Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.2, p. 263–273.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_2_263_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 263-273

Gysin Map and Atiyah-Hirzebruch Spectral Sequence

FABIO FERRARI RUFFINO

Abstract. – *We discuss the relations between the Atiyah-Hirzebruch spectral sequence and the Gysin map for a multiplicative cohomology theory, on spaces having the homotopy type of a finite CW-complex. In particular, let us fix such a multiplicative cohomology theory h^* and let us consider a smooth manifold X of dimension n and a compact submanifold Y of dimension p, satisfying suitable hypotheses about orientability. We prove that, starting the Atiyah-Hirzebruch spectral sequence with the Poincaré dual of Y in X, which, in our setting, is a simplicial cohomology class with coefficients in $h^0\{*\}$, if such a class survives until the last step, it is represented in $E_\infty^{n-p,0}$ by the image via the Gysin map of the unit cohomology class of Y. We then prove the analogous statement for a generic cohomology class on Y.*

1. – Introduction

Given a multiplicative cohomology theory, under suitable hypotheses we can define the Gysin map, which is a natural pushfoward in cohomology. Moreover, for a finite CW-complex or any space homotopically equivalent to it, we can construct the Atiyah-Hirzebruch spectral sequence, which relates the cellular cohomology with the fixed cohomology theory. In particular, the groups of the starting step of the spectral sequence $E_1^{p,q}(X)$ are canonically isomorphic to the groups of cellular cochains $C^p(X, h^q\{*\})$, for $\{*\}$ a fixed space with one point. Since the first coboundary $d_1^{p,q}$ coincides with the cellular coboundary, the groups $E_2^{p,q}(X)$ are canonically isomorphic to the cellular cohomology groups $H^p(X, h^q\{*\})$. The sequence stabilizes to $E_\infty^{p,q}(X)$ and, denoting by X^p the p-skeleton of X, there is a canonical isomorphism:

$$E_\infty^{p,q}(X) \simeq \frac{\mathrm{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^{p-1}))}{\mathrm{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^p))} \tag{1}$$

i.e. $E_\infty^{p,q}$ can be described as the group of $(p+q)$-classes on X which are 0 when pulled back to X^{p-1}, up to classes which are 0 when pulled back to X^p. Let us now consider an n-dimensional smooth manifold X and a compact p-dimensional submanifold Y. For $i : Y \to X$ the embedding, we can define the Gysin map:

$$i_! : h^*(Y) \longrightarrow \tilde{h}^{*+n-p}(X)$$

which in particular gives a map $i_! : h^0(Y) \longrightarrow \tilde{h}^{n-p}(X)$. We assume that we have an oriented triangulation of X restricting to a triangulation of Y (this is always possible for X orientable [8]): we require that Y is a cycle in $C_p(X, h^0\{*\})$, identifying each simplex σ of the triangulation with $\sigma \otimes_{\mathbb{Z}} 1$, for $1 \in h^0\{*\}$. Then, for $\eta \in h^0\{*\}$ and $P : Y \to \{*\}$, we prove that $i_!(P^*\eta)$ represents an element of $\mathrm{Ker}(h^{p+q}(X) \to h^{p+q}(X^{p-1}))$ (the latter being the numerator of (1)) and, if the Poincaré dual $\mathrm{PD}_X[Y \otimes \eta] \in H^{n-p}(X, h^0\{*\})$ survives until the last step, its class in $E_\infty^{n-p,0}$ is represented exactly by $i_!(P^*\eta)$. More generally, without assuming $q = 0$, if $Y \otimes a$ is a cycle in $C_p(X, h^q\{*\})$ for $a \in h^q\{*\}$, and if $\mathrm{PD}_X[Y \otimes a] \in H^{n-p}(X, h^q\{*\})$ survives until $E_\infty^{n-p,q}$, then its class in (1) is represented by $i_!(P^*a)$. All the classes on Y considered in these examples are pull-back of classes in $h^*\{*\}$: we will see that all the other classes give no more information.

The study of the relations between Gysin map and Atiyah-Hirzebruch spectral sequence was treated in [6] for K-theory, arising from the physical problem of relating two different classifications of D-brane charges in string theory. In this article we generalize the statement to any multiplicative cohomology theory.

