is subnormal, we have that all finite homomorphic images of $\langle a, x \rangle$ are nilpotent and so $\langle a, x \rangle$ itself is nilpotent (see [16] Part 2, Theorem 10.51). On the other hand, $\langle a, x \rangle$ is subnormal in G, so that also $\langle x \rangle$ is a subnormal subgroup of G, and G is a Baer group. □

Let G be a group and let X be any subgroup of G. For each non-negative integer i, we shall denote by $X^{G,i}$ the i-th term of the *series of normal closures* of X in G, for each non-negative integer i. In particular, $X^{G,0} = G$ and $X^{G,1} = X^G$, while a subgroup X is subnormal in G if and only if there exists a positive integer k such that $X^{G,k} = X$.

COROLLARY 2.3. – *Let G be a non-periodic group in which all non-periodic subgroups are subnormal. Then G is soluble.*

PROOF. – Let a be an element of infinite order of G, and let k be the defect of the subnormal subgroup $\langle a\rangle$ of G. For each non-negative integer $i \leq k$, all subgroups of the factor group $\langle a\rangle^{G,i}/\langle a\rangle^{G,i+1}$ are subnormal, and hence $\langle a\rangle^{G,i}/\langle a\rangle^{G,i+1}$ is soluble by Möhres' theorem. As $\langle a\rangle^{G,k} = \langle a\rangle$, it follows that G itself is a soluble group. □

A subgroup X of a group G is said to be permutable (or *quasinormal*) in G if $XH = HX$ for each subgroup H of G; of course, all normal subgroups are permutable and it is well known that permutable subgroups of arbitrary groups are ascendant, so that in particular every permutable subgroup of a finite group is subnormal. A group G is called *quasihamiltonian* if all its subgroups are permutable. The structure of quasihamiltonian groups has been completely described by K. Iwasawa (see for instance [21], Chapter 2). Recall also that a subgroup X of a group G is *permodular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G and the index $|\langle g, X\rangle : Y|$ is finite, whenever g is an element of G and Y is a subgroup of $\langle g, X\rangle$ containing X for which the interval $[\langle g, X\rangle/Y]$ is a finite lattice. Thus permutable subgroups of arbitrary groups are permodular, while in any finite group modular and permodular subgroups coincide. It is also easy to prove that a periodic group whose cyclic subgroups are permodular is locally finite (see for instance [21], Lemma 6.2.8).

THEOREM 2.4. – *Let G be a non-periodic group in which all non-periodic subgroups are permodular. Then G is a quasihamiltonian group.*

PROOF. – Clearly, the class of quasihamiltonian group is local, and so without loss of generality it can be assumed that the group G is finitely generated. Let W be a finite set of generators of G, and let a be any element of infinite order of W. Then the subgroup $\langle a\rangle$ is permodular in G and hence the index $|\langle a\rangle^G : \langle a\rangle|$ is finite (see [21], Lemma 6.2.8). It follows that there exists an infinite polycyclic-by-finite normal subgroup N of G such that the factor group G/N is generated by finitely many elements of finite order; moreover, all subgroups of G/N are permodular since N is not periodic. Therefore G/N is periodic (see [21], Lemma 2.4.8) and so even finite. Thus G is polycyclic-by-finite and all its finite homomorphic images have modular subgroup lattice, so that also the subgroup lattice of G is modular (see [12]) and G is a quasihamiltonian group (see [21], Theorem 2.4.11). □

