Bollettino
Unione Matematica Italiana

Elena Angelini

Higher Secants of Spinor Varieties

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.2, p. 213–235.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_2_213_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 213-235

Higher Secants of Spinor Varieties

ELENA ANGELINI

Abstract. – *Let S_h be the even pure spinors variety of a complex vector space V of even dimension $2h$ endowed with a non degenerate quadratic form Q and let $\sigma_k(S_h)$ be the k-secant variety of S_h. We decribe an algorithm which computes the complex dimension of $\sigma_k(S_h)$. Then, by using an inductive argument, we get our main result: $\sigma_3(S_h)$ has the expected dimension except when $h \in \{7, 8\}$. Also we provide theoretical arguments which prove that S_7 has a defective 3-secant variety and S_8 has defective 3-secant and 4-secant varieties.*

1. – Introduction

In this paper we study the higher secant varieties of spinor varieties.

We consider a complex $2h$-dimensional vector space V and a non degenerate quadratic form Q defined on it. The *space of spinors* associated to (V, Q) can be identified with the space of the spin representation of $Cl(V, Q)$, the *Clifford algebra* generated by V. In particular, *pure spinors* represent, from a geometrical point of view, the set of all maximal totally isotropic vector subspaces of V, which is a projective variety, called *spinor variety*. For simplicity, we consider one of its two irreducible isomorphic components, i.e. the *even pure spinors variety*, which we denote by S_h.

Let X be a non-degenerate projective variety in $\mathbb{P}^N(\mathbb{C})$; then $\sigma_k(X)$ indicates the *k-secant variety* of X, that is the Zariski closure of the union of all linear spaces spanned by k points of X, see [16] and [13] for several applications. It's easy to check the following inequality:

$$\dim_{\mathbb{C}} \sigma_k(X) \leq \min\{k \dim_{\mathbb{C}} X + k - 1, N\}.$$

If the equality holds, then we say that $\sigma_k(X)$ has the *expected dimension*, otherwise X is said to be *k- defective* and

$$\delta_k = \min\{k \dim_{\mathbb{C}} X + k - 1, N\} - \dim_{\mathbb{C}} \sigma_k(X)$$

is its *k-defect*. The problem of determining the complex dimension of $\sigma_k(X)$ is called the *defectivity problem* for X. If $v_d(\mathbb{P}^n(\mathbb{C}))$ is the Veronese variety then $\sigma_k(v_d(\mathbb{P}^n(\mathbb{C})))$ has the expected dimension except in some particular cases, ([3]), ([7]). Concerning Grassmannians and Segre varieties, this problem has been studied by several authors but it's still open, as we can see, respectively, in [6] and [2]; for related results see also [1], [5] and [10]. At the best of my knowledge, the case of spinor varieties is almost absent in the mathematical literature; it's known that $\sigma_2(S_h)$ has always the expected dimension ([11]), but for $k \geq 3$ the problem was completely open.

By using *Macaulay2* software system, we construct an algorithm which allows us to compute the dimension of $\sigma_k(S_h)$ by studying the span of the tangent spaces at k chosen random points, for $h \leq 12$. Afterwards, by using induction, we get our main result:

THEOREM 1.1. – (i) $\sigma_3(S_h)$ *has the expected dimension, except when* $h \in \{7, 8\}$. (ii) S_7 *has a defective* 3*-secant variety and* S_8 *has defective* 3*-secant and* 4*-secant varieties. In particular* $\dim_{\mathbb{C}} \sigma_3(S_7) = 58$, $\dim_{\mathbb{C}} \sigma_3(S_8) = 85$ *and* $\dim_{\mathbb{C}} \sigma_4(S_8) = 111$.

We remark that the main tool of our investigation is the parametrization of S_h with all *principal sub-Pfaffians* of a skew symmetric matrix of size h.

The paper is organized in six sections. In the second one we introduce Clifford algebras and spinor varieties, following [8], [15] and [4]; in the third we recall the main definitions and properties of higher secant varieties, ([13]), ([16]). Finally, sections four, five and six are devoted to our main results.

This article is based upon the author's laurea thesis and the main result confirms its final conjectures, ([4]). Thanks are due especially to Giorgio Ottaviani for his guidance and insight.

2. – Clifford algebras and spinors

Let V be a vector space over $\mathbb{C}$ of even dimension $n = 2h > 0$. Let Q be a quadratic form on V such that the corresponding symmetric bilinear form B is non degenerate.

We denote by $Cl(V, Q) = T(V)/I_Q(V)$ the *Clifford algebra associated to* (V, Q), where $T(V)$ is the tensor algebra of V and $I_Q(V) \subset T(V)$ is the two-sided ideal generated by the elements

$$v \otimes v - Q(v) \cdot 1$$

with $v \in V$.

Let

$$Cl(V, Q)_{\pm} = T(V)_{\pm}/I_Q(V) \cap T(V)_{\pm}$$

where $T(V)_+$ and $T(V)_-$ denote the set of even and odd tensors, respectively. We call *even* the elements of $Cl(V, Q)_+$ and *odd* those of $Cl(V, Q)_-$.

Let E and F be maximal totally isotropic vector subspaces of V such that $V = E \oplus F$, let f be the product in $Cl(V, Q)$ of the elements of a basis of F. The *spin representation* of $Cl(V, Q)$ is the irreducible representation of $Cl(V, Q)$ and its representation space, $S(V, Q)$, is the *space of spinors* of (V, Q). We denote by $S(V, Q)_+$ (respectively: $S(V, Q)_-$) the *space of even* (respectively: *odd*) *spinors* of (V, Q) .

Inside the space of spinors, the subset of *pure spinors* has a very important geometrical meaning, as we describe in the following.

Let W be a maximal totally isotropic subspace of V and let f_W be the product of the vectors in a basis of W (f_W is well defined up to a non zero scalar).

It's not hard to show that $Cl(V, Q)f \cap f_W Cl(V, Q)$ is a complex vector space of dimension 1. So we can pose

$$Cl(V, Q)f \cap f_W Cl(V, Q) = S(V, Q)_W f$$

where $S(V, Q)_W$ denotes a vector subspace of $S(V, Q)$ of dimension 1 .

DEFINITION 2.1. – *Any element of $S(V, Q)_W \setminus \{0\}$ is called representative spinor of W. Moreover, we call pure spinor any element of $S(V, Q)_W \setminus \{0\}$, for some maximal totally isotropic vector subspace W of V.*

It's easy to check that the subset of pure spinors is a projective variety, called *spinor variety*, and that it is in $1-1$ correspondence with the variety of maximal totally isotropic vector subspaces of V. Furthermore, the spinor variety has two isomorphic irreducible components, called *even* and *odd pure spinors variety*. From now on we focus our attention on the first one, which we denote by S_h.

Let $\mathcal{B} = \{e_1, \ldots, e_h, f_1, \ldots, f_h\}$ be a basis of $V = E \oplus F$, where $\{e_1, \ldots, e_h\}$ is a basis of E and $\{f_1, \ldots, f_h\}$ is a basis of F, such that $B(e_i, f_j) = \dfrac{\delta_{ij}}{2}$, for all $i, j \in \{1, \ldots, h\}$. We remark that the matrix $\mathfrak{B}$ of the form B with respect to $\mathcal{B}$ is

$$\mathfrak{B} = \begin{bmatrix} O_h & \frac{1}{2} I_h \\ \frac{1}{2} I_h & O_h \end{bmatrix}$$

where O_h and I_h are the null matrix and the identity matrix of size h, respectively. Moreover, we pose $f = f_1 \cdot \ldots \cdot f_h$.

