BOLLETTINO
UNIONE MATEMATICA ITALIANA

FRANCESCO ALTOMARE

Corrigendum to "Asymptotic Formulae for Bernstein-Schnabl Operators and Smoothness" (Boll. U.M.I (9) II (2009), 135-150)

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.2, p. 259–262.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_2_259_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 259-262

Corrigendum to "Asymptotic Formulae for Bernstein-Schnabl Operators and Smoothness" (Boll. U.M.I (9) II (2009), 135-150)

FRANCESCO ALTOMARE

Contrary to what is asserted in Theorem 7 of [1], properties (*ii*) and (*iii*) in the statement of that theorem are not equivalent to the remaining ones. Accordingly, Corollaries 9 and 10 as well as Theorem 13 of [1], that depends on this wrong equivalence, require to be put right.

In this Note we fill this gap by stating the right condition that is equivalent to the above mentioned properties (*ii*) and (*iii*) and, accordingly, we properly modify Corollaries 9 and 10 and Theorem 13 of [1].

Consider the sequence $(B_n)_{n\geq 1}$ of Bernstein-Schnabl operators on $C([0,1])$ associated with a continuous selection $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ ([1, Section 2, formula (2.7)]) and denote by $(T(t))_{t\geq 0}$ the associated Feller semigroup on $C([0,1])$ ([1, Section 4, formula (4.3)]). We also assume that properties (2.3) and (2.4) of [1] are fulfilled. Set

$$\alpha(x) := \frac{1}{2}\left(\int_0^1 e_2 d\mu_x - x^2\right) \quad (0 \leq x \leq 1) \tag{1}$$

and, for $M \geq 0$, denote by $C^2_{M,\alpha}([0,1])$ the subset of all functions $f \in C^1([0,1])$ such that f' is absolutely continuous and

$$\alpha(x) \mid f''(x) \mid \leq M \quad \textit{for a.e. } x \in [0,1]. \tag{2}$$

Theorem 7 of [1] is to be replaced by the following two statements.

THEOREM 7.1. – *Given* $f \in C([0,1])$ *and* $M \geq 0$, *the following statements are equivalent:*

(i) For every $n \geq 1$ *and* $x \in [0,1]$

$$\mid B_n(f)(x) - f(x) \mid \leq \frac{M\alpha(x)}{n}.$$

(ii) $f \in C^1([0,1])$ *and* f' *is Lipschitz continuous with Lipschitz constant M.*

If, in addition, α *is concave, then statements (i) and (ii) are also equivalent to*

(*iii*) *For every* $t \geq 0$ *and* $x \in [0,1]$

$$\mid T(t)(f)(x) - f(x) \mid \leq Mt\alpha(x).$$

THEOREM 7.2. – *Given* $f \in C([0,1])$ *and* $M \geq 0$, *the following statements are equivalent:*

(i) For every $n \geq 1$

$$\| B_n f - f \|_\infty \leq \frac{M}{n}.$$

(ii) For every $t \geq 0$

$$\| T(t)f - f \|_\infty \leq M\, t.$$

(iii) $f \in C^2_{M,\alpha}([0,1])$.

PROOF OF THEOREM 7.1. – The equivalence $(i) \Leftrightarrow (ii)$ follows from [5, Satz 3.10] taking Proposition 1 of [1] into account.

$(i) \Rightarrow (iii)$. The proof is the same we used to show the implication $(i) \Rightarrow (v)$ of Theorem 7 of [1].

$(iii) \Rightarrow (ii)$. By arguing as in the proof of the implication $(iii) \Rightarrow (iv)$ of Theorem 7 of [1], we get that, if a function $u \in C([0,1])$ is differentiable in a neighborhood of a point x_0 of $]0,1[$ and if, in addition, it is two times differentiable at it, then

$$\lim_{t \to 0^+} \frac{T(t)u(x_0) - u(x_0)}{t} = \alpha(x_0)u''(x_0). \tag{3}$$

Therefore, by applying again Satz 3.10 of [5] to the sequence of positive linear operators $(T(1/n))_{n \geq 1}$, the result follows. □

PROOF OF THEOREM 7.2. – The proof of the implication $(i) \Rightarrow (ii)$ is the same as the one of the implication $(ii) \Rightarrow (iii)$ of Theorem 7 of [1]. Assume now that property (ii) is fulfilled. Then, setting $L_n := T(1/n)$ for each $n \geq 1$, we get

$$\sup_{n \geq 1} n \| L_n(f) - f \| \leq M.$$

Moreover, by (3),

$$\lim_{n \to \infty} n(L_n(u)(x_0) - u(x_0) = \alpha(x_0)u''(x_0)$$

for every $x_0 \in\,]0,1[$ and for every function $u \in C([0,1])$ that is differentiable in a neighborhood of x_0 and two times differentiable at it. Therefore $f \in C^2_{M,\alpha}([0,1])$ because of Theorem 4.3 of [4] applied with $\rho = \mathbf{1}$, $\lambda_n = n$, $w_0 = w_1 = \mathbf{1}$ and $w_2 = 1/\alpha$ (see also Section 6 of [4] and, in particular, formulae (6.2) and (3.3)).