A subgroup X of a group G is said to be *pronormal* if the subgroups X and X^g are conjugate in $\langle X, X^g\rangle$ for every element g of G. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups; moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. The concept of a pronormal subgroup was introduced by P. Hall, and later studied by several authors. It is easy to show that a subgroup of a group G is normal if and only if it is ascendant and pronormal in G. In particular, if G is a group in which all subgroups are pronormal, then every subgroup of G is a T-group, i.e. in any subgroup of G normality is a transitive relation (for the structure of soluble T-groups we refer to [15]). Periodic locally soluble groups in which all subgroups are pronormal have been completely described by N. F. Kuzennyi and I. Y. Subbotin [10]. Moreover, it is known that a non-periodic group whose subgroups are pronormal is abelian, provided that it has no infinite simple sections (see [6], Lemma 3.3). In order to avoid torsion-free Tarski groups and other similar pathologies, we work here within the universe of $\mathcal{W}$-groups. Recall that a group G is a *$\mathcal{W}$-group* if every finitely generated non-nilpotent subgroup of G has a finite non-nilpotent homomorphic image; the class of $\mathcal{W}$-groups is quite large, and it contains in particular all locally (soluble-by-finite) groups (see for instance [16] Part 2, Theorem 10.51).

In our proof we will need the following result on soluble groups whose homomorphic images have a transitive normality relation, that was proved by D.J.S. Robinson [17].

LEMMA 2.5. – *Let G be a finitely generated soluble group whose finite homomorphic images are T-groups. Then G is a T-group.*

THEOREM 2.6. – *Let G be a non-periodic $\mathcal{W}$-group in which all non-periodic subgroups are pronormal. Then G is abelian.*

PROOF. – Assume for a contradiction that the statement is false, so that G contains a finitely generated non-periodic subgroup E which is not abelian. Clearly, every subgroup of finite index of E is pronormal, so that in particular any finite homomorphic image of E is a T-group with all its subgroups and hence it is metabelian. Moreover, every soluble homomorphic image of E is a T-group by Lemma 2.5, and so it is either finite or abelian (see [15]). On the other hand, E has a finite non-nilpotent homomorphic image since G is a $\mathcal{W}$-group, so that the factor group E/E'' is not abelian and hence it must be finite. It follows that E'' is a finitely generated unsoluble group, and so it has a finite non-nilpotent homomorphic image E''/K. Then the index $|E : K|$ is finite, and hence also the core N of K in E has finite index in E. But E/N is not metabelian, and this contradiction completes the proof. □

In the last part of this section we deal with a criterion for a non-periodic group to be metahamiltonian. Recall that a group G is *metahamiltonian* if all its non-abelian subgroups are normal; metahamiltonian groups were introduced and investigated by G.M. Romalis and N.F. Sesekin ([18],[19],[20]), who proved in particular that any metahamiltonian group has finite commutator subgroup, provided that it is locally graded; here a group is *locally graded* if all its finitely generated non-trivial subgroups have a finite non-trivial homomorphic image (in particular, all $\mathcal{W}$-groups are locally graded). Further informations on the structure of metahamiltonian groups have later been obtained by other authors (see for instance [3] and [9]).

THEOREM 2.7. – *Let G be a non-periodic group whose finitely generated subgroups are hyper-(abelian or finite). If every non-periodic subgroup of G is either abelian or normal, then G is metahamiltonian.*

PROOF. – Let E be any finitely generated non-periodic subgroup of G. Then every subgroup of finite index of E is either abelian or normal, and hence in particular all finite homomorphic images of E are metahamiltonian. It follows that E itself is metahamiltonian (see [3], Theorem A). Therefore every finitely generated subgroup of G is metahamiltonian, and hence G is a metahamiltonian group (see [3], Lemma 3.1). □

3. – Finiteness normality conditions

If G is any group, the subgroup consisting of all elements of G having only finitely many conjugates is called the *FC-centre* of G, and G is an *FC-group* if it coincides with its FC-centre. Recall that a subgroup X of a group G is said to be *almost normal* if it has only finitely many conjugates in G, or equivalently if its normalizer $N_G(X)$ has finite index in G. Thus an element x of a group G has only finitely many conjugates if and only if the cyclic subgroup $\langle x \rangle$ is almost normal, and hence a group has the property FC if and only if all its cyclic subgroups are almost normal. A famous theorem of B. H. Neumann [13] states that every subgroup of a group G is almost normal if and only if the centre $Z(G)$ has finite index in G. In the first part of this section we consider groups in which almost normality is imposed only to non-periodic subgroups. Observe here that groups in which all infinite subgroups are almost normal have been described in [8].