Let W be a vector subspace of V such that $\dim_{\mathbb{C}} W = h$, i.e. $W \in Gr(h, 2h)$, the usual Grassmannian. Thus, we can associate to W the h by $2h$ matrix

$$P = [C_W | D_W]$$

where $C_W, D_W \in M(h, \mathbb{C})$. In particular, if C_W is invertible, then we can assume that

$$P = [I_h | U_W]$$

where $U_W = C_W^{-1} D_W$. So, we have that W is totally isotropic if and only if

$$P \cdot \mathfrak{B} \cdot P^t = O_h,$$

in other words if and only if

$$U_W = -U_W^t.$$

We immediately get the following:

THEOREM 2.1. – *The generic element of S_h can be represented in blocks matrix form as $[I_h|U]$, where $U \in M(h, \mathbb{C})$ is skew symmetric.*

Now, let $U = \{u_{ij}\}$ be a skew symmetric matrix of size h with complex entries and let

$$s(U) = \left(e_1 + \sum_{j=1}^{h} u_{1j} f_j\right) \cdot \left(e_2 + \sum_{j=1}^{h} u_{2j} f_j\right) \cdot \ldots \cdot \left(e_h + \sum_{j=1}^{h} u_{hj} f_j\right)$$

be an element of S_h in a neighborhood of

$$s_0 = e_1 \cdot \ldots \cdot e_h.$$

We remark that $s(U)$ and s_0 are representative spinors of

$$W(U) = \left\langle e_1 + \sum_{j=1}^{h} u_{1j} f_j, e_2 + \sum_{j=1}^{h} u_{2j} f_j, \ldots, e_h + \sum_{j=1}^{h} u_{hj} f_j \right\rangle$$

and of $E = W(O_h)$ respectively. By computing $s(U)f$ we get the following formula, [4] and [15]:

$$s(U) = \sum_K Pf_K(U) e_{K^c}$$

where K denotes any sequence of integers between 1 and h of even lenght, $K^c = \{1, \ldots, h\} \backslash K$, $Pf_K(U)$ is the Pfaffian of the submatrix of U made up by rows and columns indexed by K, and e_{K^c} is the Clifford product of the e_i's, $i \in K^c$.

In this way we get one of the main tools for our investigations:

THEOREM 2.2. – *All the principal sub-Pfaffians of a generic skew symmetric matrix of size h parametrize a generic element of S_h in $\mathbb{P}^{2^{h-1}-1}(\mathbb{C})$.*

Before closing this section we remark that, given

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in SO(2h, Q)$$

where $g_{ij} \in M(h, \mathbb{C})$, $i,j \in \{1,2\}$ and

$$P = [I_h|U] \in S_h$$

where $U \in M(h, \mathbb{C})$ is skew symmetric, g acts on P as follows:

$$g(P) = \left[I_h \middle| \left(g_{11}^t + U_h g_{12}^t\right)^{-1} \left(g_{21}^t + U_h g_{22}^t\right)\right],$$

when $\left(g_{11}^t + U_h g_{12}^t\right)^{-1}$ is defined. As we can see in [15], this action is *generically* 3-*transitive* , i.e. $Spin(2h, Q)$ has an open orbit in $S_h \times S_h \times S_h$. In order to prove theorem 1.1 part (ii), in section 5 we provide a proof of this statement based on a new argument: namely we consider 3 points of S_h that are in the same parametrization (see theorem 5.1).

3. – Higher secant varieties

Let $X \subseteq \mathbb{P}^N(\mathbb{C})$ be a d-dimensional projective variety.
We pose the following:

DEFINITION 3.1. – *The k-secant variety $\sigma_k(X)$ is the Zariski closure of the union of all linear spaces spanned by k points of X, that is*

$$\sigma_k(X) = \overline{\bigcup_{x_1,\dots,x_k \in X} \langle x_1, \dots, x_k \rangle}.$$

If $X \subseteq \mathbb{P}^N(\mathbb{C})$ is non-degenerate, i.e. it is not contained in any hyperplane, then we have the following estimate on the dimension of $\sigma_k(X)$:

$$\dim_{\mathbb{C}} \sigma_k(X) \leq \min\{kd + k - 1, N\}.$$

The problem of determining when the dimension of the secant variety $\sigma_k(X)$ reaches this upper bound is called *defectivity problem* for X. In this sense we have the following:

DEFINITION 3.2. – *Let $X \subseteq \mathbb{P}^N(\mathbb{C})$ be a non-degenerate projective variety of dimension d .*

1. *If* $\dim_{\mathbb{C}} \sigma_k(X) = \min\{kd + k - 1, N\}$ *then we say that $\sigma_k(X)$ has the expected dimension.*
2. *If* $\dim_{\mathbb{C}} \sigma_k(X) < \min\{kd + k - 1, N\}$ *then we say that X has a defective k - secant variety and that*

$$\delta_k = \min\{k \dim_{\mathbb{C}} X + k - 1, N\} - \dim_{\mathbb{C}} \sigma_k(X)$$

is its k-defect.

3. *If there's a k such that X is k-defective then we say that X is defective.*

Now we recall the main tool to compute the dimensions of higher secant varieties:

LEMMA 3.1 (Terracini, 1911). – *Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a projective variety and let z be a generic point of $\sigma_k(X)$. Then the projective tangent space to $\sigma_k(X)$ at z is given by*

$$\widetilde{T}_z\sigma_k(X) = \left\langle \widetilde{T}_{x_1}X, \ldots, \widetilde{T}_{x_k}X \right\rangle$$

where $x_1, \ldots, x_k$ are generic points of X such that $z \in \langle x_1, \ldots, x_k \rangle$ and $\widetilde{T}_{x_i}X$ denotes the projective tangent space to X at x_i.

By upper semicontinuity, we immediately get an argument to prove that a variety isn't defective:

COROLLARY 3.2. – *Let $x_1, \ldots, x_k \in X$ be smooth points such that $\widetilde{T}_{x_1}X, \ldots, \widetilde{T}_{x_k}X$ are linearly independent, or else*

$$\left\langle \widetilde{T}_{x_1}X, \ldots, \widetilde{T}_{x_k}X \right\rangle = \mathbb{P}^N(\mathbb{C}).$$

Then $\sigma_k(X)$ has the expected dimension.

Terracini's lemma also provides a method to show that X has a defective k-secant variety. More precisely, we have the following:

COROLLARY 3.3 ([9]). – *Let $d = \dim_{\mathbb{C}} X$ and let us suppose that*

$$kd + k - 1 \leq N.$$

If there exists an irreducible curve of X, embedded in $\mathbb{P}^{2k-2}(\mathbb{C})$ and containing k general points of X, then $\sigma_k(X)$ hasn't the expected dimension.

4. – An algorithm for the secant defect of spinor varieties

To deal with our problem, we constructed an algorithm through the *Macaulay2* computation system, ([12]).