The implication $(iii) \Rightarrow (i)$ can be proved similarly by applying again Theorem 4.3 of [4] to the sequence $(B_n)_{n \geq 1}$ and by taking into account that the remainder $o(1)$ that appears in formula (4.5) of Theorem 4.3 of [4], is zero in this case because the operators B_n leave invariant the affine functions (see the proof of Lemma 4.4 of [4]). □

Consider $\alpha \in C([0,1])$ such that

(4) $$\alpha(0) = \alpha(1) = 0, \quad 0 < \alpha(x) \quad \text{for every } 0 < x < 1,$$

and

(5) $$\sup_{0<x<1} \frac{\alpha(x)}{x(1-x)} < +\infty.$$

In [2, Theorem 3.10] it was shown that all the functions α satisfying (4) and (5) can be written as $\alpha = M\tilde{\alpha}$ where $M \geq 0$ and $\tilde{\alpha}$ is given by (1) for a suitable continuous selection of probability Borel measures on $[0,1]$.

Consider the differential operator $A : D_V(A) \to C([0,1])$ defined by setting for every $u \in D_V(A)$

(6) $$A(u)(x) := \begin{cases} \alpha(x)u''(x) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, 1, \end{cases}$$

where $D_V(A)$ denotes the linear subspace of all functions $u \in C([0,1] \cap C^2(]0,1[)$ such that

(7) $$\lim_{x \to 0^+} \alpha(x)u''(x) = \lim_{x \to 1^-} \alpha(x)u''(x) = 0.$$

Let $\mathcal{T} = (T(t))_{t \geq 0}$ be the Feller semigroup on $C([0,1])$ generated by $(A, D_V(A))$ (see [1, p. 147]) and denote by $Fav(\mathcal{T})$ its Favard class, i.e., the linear subspace of all functions $f \in C([0,1])$ such that $\sup\limits_{0<t} \dfrac{\| T(t)f - f \|_\infty}{t} < +\infty$.

Corollaries 9 and 10 of [1] are to be modified as follows.

COROLLARY. – *Under the above assumptions,*

$$Fav(\mathcal{T}) = \bigcup_{M \geq 0} C^2_{M,\alpha}([0,1]).$$

Moreover, for every $M \geq 0$ *and* $t \geq 0$,

$$T(t)(C^2_{M,\alpha}([0,1])) \subset C^2_{M,\alpha}([0,1]).$$

The proof of the Corollary is similar to the ones of Corollaries 9 and 10 of [1] on the light of Theorem 7.1 and 7.2 of this Note.

Finally, Theorem 13 of [1] is to be replaced by the next one.

THEOREM 13. – *Given* $u \in C([0,1])$, *the following statements are equivalent:*
(i) There exists $\lim_{n \to \infty} n(B_n(u) - u)$ *uniformly on* $[0,1]$.
(ii) $u \in D_V(A)$, *i.e.,* $u \in C^2(]0,1[)$ *and*

$$\lim_{x \to 0^+} \alpha(x)u''(x) = \lim_{x \to 1^-} \alpha(x)u''(x) = 0.$$

PROOF. – $(i) \Rightarrow (ii)$. It is an immediate consequence of Theorem 11 of [1].

$(ii) \Rightarrow (i)$. On the subspace $D_V(A)$ consider the graph norm

$$\| u \|_A := \| u \|_\infty + \| A(u) \|_\infty \qquad (u \in D_V(A)).$$

Since $(A, D_V(A))$ is closed, $(D_V(A), \| \cdot \|_A)$ is a Banach space. Moreover, by Theorem 3.4 of [2] (see also the note added in proof), $C^2([0,1])$ is a core for $(A, D_V(A))$ and hence it is dense in $(D_V(A), \| \cdot \|_A)$. Note that, if $v \in D_V(A)$, then $v \in C^2_{M,\alpha}([0,1]$, where $M := \sup_{0<x<1} \alpha(x) \mid v''(x) \mid$, and hence, by Theorem 7.2, $\sup_{n\geq 1} n \| B_n v - v \|_\infty < +\infty$. Therefore, the sequence of bounded linear operators $(n(B_n - I))_{n\geq 1}$ (I denoting the identity operator) is pointwise bounded on $D_V(A)$ and hence, by the uniform boundedness principle, it is equibounded. On the other hand, if $v \in C^{(2)}([0,1])$, then

$$\lim_{n\to\infty} n(B_n(v) - v) = A(v) \qquad \text{uniformly on } [0,1]$$

([2, Theorem 3.1]); consequently, by a density argument, we obtain a similar formula for every $u \in D_V(A)$ and so the result follows. □

REMARK. – For the classical Bernstein operators, Theorem 13 has been also obtained with different methods in ([3, Section 4, Remark to Lemma 4]).

Acknowledgements. We would like to thank Dr. Cristian Tacelli for raising a question that led us to seek and to find out the gap in the proof of Theorem 7 of [1].

REFERENCES

[1] F. ALTOMARE, *Asymptotic formulae for Bernstein-Schnabl operators and smoothness*, Boll. U.M.I. (9) **II** (2009), 139-150.

[2] F. ALTOMARE - V. LEONESSA - I. RAŞA, *On Bernstein-Schnabl operators on the unit interval,* J. for Anal. and its Appl., **27**, no. 3 (2008), 353-379.

[3] H. BERENS - G. G. LORENTZ, *Inverse theorems for Bernstein polynomials*, Indiana Univ. Math. Journal, **21**, no. 8 (1972), 693-708.

[4] G. G. LORENTZ - L. L. SCHUMAKER, *Saturation of positive operators*, J. Approx. Theory, **5** (1972), 413-424.

[5] G. MÜLBACH, *Operatoren vom Bernsteinschen Typ*, J. Approx. Theory, **3** (1970), 274-292.

Dipartimento di Matematica, Università degli Studi di Bari "A. Moro"
Campus Universitario, Via E. Orabona, 4, 70125 Bari - Italia
E-mail: altomare@dm.uniba.it

Received December 21, 2010 and in revised form January 20, 2011