LEMMA 3.1. – *Let G be a non-periodic group in which every element of infinite order has only finitely many conjugates. Then the FC-centre of G has finite index.*

PROOF. – Let a be any element of infinite order of G. If x is an element of finite order of the centralizer $C_G(a)$, the product ax has likewise infinite order and hence its centralizer $C_G(ax)$ has finite index in G. Clearly, the intersection $C_G(a) \cap C_G(ax)$ is contained in $C_G(x)$, so that the index $|G : C_G(x)|$ is finite and x has only finitely many conjugates in G. Therefore $C_G(a)$ is contained in the FC-centre F of G, and hence F has finite index in G. □

THEOREM 3.2. – *Let G be a non-periodic group. Then all non-periodic subgroups of G are almost normal if and only if either $G/Z(G)$ is finite or G satisfies the following conditions:*

(i) *G is abelian-by-finite;*
(ii) *the commutator subgroup G' of G is finitely generated;*
(iii) *there exists a periodic normal subgroup T of G such that*

$$G/T = \langle gT \rangle \ltimes A/T,$$

where the coset gT has prime order p and A/T is a free abelian normal subgroup of rank $p-1$ of G/T on which g acts rationally irreducibly.

PROOF. – Suppose that all non-periodic subgroups of G are almost normal, but G is not central-by-finite. Let a be an element of infinite order of G, and let X be any periodic subgroup of the centralizer $C_G(a)$. Then $\langle a, X \rangle = \langle a \rangle \times X$, so that X is a characteristic subgroup of $\langle a, X \rangle$. On the other hand, $\langle a, X \rangle$ is almost normal in G, and so also X is an almost normal subgroup of G. In particular, all subgroups of $C_G(a)$ are almost normal and hence the index $|C_G(a) : Z(C_G(a))|$ is finite by Neumann's theorem. As $C_G(a)$ has finite index in G, it follows that G is abelian-by-finite. Moreover, since $\langle a \rangle$ has only finitely many conjugates in G, the normal subgroup $\langle a \rangle^G$ is finitely generated and hence it satisfies the maximal condition on subgroups. On the other hand, every subgroup of the factor group $G/\langle a \rangle^G$ is almost normal, and so $G/\langle a \rangle^G$ is central-by-finite by Neumann's theorem. In particular, $G'/G' \cap \langle a \rangle^G$ is finite and then G' is finitely generated.

Let T be the largest periodic normal subgroup of G, and put $\bar{G} = G/T$. Clearly, the FC-centre $\bar{F}$ of $\bar{G}$ is torsion-free abelian and the factor group $\bar{G}/\bar{F}$ is finite. Thus every infinite subgroup of $\bar{G}$ is non-periodic and so it is almost normal. Assume that $\bar{G}/Z(\bar{G})$ is finite, so that $\bar{G}$ is abelian by Schur's theorem and G' is periodic; then G' is finite, a contradiction since G is abelian-by-finite and the index $|G : Z(G)|$ is infinite. Therefore $\bar{G}$ is not central-by-finite, and hence

$$\bar{G} = \langle gT \rangle \ltimes A/T,$$

where gT has prime order p and A/T is a free abelian normal subgroup of rank $p-1$ on which g acts rationally irreducibly (see [8]).