The script of the algorithm is given below:

```
h = value read "h?"
k = value read "k?"
p = floor(h*(h-1)/2)
R = QQ[x_0..x_(p-1)]
```

```
X = vars R
M = X -> genericSkewMatrix(R,x_0,h)
par = X -> apply(floor(h/2)+1,i- >generators pfaffians(2*i,M(X)))
f = l -> (a=l#0;for i from 1 to #(l)-1 do(a=a|(l#i););a)
S = f(par(X))
J = jacobian S
g = l -> (a=l#0;for i from 1 to #(l)-1 do(a=a||(l#i););a)
punti = apply(k,i->for j from 1 to p list random(1000))
puntibis = apply(k,i-> matrix{punti#i})
Spunti = apply(k,i-> substitute(S,matrix(R,{flatten entries puntibis#i})))
Jpunti = apply(k,i-> substitute(J,matrix(R,{flatten entries puntibis#i})))
JS = apply(k,i->(Spunti#i)||(Jpunti#i))
JJS = g(JS)
rank JJS.
```

This algorithm is based on *Terracini's lemma* and on the fact that *Pfaffians* parametrize S_h; moreover it was conceived for every h and k integers, where $h = \frac{1}{2}\dim_{\mathbb{C}} V$.

The main steps of our algorithm are the following:

1. *Preliminaries.*

Given h, k and further computed the dimension of S_h

$$p = \frac{h(h-1)}{2},$$

we define the polinomial ring R with rational coefficients in the variables $\{x_0, \dots, x_{p-1}\}$.

2. *Parametrization of S_h.*

In order to parametrize the variety of even pure spinors, we construct the function

$$M : \mathcal{M}_{(1,p)}(\mathbb{Q}) \to \mathcal{M}_{(h,h)}(\mathbb{Q})$$

defined by

$$X = (x_0, \dots, x_{p-1}) \to M(X) = \{m_{ij}\}$$

where

$$m_{ij} = x_{i+j+(i-1)h-\frac{(i+1)(i+2)}{2}} \text{ with } 1 \le i < j \le h$$

and $m_{ij} = -m_{ji}$. Then we compute the principal sub-Pfaffians of this matrix by using the function

$$par : \mathcal{M}_{(1,p)}(\mathbb{Q}) \to \mathcal{M}_{(1,2^{h-1})}(\mathbb{Q})$$

such that

$$X = \left(x_0, \ldots, x_{p-1}\right) \to par(X) = (\text{principal sub} - \text{Pfaffians of } M(X)).$$

3. *Definition of* S_h.

From the theorem 2.2 we obtain that S_h is the image of the function *par*, i.e. it belongs to $\mathcal{M}_{(1,2^{h-1})}(\mathbb{Q})$:

$$S = par(X) = (s_i)_{i=0,\ldots,2^{h-1}-1}.$$

We observe that *par*, being defined through *apply*, produces a list of $\left\lfloor \dfrac{h}{2} \right\rfloor + 1$ row matrices; by means of the function *f* we juxtapose all Pfaffians in one row matrix.

4. *Computation of the jacobian matrix of the parametrization.*

Applying *jacobian* to S we get the following p by 2^{h-1} matrix:

$$J = \left(\partial_j s_i\right)_{i=0,\ldots,2^{h-1}-1;\, j=0,\ldots,p-1}.$$

5. *Choice of k random points in* S_h *and computation of their coordinates.*

In order to study $\sigma_k(S_h)$, we have to choose k elements of S_h: so, we consider a list of k sets (*punti*) of p random rational numbers and we construct the corresponding skew symmetric h by h matrices; then we compute the principal sub-Pfaffians of these matrices. In this way we get a list (*Spunti*) composed of the parametric coordinates of the k selected points:

$$\begin{aligned}
punti &= \{punti_0, \ldots, punti_{k-1}\} \\
punti_i &= \left(q_0^i, \ldots, q_{p-1}^i\right),\ q_j^i \in \mathbb{Q} \text{ random, } q_j^i \leq 1000 \\
Spunti &= \{S(punti_0), \ldots, S(punti_{k-1})\} = \{P_0, \ldots, P_{k-1}\}.
\end{aligned}$$

6. *Construction of the affine tangent spaces to* S_h *at the k points.*

Now we evaluate the jacobian matrix J at the points under consideration. Thus we obtain a list (*Jpunti*) of matrices whose images correspond to the vector tangent spaces to S_h; placing the row made up of the coordinates of one point before the corresponding jacobian matrix we get the affine tangent space to S_h at such point:

$$\begin{aligned}
Jpunti &= \left\{J|_{X=punti_0}, \ldots, J|_{X=punti_{k-1}}\right\} = \{J_0, \ldots, J_{k-1}\} \\
JS &= \{P_0|J_0, \ldots, P_{k-1}|J_{k-1}\} = \{JS_0, \ldots, JS_{k-1}\}.
\end{aligned}$$

7. *Computation of the dimension of* $\sigma_k(S_h)$.

Finally, we arrange in columns the $(p+1)$ by 2^{h-1} matrices $JS_0, \ldots, JS_{k-1}$ and we obtain the $k(p+1)$ by 2^{h-1} matrix JJS associated with the span of the affine tangent spaces. From *Terracini's Lemma* we get that the rank of JJS produces the affine dimension of $\sigma_k(S_h)$; subtracting 1 to the output we get the required dimension:

$$\begin{array}{lcccl} g & : & \{\text{lists of matrices}\} & \to & \{\text{matrices}\} \\ & & B = \{B_1, B_2, \ldots\} & \to & g(B) = (B_1|B_2|\ldots)^t \end{array}$$

$$g(JS) = \begin{pmatrix} JS_0 \\ \vdots \\ \vdots \\ JS_{k-1} \end{pmatrix} = JJS$$

OUTPUT $rank(JJS)$.

REMARK 4.1. – If the achieved value coincides with the expected dimension of $\sigma_k(S_h)$, i.e. if JJS has maximum rank, then we can be sure that the actual dimension is that value (corollary 3.2); otherwise we can only guess S_h is k -defective.

It's not hard to check, by direct computations, that, if $h \leq 5$, then S_h isn't defective, [4] and [11]. So we used this algorithm from the stage $(h,k) = (6,2)$ to the stage $(h,k) = (9,5)$: beyond these values the memory of the computer was used up.

Our results are summarized as follows.

$\boxed{\mathbf{k = 2}}$

h	p	N	exp dim$\sigma_k(S_h)$	dim$\sigma_k(S_h)$	defective
6	15	31	31	31	NO
7	21	63	43	43	NO
8	28	127	57	57	NO
9	36	255	73	73	NO
10	45	511	91	91	NO
11	55	1023	111	111	NO

k = 3

h	p	N	$\exp\dim\sigma_k(S_h)$	$\dim\sigma_k(S_h)$	defective
7	21	63	63	58	YES ([1])
8	28	127	86	85	YES ([2])
9	36	255	110	110	NO
10	45	511	137	137	NO
11	55	1023	167	167	NO
12	66	2047	200	200	NO

k = 4

h	p	N	$\exp\dim\sigma_k(S_h)$	$\dim\sigma_k(S_h)$	defective
7	21	63	63	63	NO
8	28	127	115	111	YES ([3])
9	36	255	147	147	NO
10	45	511	183	183	NO

k = 5

h	p	N	$\exp\dim\sigma_k(S_h)$	$\dim\sigma_k(S_h)$	defective
8	28	127	127	127	NO
9	36	255	184	184	NO

The last three tables provide a proof of theorem 1.1 part (i) till $h = 12$ and even some cases more.

In the first table we can see that, if $6 \leq h \leq 11$, then $\sigma_2(S_h)$ has the expected dimension; this fact agrees with already known theoretical results, ([11]).

However, we found some "anomalies" when $(h, k) \in \{(7, 3), (8, 3), (8, 4)\}$. So, we supposed that actually these varieties haven't the expected dimension. Indeed, in the next section we explain, from a theoretical point of view, that S_8 has a defective 3-secant variety and a defective 4-secant variety and that S_7 has a defective 3-secant variety. Hence we get a proof of theorem 1.1 part (ii).

5. – The defective cases

In order to prove that $\sigma_3(S_8)$ and $\sigma_4(S_8)$ haven't the expected dimension, we proceed as follows.

Let assume that h is an even number, $h = 2m$. With the notations of section 2,

([1]) see theorem 5.5.
([2]) see theorem 5.3.
([3]) see corollary 5.4.

let

$$s_0 = e_1 \cdot \ldots \cdot e_h,\ s_1 = \prod_{i=1}^{m}(1 + e_{2i-1} \cdot e_{2i}),\ s_2 = \prod_{i=1}^{m}(1 - e_{2i-1} \cdot e_{2i})$$

be elements of S_h: they are *representative spinors* of the maximal totally isotropic subspaces

$$\begin{aligned} E =& \langle e_1, \ldots, e_h \rangle \\ G =& \langle e_{2i-1} + f_{2i}, e_{2i} - f_{2i-1}, 1 \leq i \leq m \rangle \\ H =& \langle e_{2i-1} - f_{2i}, e_{2i} + f_{2i-1}, 1 \leq i \leq m \rangle \end{aligned}$$

respectively. Their corresponding h by $2h$ matrices are

$$\begin{aligned} P_0 =& [I_h | O_h] \\ P_1 =& [I_h | J_m] \\ P_2 =& [I_h | -J_m] \end{aligned}$$

where J_m denotes the skew symmetric matrix of size h made up of m diagonal blocks like $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

THEOREM 5.1. – *The orbit of* (P_0, P_1, P_2) *is open in* $S_h \times S_h \times S_h$.

PROOF. – Let consider the function

$$\begin{aligned} f \quad : \quad SO(2h, Q) \quad &\rightarrow \quad S_h \times S_h \times S_h \\ g \quad &\rightarrow \quad (g(P_0), g(P_1), g(P_2)) \end{aligned}$$

where

$$\begin{aligned} g(P_0) =& \left[I_h \middle| (g_{11}^t)^{-1} g_{21}^t \right] \\ g(P_1) =& \left[I_h \middle| (g_{11}^t + J_m g_{12}^t)^{-1} (g_{21}^t + J_m g_{22}^t) \right] \\ g(P_2) =& \left[I_h \middle| (g_{11}^t - J_m g_{12}^t)^{-1} (g_{21}^t - J_m g_{22}^t) \right]; \end{aligned}$$

we remark that

$$\operatorname{Im} f = \{(g(P_0), g(P_1), g(P_2)) | g \in SO(2h, Q)\}$$

is the orbit of (P_0, P_1, P_2). Taking $g = I_{2h}$, the tangent map of f at the point g is:

$$df_{I_{2h}} : so(2h, Q) \rightarrow T_{(P_0, P_1, P_2)}[S_h \times S_h \times S_h],$$

where $so(2h,Q)$ is the Lie algebra of $SO(2h,Q)$, that is:

$$so(2h,Q)=\{A\in SO(2h,Q)|A^t\mathfrak{B}=-\mathfrak{B}A\}.$$

We have that $\operatorname{Im} df_{I_{2h}}$ is the tangent space to the orbit of (P_0,P_1,P_2) at (P_0,P_1,P_2). Our aim is to show that $df_{I_{2h}}$ is surjective, or that

$$\begin{aligned}\dim_{\mathbb{C}}\ker df_{I_{2h}} &= \dim_{\mathbb{C}} so(2h,Q)-\dim_{\mathbb{C}}\operatorname{Im} df_{I_{2h}}\\ &=\frac{2h(2h-1)}{2}-\frac{3h(h-1)}{2}\\ &=\frac{h(h+1)}{2}.\end{aligned}$$

In order to study $\ker df_{I_{2h}}$, we use the first-order Taylor expansion of $f=(f_1,f_2,f_3)$ about I_{2h}. So, let $H\in so(2h,Q)$, i.e.

$$H=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}$$

with $H_{ij}\in M(h,\mathbb{C})$, $i,j\in\{1,2\}$, such that $H_{11}^t=-H_{22}$ and H_{12}, H_{21} are skew symmetric; we get that

$$\begin{aligned}f_1(I_{2h}+H) &= \left[I_h\middle|\left(I_h+H_{11}^t\right)^{-1}H_{21}^t\right]=\left[I_h|H_{21}^t+\dots\right]\\ f_2(I_{2h}+H) &= \left[I_h\middle|\left(I_h+H_{11}^t+J_mH_{12}^t\right)^{-1}\left(H_{21}^t+J_m\left(I_h+H_{22}^t\right)\right)\right]\\ &= \left[I_h|J_m+H_{21}^t+J_mH_{22}^t-H_{11}^tJ_m-J_mH_{12}^tJ_m+\dots\right]\\ f_3(I_{2h}+H) &= \left[I_h\middle|\left(I_h+H_{11}^t-J_mH_{12}^t\right)^{-1}\left(H_{21}^t-J_m\left(I_h+H_{22}^t\right)\right)\right]\\ &= \left[I_h|-J_m+H_{21}^t-J_mH_{22}^t+H_{11}^tJ_m-J_mH_{12}^tJ_m+\dots\right]\end{aligned}$$

and then we have that

$$\ker df_{I_{2h}}=\left\{H\in so(2h,Q)\middle|\begin{array}{l}H_{21}^t=0\\ H_{21}^t+J_mH_{22}^t-H_{11}^tJ_m-J_mH_{12}^tJ_m=0\\ H_{21}^t-J_mH_{22}^t+H_{11}^tJ_m-J_mH_{12}^tJ_m=0\end{array}\right\}.$$

A direct computation shows that

$$\ker df_{I_{2h}}=\left\{H\in so(2h,Q)\middle|\begin{array}{l}H_{21}^t=H_{12}^t=0\\ J_mH_{22}^t=\left(J_mH_{22}^t\right)^t\end{array}\right\},$$

thus

$$\dim_{\mathbb{C}} \ker df_{I_{2h}} = \dim_{\mathbb{C}} \left\{ A \in M(h, \mathbb{C}) \middle| \ J_m A = (J_m A)^t \right\}.$$

Since the set in question is the pull-back via multiplication by J_m^{-1} (which is a diffeomorphism) of the set of symmetric matrices, we get that

$$\dim_{\mathbb{C}} \left\{ A \in M(h, \mathbb{C}) \middle| \ J_m A = (J_m A)^t \right\} = \frac{h(h+1)}{2}$$

which concludes the proof. □

COROLLARY 5.2. – *If $h = 2m$ then*

$$s_0 = e_1 \cdot \ldots \cdot e_h,\ s_1 = \prod_{i=1}^{m} (1 + e_{2i-1} \cdot e_{2i}),\ s_2 = \prod_{i=1}^{m} (1 - e_{2i-1} \cdot e_{2i})$$

are general points of S_h.