Conversely, suppose that G satisfies the conditions of the statement. Then the commutator subgroup of A is periodic, and so also finite as G is abelian-by-finite and G' is finitely generated. It follows that also $A/Z(A)$ is finite, and hence all subgroups of A are almost normal in G. Let X be any non-periodic subgroup of G which is not contained in A. Then $XA = G$ and $XT \cap A$ is a non-periodic normal subgroup of G, so that the index $|A : XT \cap A|$ is finite. In particular, the subgroup XT has finite index in G. Let Y be a finitely generated subgroup of X such that $XT = YT$. The subgroup $[Y, T]$ is contained in $G' \cap T$, so that it is finite and hence

Y has finite index in the normal closure Y^T. Thus there exists a positive integer n such that $(Y^T)^n$ is contained in Y; moreover, Y^T is finitely generated, so that $Y^T/(Y^T)^n$ is finite and hence Y is almost normal in XT and so also in G. Therefore the subgroup $X = Y(T \cap X)$ is almost normal in G, and the proof of the theorem is complete. □

Let $A = \langle a, b\rangle$ be a free abelian group of rank 2, and let g be the automorphism of A defined by $a^g = b$ and $b^g = a^{-1}b^{-1}$. Then g has order 3 and acts rationally irreducibly on A, so that the semidirect product $G = \langle g\rangle \ltimes A$ satisfies conditions (i),(ii) and (iii) of Theorem 3.2 and hence all its non-periodic subgroups are almost normal.

Recall that a subgroup X of a group G is *nearly normal* if X has finite index in its normal closure X^G; thus a group G is an FC-group if and only if every cyclic subgroup of G is nearly normal. It was proved by B.H. Neumann [13] that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite. Thus Schur's theorem yields that if all subgroups of a group are almost normal, then they are also nearly normal. The structure of groups in which all non-periodic subgroups are nearly normal has been described in [2]. It follows from such description and from Theorem 3.2 that there exists a non-periodic group G such that all non-periodic subgroups of G are almost normal, but G contains some non-periodic subgroup which is not nearly normal. On the other hand, it turns out that if G is a non-periodic abelian-by-finite group with finitely generated commutator subgroup and every non-periodic subgroup of G is nearly normal, then all non-periodic subgroups of G are also almost normal. Moreover, if G is a non-periodic group in which all non-periodic subgroups are almost normal, then all non-periodic subgroups of G are nearly normal if and only if either $G/Z(G)$ is finite or G satisfies the conditions of the statement of Theorem 3.2 with $p = 2$.

A subgroup X of a group G is said to be *normal-by-finite* if the core X_G of X in G has finite index in X, and a group G is a *CF-group* if all its subgroups are normal-by-finite. It has been proved by H. Smith and J. Wiegold [23] that any locally soluble CF-group is abelian-by-finite. Thus the last result of this section shows in particular that any non-periodic locally soluble group in which all non-periodic subgroup are normal-by-finite is abelian-by-finite.

THEOREM 3.3. – *Let G be a non-periodic group in which all non-periodic subgroups are normal-by-finite. Then all subgroups of G are normal-by-finite. Moreover, if $|X/X_G| \leq k$ for some positive integer k and for every non-periodic subgroup X of G, then $|H/H_G| \leq 2k$ for each subgroup H of G.*

PROOF. – Let a be an element of infinite order of G. As $\langle a\rangle$ is normal-by-finite, its core $\langle b\rangle$ is an infinite cyclic normal subgroup of G. Consider any periodic subgroup H of G, and let N be the core of $\langle b, H\rangle$ in G. Clearly, the subgroup $\langle b, H\rangle$ is normal-by-finite, and so the index $|\langle b, H\rangle : N|$ is finite. Moreover,

$$C_N(b) = \langle b\rangle \times C_{H\cap N}(b)$$

and so the centralizer $C_{H\cap N}(b)$ is a normal subgroup of G. On the other hand, the factor group $G/C_G(b)$ has order at most 2, so that

$$|H \cap N : C_{H\cap N}(b)| \leq 2$$

and hence the index of $C_{H\cap N}(b)$ in H is finite. Therefore the subgroup H is normal-by-finite. Finally, note that if $|\langle b, H\rangle : N| \leq k$ for some k, then $|H : C_{H\cap N}(b)| \leq 2k$ and so in particular $|H : H_G| \leq 2k$. □

4. – Some further conditions

The first lemma of this section shows that in many cases non-periodic groups in which every proper non-periodic subgroup belongs to a given group class $\mathfrak{X}$ actually are minimal non-$\mathfrak{X}$ groups.