Now we are ready to prove the following:

THEOREM 5.3. – *The variety S_8 is 3-defective and $\delta_3 = 1$.*

PROOF. – From corollary 5.2 we get that

$$s_0 = e_1 \cdot \ldots \cdot e_8,\ s_1 = \prod_{i=1}^{4} (1 + e_{2i-1} \cdot e_{2i}),\ s_2 = \prod_{i=1}^{4} (1 - e_{2i-1} \cdot e_{2i})$$

are general points of S_8; their corresponding 8 by 16 matrices are:

$$P_0 = [I_8 | O_8]$$
$$P_1 = [I_8 | J_4]$$
$$P_2 = [I_8 | -J_4].$$

Let C be the rational normal curve defined by

$$C(t) = [I_8 | tJ_4].$$

We have that C is embedded in $\mathbb{P}^4(\mathbb{C})$, it's contained in S_8 and

$$C(0) = P_0,\ C(1) = P_1,\ C(-1) = P_2.$$

Since

$$3 \dim_{\mathbb{C}} S_8 + 2 = 86 < 2^{8-1} - 1 = 127,$$

we may apply corollary 3.3 and we get that $\sigma_3(S_8)$ hasn't the expected dimension, as desired. □

REMARK 5.1. – Same argument says that, for all $h = 2m$, there exists a rational normal curve of degree m in S_h through three general points.

Theorem 5.3 implies that four projective tangent spaces to S_8 are always linearly dependent. Hence the following holds, see also the table for $k = 4$ in previous section:

COROLLARY 5.4. – *The variety S_8 is 4-defective and $\delta_4 = 4$.*

In the case of $h = 7$ we can't apply corollary 5.2. Nevertheless we have the following:

THEOREM 5.5. – *The variety S_7 is 3-defective and $\delta_3 = 5$.*

PROOF. – Let $X_1, X_2, X_3 \in S_7$ represented in blocks matrix form and let

$$f : SO(14, Q) \to S_7 \times S_7 \times S_7$$

be the function defined by

$$f(g) = (g(X_1), g(X_2), g(X_3)), \text{ for all } g \in SO(14, Q).$$

Taking $g = I_{14}$, the tangent map of f at the point g is:

$$df_{I_{14}} : so(14, Q) \to T_{(X_1,X_2,X_3)}[S_7 \times S_7 \times S_7].$$

To complete the proof it suffices to find $X_1 = [I_7|U_1],\ X_2 = [I_7|U_2],\ X_3 = [I_7|U_3] \in S_7$ such that:

1. the orbit of (X_1, X_2, X_3) is open in $S_7 \times S_7 \times S_7$;
2. $\dim_{\mathbb{C}}\langle T_{X_1}S_7, T_{X_2}S_7, T_{X_3}S_7\rangle = 59$ (we recall that 59 is the value we got by applying our algorithm at the stage $(h, k) = (7, 3)$).

In order that X_1, X_2, X_3 may satisfy the first property, the rank of the 91 by 63 matrix corresponding to $df_{I_{14}}$ has to be maximum.

So, we use the first-order Taylor expansion of $f = (f_1, f_2, f_3)$ about I_{14}. If

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in so(14, Q),$$

with $H_{ij} \in M(7, \mathbb{C})$, $i, j \in \{1, 2\}$, we have that, for $i \in \{1, 2, 3\}$,

$$\begin{aligned} f_i(I_{14} + H) &= \Big[I_7 \Big| \big(I_7 + H_{11}^t + U_i H_{12}^t\big)^{-1}\big(H_{21}^t + U_i(I_7 + H_{22}^t)\big)\Big] \\ &= \big[I_7 \big| U_i + H_{21}^t + U_i H_{22}^t - H_{11}^t U_i - U_i H_{12}^t U_i + \ldots\big]. \end{aligned}$$

Since $H \in so(14, Q)$, it's not hard to show ([4]) that, for $i \in \{1, 2, 3\}$,

$$A_i = H_{21}^t + U_i H_{22}^t - H_{11}^t U_i - U_i H_{12}^t U_i^t$$

is a skew symmetric matrix. By computing the jacobian of Pfaffians of size 2 of A_i, $i \in \{0, 1, 2\}$, we get the matrix corresponding to $df_{I_{14}}$.

In order to find such points we employed the Macaulay2 software system, ([4]); in particular $U_1 = O_7$ whereas U_2 and U_3 are made of random rational entries. With these choices the above conditions 1. and 2. are satisfied. □

REMARK 5.2. – The result of theorem 5.5 agrees with the fact that the ideal of $\sigma_2(S_7)$ is generated in degree 4, as we can see in [14].

6. – Non defective spinor varieties

In this section, by using induction, we get our main result.

First of all we have the following:

THEOREM 6.1. – *For all $h \geq 12$, the affine tangent spaces to S_h at*

$$P_0^h = \left[\begin{array}{cc|cc} I_{12} & O_{12\times(h-12)} & O_{12} & O_{12\times(h-12)} \\ O_{(h-12)\times 12} & I_{h-12} & O_{(h-12)\times 12} & O_{h-12} \end{array}\right]$$

$$P_1^h = \left[\begin{array}{cc|cc} I_{12} & O_{12\times(h-12)} & J_6 & O_{12\times(h-12)} \\ O_{(h-12)\times 12} & I_{h-12} & O_{(h-12)\times 12} & O_{h-12} \end{array}\right]$$

$$P_2^h = \left[\begin{array}{cc|cc} I_{12} & O_{12\times(h-12)} & K_6 & O_{12\times(h-12)} \\ O_{(h-12)\times 12} & I_{h-12} & O_{(h-12)\times 12} & O_{h-12} \end{array}\right]$$

where J_6 is the standard skew symmetric matrix of size 12 already used before and K_6 is the skew symmetric matrix of size 12 with six diagonal blocks of type

$$\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix},\ t \in \{2, 3, \ldots, 7\},$$

are linearly independent.

PROOF. – We proceed by using induction on h.

If $h = 12$, a slight modification of our algorithm in step 5 allows us to check the statement.

Therefore, we assume that the theorem holds for all h such that $12 \leq h \leq s$, we want to prove it also for $s + 1$.

First of all we remark that S_s is embedded in S_{s+1} as follows:

$$(1)\qquad [I_s U] \in S_s \overset{i}{\hookrightarrow} \left[\begin{array}{cc|cc} I_s & O_{s\times 1} & U & O_{s\times 1} \\ O_{1\times s} & 1 & O_{1\times s} & 0 \end{array}\right] = \left[I_{s+1} \middle| \widetilde{U}\right] \in S_{s+1}$$

where $U \in M(s,\mathbb{C})$ is skew symmetric.