LEMMA 4.1. – *Let $\mathfrak{X}$ be a subgroup closed group class consisting of locally (polycyclic-by-finite) groups and such that in any $\mathfrak{X}$-group the elements of finite order form a subgroup. If G is a locally graded non-periodic group whose proper non-periodic subgroups belong to $\mathfrak{X}$, then every proper subgroup of G is an $\mathfrak{X}$-group.*

PROOF. – Suppose first that the group G is finitely generated, so that it contains a proper normal subgroup N of finite index. Then N is a finitely generated $\mathfrak{X}$-group, so that it is polycyclic-by-finite. Therefore G itself is polycyclic-by-finite and contains a torsion-free abelian non-trivial normal subgroup A. If H is any finite subgroup of G, we have that HA^p is a proper subgroup of G for each prime number p, so that HA^p is an $\mathfrak{X}$-group and H belongs to $\mathfrak{X}$. On the other hand, every infinite proper subgroup of G is not periodic, and hence it also is an $\mathfrak{X}$-group.

Suppose now that G is not finitely generated, so that all finitely generated subgroups of G belong to $\mathfrak{X}$, and in particular the elements of finite order of G form a characteristic subgroup T. Let a be an element of infinite order of G, and let p be a prime number. Then $\langle a^p, T\rangle \neq \langle a, T\rangle$ and so $\langle a^p, T\rangle$ is an $\mathfrak{X}$-group. It follows that all periodic subgroups of G belong to $\mathfrak{X}$, and hence also in this case every proper subgroup of G is an $\mathfrak{X}$-group. □

If $\mathfrak{X}$ is a class of groups satisfying the conditions of Lemma 4.1 and such that all locally graded minimal non-$\mathfrak{X}$ groups are periodic, it follows from the above lemma that any locally graded non-periodic group whose proper non-periodic subgroups belong to $\mathfrak{X}$ is likewise an $\mathfrak{X}$-group. This is the case for many relevant group classes. In the following statement, recall that a group G is a *CC-group* if $G/C_G(\langle x\rangle^G)$ is a Černikov group for each element x of G; groups with the property CC generalize of course FC-groups, and have received much attention in recent years.

COROLLARY 4.2. – *Let G be a locally graded non-periodic group.*

(a) *If all proper non-periodic subgroups of G are abelian, then G is abelian.*

(b) *If all proper non-periodic subgroups of G are nilpotent, then G is nilpotent.*

(c) *If all proper non-periodic subgroups of G are FC-groups, then G is an FC-group.*

(d) *If all proper non-periodic subgroups of G have finite commutator subgroup, then G has finite commutator subgroup.*

(e) *If all proper non-periodic subgroups of G are CC-groups, then G is a CC-group.*

(f) *If all proper non-periodic subgroups of G are metahamiltonian, then G is metahamiltonian.*

(g) *If all proper non-periodic subgroups of G have modular subgroup lattice, then G has modular subgroup lattice.*

PROOF. – All the group classes involved in the statement satisfy the hypotheses of Lemma 4.1. Moreover, it is well known that locally graded minimal non-abelian groups are finite, while locally graded minimal non-nilpotent groups must be periodic (see for instance [22]) and the same property holds for locally graded minimal non-FC groups (see [24], Lemma 8.14). Then part (d) follows directly from (c), as any FC-group whose proper subgroups are finite-by-abelian is likewise finite-by-abelian. Again, it was proved in [14] that locally graded minimal non-CC groups are periodic, while locally graded minimal non-metahamiltonian groups are known to be finite (see [3], Lemma 4.2). Finally, an infinite locally graded group has modular subgroup lattice, provided that all its proper subgroups have the same property (see [4]). □

LEMMA 4.3. – *Let G be a locally graded non-periodic group in which all proper non-periodic subgroups are locally soluble. Then G is locally soluble.*

PROOF. – It can obviously be assumed that the group G is finitely generated, so that it contains a proper subgroup of finite index and hence G is soluble-by-finite. Thus there exists a soluble normal subgroup H of G such that G/H is finite and H/H' is infinite. Let p be a prime number which does not divide the index of H in G, and put $K/H' = (H/H')^p$. Then K is a normal subgroup of finite index of G

which is properly contained in H, and it follows from the theorem of Schur and Zassenhaus that there exists a subgroup L of G such that $G = HL$ and $H \cap L = K$. Clearly, L is a proper subgroup of finite index of G, and hence it is soluble. Therefore also G is soluble. □

A group G is said to be *minimax* if it has a series of finite length whose factors either satisfy the minimal or the maximal condition on subgroups. It is well known that if G is a soluble minimax group, then the finite residual J of G is the direct product of finitely many Prüfer subgroups, the Fitting subgroup F/J of G/J is nilpotent and the factor group G/F is polycyclic and abelian-by-finite (see [16] Part 2, Theorem 10.33). It follows that if G is a soluble minimax group, the set $Spec(G)$ of all prime numbers p such that G has a section of type p^∞ is finite, and it is an invariant of G, called the *spectrum* of G; if $Spec(G) = \pi$, the group G is *π-minimax*. In particular, soluble minimax groups with empty spectrum are precisely the polycyclic groups. Note also that any soluble minimax group of finite exponent is finite.

THEOREM 4.4. – *Let G be a locally graded non-periodic group in which every proper non-periodic subgroup is a soluble π-minimax group. Then G is a soluble π-minimax group.*

PROOF. – Assume for a contradiction that the statement is false. As the group G is locally soluble by Lemma 4.3, it has no proper subgroups of finite index and in particular G cannot be finitely generated. Let T be the largest periodic normal subgroup of G; clearly, there exists an element of infinite order a of G such that $\langle a, T\rangle$ is properly contained in G. Thus the subgroup $\langle a, T\rangle$ is π-minimax and so T is a soluble Černikov group. Let N be any proper normal subgroup of G. Then N is a soluble π-minimax group and so its finite residual D is generated by its finite characteristic subgroups. Since G has no proper subgroups of finite index, it follows that D is contained in the centre $Z(G)$ of G. Moreover, the factor group N/D is residually finite, and it even has a collection of G-invariant subgroups of finite index with trivial intersection. Thus N/D lies in $Z(G/D)$, and hence N is contained in $Z_2(G)$. Therefore the factor group $G/Z_2(G)$ is simple, and so $G = Z_2(G)$ is nilpotent, as it is locally soluble. The factor group G/G' is a non-periodic divisible abelian group, so that it has a subgroup which is isomomorphic to the additive group of rational numbers, and this contradiction completes the proof of the theorem. □

COROLLARY 4.5. – *Let G be a locally graded non-periodic group in which all proper non-periodic subgroups are polycyclic. Then G is polycyclic.*

Our last result deals with the class of supersoluble groups.

COROLLARY 4.6. – *Let G be a locally graded non-periodic group in which all proper non-periodic subgroups are supersoluble. Then G is supersoluble.*

PROOF. – The group G is polycyclic by Corollary 4.5, so that in particular it contains a normal subgroup of finite index H admitting an infinite cyclic homomorphic image. Thus G contains a normal subgroup K such that G/K is finite and the index $|G : K|$ is divisible by at least four different prime numbers. Let N be any normal subgroup of G such that G/N is finite. Then the index $|G : K \cap N|$ is also finite and so all proper subgroups of G/N are supersoluble. On the other hand, the factor group $G/K \cap N$ cannot be minimal non-supersoluble (see [5], Satz 2), so that it must be supersoluble and hence G/N is likewise supersoluble. Therefore all finite homomorphic images of G are supersoluble, and so G itself is supersoluble (see [1]). □

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