Now, let

$$P = \left[I_{s+1} \middle| \widetilde{U}\right] = \left[\begin{array}{c|cccc} & & & & y_1 \\ I_{s+1} & & U & & \vdots \\ & & & & y_s \\ & -y_1 & \cdots & -y_s & 0 \end{array}\right] \in S_{s+1},$$

with $U = \{u_{ij}\}$ skew symmetric of size s; we can parametrize P in $\mathbb{P}^{2^{(s+1)-1}-1}(\mathbb{C})$ in such a way that the first coordinates correspond to the principal sub-Pfaffians of U and the last one to those of $\widetilde{U}$ that involve the last column. Moreover, if $P \in S_s$, then, because of (1), the affine tangent space to S_{s+1} at $i(P)$ can be represented by the following $\dfrac{(s+1)s}{2} + 1 \times 2^{(s+1)-1}$ matrix M^{s+1}, whose blocks form is:

$$M^{s+1} = \left(C_1 \quad C_2 \right)$$

where

$$C_1 = \begin{array}{|cccccc|} \hline 1 & Pf_2(U) & Pf_4(U) & \cdots & Pf_l(U) & \cdots \\ \hline O_{\frac{(s-1)s}{2}\times 1} & \frac{\partial}{\partial u_{ij}} Pf_2(U) & \frac{\partial}{\partial u_{ij}} Pf_4(U) & \cdots & \frac{\partial}{\partial u_{ij}} Pf_l(U) & \cdots \\ \hline \multicolumn{6}{|c|}{O_{s\times 2^{s-1}}} \\ \hline \end{array}$$

and

$$C_2 = \begin{array}{|c|c|c|} \hline \multicolumn{3}{|c|}{O_{1\times 2^{s-1}}} \\ \hline \multicolumn{3}{|c|}{O_{\frac{(s-1)s}{2}\times 2^{s-1}}} \\ \hline I_s & A^{s+1} & * \\ \hline \end{array};$$

$Pf_l(U)$ is the set of the principal sub-Pfaffians of U of size l, A^{s+1} is the $s \times \binom{s}{3}$ matrix made up of the derivatives, with respect to $y_1,\dots,y_s$, of the principal sub-Pfaffians of $\widetilde{U}$ of size 4 that involve the last column and the entries of $*$ are the derivatives, with respect to $y_1,\dots,y_s$, of the principal sub-Pfaffians of $\widetilde{U}$ of order $r \geq 6$ that involve the last column.

We remark that the first two blocks of C_1

$$\begin{array}{|cccccc|} \hline 1 & Pf_2(U) & Pf_4(U) & \cdots & Pf_l(U) & \cdots \\ \hline O_{\frac{(s-1)s}{2}\times 1} & \frac{\partial}{\partial u_{ij}} Pf_2(U) & \frac{\partial}{\partial u_{ij}} Pf_4(U) & \cdots & \frac{\partial}{\partial u_{ij}} Pf_l(U) & \cdots \\ \hline \end{array}$$

represent the affine tangent space to S_s at P.

A direct computation shows that A^{s+1} has the following blocks structure:

$$\begin{pmatrix} D_1 & D_2 & \cdots & D_{s-2} \end{pmatrix}$$

where D_i's entries, $i \in \{1, \ldots, s-2\}$, are the derivatives, with respect to $y_1, \ldots, y_s$, of the principal sub-Pfaffians of $\widetilde{U}$ of size 4 whose first row is the i-th. For our aim, we need only the first four blocks of A^{s+1}, i.e.:

$$D_1 = \begin{array}{|c|c|c|c|}
\hline
u_{23}\ u_{24}\ \cdots\ u_{2s} & u_{34}\ u_{35}\ \cdots\ u_{3s} & \cdots & u_{(s-1)s} \\
\hline
-u_{13}\ -u_{14} \cdots -u_{1s} & 0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
u_{12} I_{s-2} & -u_{14}\ -u_{15} \cdots -u_{1s} & & 0 \\
\cline{2-2}\cline{4-4}
 & u_{13} I_{s-3} & \ddots & \\
 & & \ddots & \vdots \\
\cline{4-4}
 & & & 0 \\
\cline{4-4}
 & & & -u_{1s} \\
\cline{4-4}
 & & & u_{1(s-1)} I_1 \\
\hline
\end{array}$$

$$D_2 = \begin{array}{|c|c|c|c|}
\hline
0 \cdots\cdots\cdots 0 & 0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
u_{34}\ u_{35}\ \cdots\ u_{3s} & u_{45}\ u_{46}\ \cdots\ u_{4s} & \cdots & u_{(s-1)s} \\
\hline
-u_{24}\ -u_{25} \cdots -u_{2s} & 0 \cdots\cdots\cdots 0 & & 0 \\
\hline
u_{23} I_{s-3} & -u_{25}\ -u_{26} \cdots -u_{2s} & \ddots & \\
\cline{2-2}
 & u_{24} I_{s-4} & \ddots & \vdots \\
\cline{4-4}
 & & & 0 \\
\cline{4-4}
 & & & -u_{2s} \\
\cline{4-4}
 & & & u_{2(s-1)} I_1 \\
\hline
\end{array}$$

$$D_3 = \begin{array}{|c|c|c|}
\hline
0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
u_{45}\ u_{46}\ \cdots\ u_{4s} & \cdots & u_{(s-1)s} \\
\hline
-u_{35}\ -u_{36} \cdots -u_{3s} & & 0 \\
\hline
u_{34} I_{s-4} & \ddots & \vdots \\
 & \ddots & \\
\cline{3-3}
 & & 0 \\
\cline{3-3}
 & & -u_{3s} \\
\cline{3-3}
 & & u_{3(s-1)} I_1 \\
\hline
\end{array}$$

$$D_4 = \begin{array}{|c|c|c|}
\hline
0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
0 \cdots\cdots\cdots 0 & \cdots & 0 \\
\hline
u_{56}\ u_{57}\ \cdots\ u_{5s} & \cdots & u_{(s-1)s} \\
\hline
-u_{46}\ -u_{47} \cdots -u_{4s} & & 0 \\
\cline{1-1}\cline{3-3}
u_{45} I_{s-5} & \ddots & \vdots \\
\cline{3-3}
 & \ddots & 0 \\
\cline{3-3}
 & & -u_{4s} \\
\cline{3-3}
 & & u_{4(s-1)} I_1 \\
\hline
\end{array}.$$

So, if instead of a generic skew symmetric $U \in M(s, \mathbb{C})$, we consider, respectively,

$$U_0^s = \begin{pmatrix} O_{12} & O_{12\times(s-12)} \\ O_{(s-12)\times 12} & O_{s-12} \end{pmatrix}$$

$$U_1^s = \begin{pmatrix} J_6 & O_{12\times(s-12)} \\ O_{(s-12)\times 12} & O_{s-12} \end{pmatrix}$$

$$U_2^s = \begin{pmatrix} K_6 & O_{12\times(s-12)} \\ O_{(s-12)\times 12} & O_{s-12} \end{pmatrix}$$

and we arrange in columns the corresponding M^{s+1} matrices, we get the span of the affine tangent spaces to S_{s+1} at $P_0^{s+1} = i(P_0^s)$, $P_1^{s+1} = i(P_1^s)$, $P_2^{s+1} = i(P_2^s)$. Reorganizing opportunely the rows, we can focus our attention on the following $\left[3\dfrac{(s+1)s}{2} + 3\right] \times 2^{(s+1)-1}$ matrix:

$$T^{s+1} = \begin{pmatrix} T^s & O_{3\frac{(s-1)s}{2}+3\times 2^{s-1}} \\ O_{3s\times 2^{s-1}} & \Omega \end{pmatrix}$$

where

$$T^s = \begin{array}{|ccccc|c|}
\hline
1 & & O_{1\times 2^{s-1}-1} & & & \cdots \\
\hline
 & & O_{\frac{(s-1)s}{2}\times 2^{s-1}-1} & & & \cdots \\
\hline
1 & Pf_2\left(U_1^s\right) & Pf_4\left(U_1^s\right) & \cdots & Pf_l\left(U_1^s\right) & \cdots \\
\hline
O_{\frac{(s-1)s}{2}\times 1} & \partial Pf_{2|U_1^s} & \partial Pf_{4|U_1^s} & \cdots & \partial Pf_{l|U_1^s} & \cdots \\
\hline
1 & Pf_2\left(U_2^s\right) & Pf_4\left(U_2^s\right) & \cdots & Pf_l\left(U_2^s\right) & \cdots \\
\hline
O_{\frac{(s-1)s}{2}\times 1} & \partial Pf_{2|U_2^s} & \partial Pf_{4|U_2^s} & \cdots & \partial Pf_{l|U_2^s} & \cdots \\
\hline
\end{array}$$

$$\Omega = \begin{array}{|c|c|c|}
\hline
I_s & O_{s\times\binom{s}{3}} & *_0 \\
\hline
I_s & A_1^{s+1} & *_1 \\
\hline
I_s & A_2^{s+1} & *_2 \\
\hline
\end{array}.$$

We want to prove that T^{s+1} has maximum rank, i.e. that

$$rankT^{s+1} = 3\frac{(s+1)s}{2} + 3.$$

By induction,

$$rankT^{s} = 3\frac{(s-1)s}{2} + 3,$$

being T^s the matrix corresponding to the span of the affine tangent spaces to S_s at P_0^s, P_1^s, P_2^s. Then we have only to prove that

$$rank\begin{pmatrix} A_1^{s+1} \\ A_2^{s+1} \end{pmatrix} = 2s. \tag{2}$$

We remark that $A_1^{s+1} = A^{s+1}_{|U_1^s}$ and $A_2^{s+1} = A^{s+1}_{|U_2^s}$; so we consider the following $2s \times \binom{s}{3}$ blocks matrix:

$$\begin{pmatrix} A_1^{s+1} \\ A_2^{s+1} \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & \cdots & B_{s-2} \end{pmatrix}$$

with $B_i = \begin{pmatrix} D_{i|U_1^s} \\ D_{i|U_2^s} \end{pmatrix}$, $i \in \{1, \dots, s-2\}$. In particular we have that:

$$B_1 = \left|\begin{array}{c|c|c|c|c|c}
0\cdots0 & 1\;0\cdots0 & 0\cdots0 & 1\;0\cdots0 & \cdots & 1/0 \\
0\cdots0 & 0\;0\cdots0 & 0\cdots0 & 0\;0\cdots0 & & 0 \\
I_{s-2} & 0\;0\cdots0 & 0\cdots0 & 0\;0\cdots0 & & \\
 & O_{s-3} & 0\cdots0 & 0\;0\cdots0 & & \vdots \\
 & & O_{s-4} & 0\;0\cdots0 & & \\
 & & & O_{s-5} & \ddots & \\
 & & & & & O_1 \\
\hline
0\cdots0 & 3\;0\cdots0 & 0\cdots0 & 4\;0\cdots0 & \cdots & 7/0 \\
0\cdots0 & 0\;0\cdots0 & 0\cdots0 & 0\;0\cdots0 & & 0 \\
2I_{s-2} & 0\;0\cdots0 & 0\cdots0 & 0\;0\cdots0 & & \\
 & O_{s-3} & 0\cdots0 & 0\;0\cdots0 & & \vdots \\
 & & O_{s-4} & 0\;0\cdots0 & & \\
 & & & O_{s-5} & \ddots & \\
 & & & & & O_1
\end{array}\right|,$$

$$B_2 = \left[\begin{array}{|c|c|c|c|c|}
\hline
0\ 0\cdots0 & 0\cdots0 & 0\ 0\cdots0 & \cdots & 0 \\
\hline
1\ 0\cdots0 & 0\cdots0 & 1\ 0\cdots0 & & 1/0 \\
\hline
0\ 0\cdots0 & 0\cdots0 & 0\ 0\cdots0 & & 0 \\
\hline
O_{s-3} & 0\cdots0 & 0\ 0\cdots0 & & \\
\cline{2-3}
& O_{s-4} & 0\ 0\cdots0 & & \vdots \\
\cline{3-3}
& & O_{s-5} & \ddots & \\
\cline{5-5}
& & & & O_1 \\
\hline
0\ 0\cdots0 & 0\cdots0 & 0\ 0\cdots0 & \cdots & 0 \\
\hline
3\ 0\cdots0 & 0\cdots0 & 4\ 0\cdots0 & & 7/0 \\
\hline
0\ 0\cdots0 & 0\cdots0 & 0\ 0\cdots0 & & 0 \\
\hline
O_{s-3} & 0\cdots0 & 0\ 0\cdots0 & & \\
\cline{2-3}
& O_{s-4} & 0\ 0\cdots0 & & \vdots \\
\cline{3-3}
& & O_{s-5} & \ddots & \\
\cline{5-5}
& & & & O_1 \\
\hline
\end{array}\right],$$

$$B_3 = \left[\begin{array}{|c|c|c|c|}
\hline
0\cdots0 & 0\ 0\cdots0 & \cdots & 0 \\
\hline
0\cdots0 & 0\ 0\cdots0 & & 0 \\
\hline
0\cdots0 & 1\ 0\cdots0 & & 1/0 \\
\hline
0\cdots0 & 0\ 0\cdots0 & & 0 \\
\hline
I_{s-4} & 0\ 0\cdots0 & & \vdots \\
\cline{2-2}
& O_{s-5} & \ddots & \\
\cline{4-4}
& & & O_1 \\
\hline
0\cdots0 & 0\ 0\cdots0 & \cdots & 0 \\
\hline
0\cdots0 & 0\ 0\cdots0 & & 0 \\
\hline
0\cdots0 & 4\ 0\cdots0 & & 7/0 \\
\hline
0\cdots0 & 0\ 0\cdots0 & & 0 \\
\hline
3I_{s-4} & 0\ 0\cdots0 & & \vdots \\
\cline{2-2}
& O_{s-5} & \ddots & \\
\cline{4-4}
& & & O_1 \\
\hline
\end{array}\right], \quad B_4 = \left[\begin{array}{|c|c|c|}
\hline
0\ 0\cdots0 & \cdots & 0 \\
\hline
0\ 0\cdots0 & & 0 \\
\hline
0\ 0\cdots0 & & 0 \\
\hline
1\ 0\cdots0 & & 1/0 \\
\hline
0\ 0\cdots0 & & 0 \\
\hline
O_{s-5} & \ddots & \vdots \\
\cline{3-3}
& & O_1 \\
\hline
0\ 0\cdots0 & \cdots & 0 \\
\hline
0\ 0\cdots0 & & 0 \\
\hline
0\ 0\cdots0 & & 0 \\
\hline
4\ 0\cdots0 & & 7/0 \\
\hline
0\ 0\cdots0 & & 0 \\
\hline
O_{s-5} & \ddots & \vdots \\
\cline{3-3}
& & O_1 \\
\hline
\end{array}\right].$$

We observe that in the case of $s = 12$ we consider the element before $/$, otherwise the element after.

By the Gauss elimination algorithm, the blocks B_1, B_2, B_3 and B_4 become, respectively:

$\overline{B_1} =$

<table>
<tr><td rowspan="5">I_{s-2}</td><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>$\cdots$</td><td>0</td></tr>
<tr><td rowspan="4">O_{s-3}</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td rowspan="4">$\ddots$</td><td rowspan="3">$\vdots$</td></tr>
<tr><td rowspan="3">O_{s-4}</td><td>$0\,0\cdots0$</td></tr>
<tr><td rowspan="2">O_{s-5}</td></tr>
<tr><td>O_1</td></tr>
<tr><td>$0\cdots0$</td><td>$1\,0\cdots0$</td><td>$0\cdots0$</td><td>$1\,0\cdots0$</td><td rowspan="2"></td><td>$1/0$</td></tr>
<tr><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>0</td></tr>
<tr><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$1\,0\cdots0$</td><td>$\cdots$</td><td>$4/0$</td></tr>
<tr><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td rowspan="7">$\ddots$</td><td>0</td></tr>
<tr><td rowspan="6">O_{s-2}</td><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td rowspan="4">$\vdots$</td></tr>
<tr><td rowspan="5">O_{s-3}</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td></tr>
<tr><td rowspan="4">O_{s-4}</td><td>$0\,0\cdots0$</td></tr>
<tr><td rowspan="3">O_{s-5}</td></tr>
<tr><td>0</td></tr>
<tr><td>O_1</td></tr>
</table>

$\overline{B_2} =$

<table>
<tr><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>$\cdots$</td><td>0</td></tr>
<tr><td rowspan="4">O_{s-3}</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td rowspan="4">$\ddots$</td><td rowspan="3">$\vdots$</td></tr>
<tr><td rowspan="3">O_{s-4}</td><td>$0\,0\cdots0$</td></tr>
<tr><td rowspan="2">O_{s-5}</td></tr>
<tr><td>O_1</td></tr>
<tr><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td></td><td>0</td></tr>
<tr><td>$1\,0\cdots0$</td><td>$0\cdots0$</td><td>$1\,0\cdots0$</td><td>$\cdots$</td><td>$1/0$</td></tr>
<tr><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>$\cdots$</td><td>0</td></tr>
<tr><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$1\,0\cdots0$</td><td rowspan="7">$\ddots$</td><td>$4/0$</td></tr>
<tr><td>$0\,0\cdots0$</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td>0</td></tr>
<tr><td rowspan="5">O_{s-3}</td><td>$0\cdots0$</td><td>$0\,0\cdots0$</td><td rowspan="3">$\vdots$</td></tr>
<tr><td rowspan="4">O_{s-4}</td><td>$0\,0\cdots0$</td></tr>
<tr><td rowspan="3">O_{s-5}</td></tr>
<tr><td>0</td></tr>
<tr><td>O_1</td></tr>
</table>

$$\overline{B_3} = \begin{array}{|c|c|c|c|}
\hline
0\cdots 0 & 1\,0\cdots 0 & \cdots & 1/0 \\
\hline
0\cdots 0 & 0\,0\cdots 0 & & 0 \\
\hline
 & 0\,0\cdots 0 & & \vdots \\
I_{s-4} & O_{s-5} & \ddots & 0 \\
 & & & O_1 \\
\hline
0\cdots 0 & 0\,0\cdots 0 & & 0 \\
\hline
0\cdots 0 & 0\,0\cdots 0 & & 0 \\
\hline
0\cdots 0 & 0\,0\cdots 0 & \cdots & 0 \\
\hline
0\cdots 0 & 0\,0\cdots 0 & & 0 \\
\hline
0\cdots 0 & 2\,0\cdots 0 & & 5/0 \\
\hline
0\cdots 0 & 0\,0\cdots 0 & & 0 \\
\hline
 & 0\,0\cdots 0 & & \vdots \\
I_{s-4} & O_{s-5} & \ddots & 0 \\
 & & & O_1 \\
\hline
\end{array}, \quad
\overline{B_4} = \begin{array}{|c|c|c|}
\hline
0\,0\cdots 0 & \cdots & 0 \\
\hline
1\,0\cdots 0 & & 1/0 \\
\hline
0\,0\cdots 0 & & 0 \\
\hline
 & & \vdots \\
O_{s-5} & \ddots & 0 \\
 & & O_1 \\
\hline
0\,0\cdots 0 & & 0 \\
\hline
0\,0\cdots 0 & & 0 \\
\hline
0\,0\cdots 0 & \cdots & 0 \\
\hline
0\,0\cdots 0 & & 0 \\
\hline
0\,0\cdots 0 & & 0 \\
\hline
2\,0\cdots 0 & & 5/0 \\
\hline
0\,0\cdots 0 & & 0 \\
\hline
 & & \vdots \\
O_{s-5} & \ddots & 0 \\
 & & O_1 \\
\hline
\end{array}.$$

Now it's easy to check that (2) holds, as desired. □

As a consequence we get immediately:

THEOREM 6.2. – *For all $h \geq 12$, $\sigma_3(S_h)$ has the expected dimension.*

REFERENCES

[1] H. ABO - G. OTTAVIANI - C. PETERSON, *Non defectivity of Grassmannians of planes*, arXiv:0901.2601v1 [math.AG], to appear on Journal of Algebraic Geometry.

[2] H. ABO - G. OTTAVIANI - C. PETERSON, *Induction for secant varieties of Segre varieties*, Trans. Amer. Math. Soc., **361**, no. 2 (2009), 767-792.

[3] J. ALEXANDER - A. HIRSCHOWITZ, *Polinomial interpolation in several variables*, Journal of Algebraic Geometry, **4**, n. 2 (1995), 201-222.

[4] E. ANGELINI, *Varietà secanti alle varietà spinoriali*, Laurea Thesis, Università di Firenze (2009).

[5] A. BORALEVI - J. BUCZYNSKI, *Secants of Lagrangian Grassmannians*, arXiv:1006.1925v1 [math.AG], to appear on Annali di Matematica.

[6] K. BAUR - J. DRAISMA - W. DE GRAAF, *Secant dimensions of minimal orbits: computations and conjectures*, Experimental Mathematics, **16**, no. 2 (2007), 239-250.

[7] M. C. BRAMBILLA - G. OTTAVIANI, *On the Alexander-Hirschowitz Theorem*, J. Pure Appl. Algebra, **212** (2008), 1229-1251.

[8] C. CHEVALLEY, *The algebraic theory of spinors*, Columbia University Press, New York (1954).

[9] L. Chiantini - C. Ciliberto, *Weakly defective varieties* , Trans. Amer. Math. Soc., **354**, no 1 (2001), 151-178.

[10] M. V. Catalisano - A. V. Geramita - A. Gimigliano, *Secant varietes of* $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ *(n- times) are NOT Defective for* $n \geq 5$", arXiv:0809.1701, to appear on Journal of Algebraic Geometry.

[11] H. Kaji, *Homogeneous projective varieties with degenerate secants*, Trans. Amer. Math. Soc., **351**, no 2 (1999), 533-545.

[12] D. R. Grayson - M. E. Stillman, *Macaulay2*, Software system available at http://www.math.uiuc.edu/Macaulay2/.

[13] J. M. Landsberg, *The geometry of Tensors with applications*, Book in preparation (2009).

[14] J. M. Landsberg - J. Weyman, *On secant varieties of Compact Hermitian Symmetric Spaces*, J. Pure Appl. Algebra, **213** (2009), 2075-2086.

[15] L. Manivel, *On spinor varieties and their secants*, SIGMA 5 (2009) 078, special volume "Elie Cartan and Differential Geometry".

[16] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of Mathematical Monographs, **127**. AMS, Providence, RI, 1993.

Dipartimento di Matematica "Ulisse Dini"
Viale G. B. Morgagni, 67/a - 50134 Firenze, Italy
e-mail: elena.angelini@math.unifi.it

Received November 17, 2010 and in revised form December 29, 2010