2. – Spectral sequences and orientability

2.1 – *Atiyah-Hirzebruch spectral sequence*

We deal with spectral sequences in the axiomatic version described in [4], chap. XV, par. 7, with the additional hypothesis of working with *finite* sequences of groups. We also take into account the presence of the grading in cohomology. For a finite simplicial complex X we consider the natural filtration:

$$\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^m = X$$

where X^i is the i-th skeleton of X. The groups and maps of the Atiyah-Hirzebruch spectral sequence of X, associated to a cohomology theory h^*, are defined as follows (for the groups $H^\bullet(p, p')$ and the map $\delta^\bullet$ we use the notation of [4], the map that we called $\psi^\bullet$ has no name in [4]):

- $H^n(p, p') = h^n(X^{p'-1}, X^{p-1})$ for $p \leq p'$;
- $\psi^n : H^n(p + a, p' + b) \to H^n(p, p')$ is induced in cohomology by the map of couples $i : (X^{p'-1}, X^{p-1}) \to (X^{p'+b-1}, X^{p+a-1})$;
- $\delta^n : H^n(p, p') \to H^{n+1}(p', p'')$ is the composition of the map $\pi^* : h^n(X^{p'-1}, X^{p-1}) \to h^n(X^{p'-1})$ induced by the map of couples $\pi : (X^{p'-1}, \emptyset) \to (X^{p'-1}, X^{p-1})$, and the Bockstein map $\beta^n : h^n(X^{p'-1}) \longrightarrow h^{n+1}(X^{p''-1}, X^{p'-1})$.

We briefly show how to link the first and the last step of the sequence. We consider the diagram:

(2)
$$\begin{array}{ccc} \tilde{h}^{p+q}(X/X^{p-1}) & \xrightarrow{\quad (f^p)^* \quad} & \tilde{h}^{p+q}(X^p) \\ & {\scriptstyle (i^{p,p-1})^*} \searrow \qquad \nearrow {\scriptstyle (\pi^{p,p-1})^*} & \\ & \tilde{h}^{p+q}(X^p/X^{p-1}) & \end{array}$$

for $\pi^{p,p-1} : X^p \to X^p/X^{p-1}$ the natural projection, $i^{p,p-1} : X^p/X^{p-1} \to X/X^{p-1}$ the natural immersion and $f^p = i^{p,p-1} \circ \pi^{p,p-1}$. The classes in $E_1^{p,q}(X) \simeq \tilde{h}^{p+q}(X^p/X^{p-1})$ surviving until the last step are the ones which belong to the image of $(i^{p,p-1})^*$, i.e. which are restrictions of a class defined on all X/X^{p-1}. For such a class a, if we denote by $\{a\}_{E_\infty^{p,q}}$ its image in the last step $E_\infty^{p,q}(X) \simeq \tilde{h}^{p+q}(X^p)$, we have:

(3) $$\{a\}_{E_\infty^{p,q}} = (\pi^{p,p-1})^*(a) \; .$$

2.2 – *Orientability and Gysin map*

Let h^* be a multiplicative cohomology theory [5]. Given a path-wise connected space X, we consider any map $p : \{*\} \to X$: by the path-wise connectedness of X two such maps are homotopic, thus the pull-back $p^* : h^*(X) \to h^*(\{*\})$ is well defined.

DEFINITION 2.1. – *For X a path-connected space we call* rank *of a cohomology class $a \in h^n(X)$ the class* $\mathrm{rk}(a) := (p^*)^n(a) \in h^n(\{*\})$ *for any map $p : \{*\} \to X$.*

Let us consider the unique map $P : X \to \{*\}$.

DEFINITION 2.2. – *We call a cohomology class $a \in h^n(X)$ trivial if there exists $\beta \in h^n\{*\}$ such that $a = (P^*)^n(\beta)$. We denote by* 1 *the class* $(P^*)^0(1)$.

It is easy to show that, for X a path-wise connected space, a trivial chomology class $a \in h^n(X)$ is the pull-back of its rank.

Let $\pi : E \to B$ be a fiber bundle with fiber F and E' a sub-bundle of E with fiber $F' \subset F$. We have a natural diagonal map $\Delta_\pi : (E, E') \to (B \times E, B \times E')$ given by $\Delta_\pi(e) = (\pi(e), e)$ so that we can define the module structure:

(4) $$h^i(B) \times h^j(E, E') \xrightarrow{\times} h^{i+j}(B \times E, B \times E') \xrightarrow{\Delta_\pi^*} h^{i+j}(E, E') \; .$$

The module structure (4) is unitary [5], i.e. $1 \cdot a = a$ for 1 defined by 2.2. More generally, for a trivial class $t = P^*(\eta)$, with $\eta \in h^*(\{*\})$, one has $t \cdot a = \eta \cdot a$.

We recall that a vector bundle $\pi : E \to B$ be of rank k is called *h-orientable* if there exists a Thom class $u \in h^k(E, E \setminus E_0)$, for E_0 the zero-section of E [9]. Let (U_a, φ_a) be a contractible local chart for E, with $\varphi_a : \pi^{-1}(U_a) \to U_a \times \mathbb{R}^k$. Let us consider the compactification $\varphi_a^+ : \pi^{-1}(U_a)^+ \to (U_a \times \mathbb{R}^k)^+$, restricting, for $x \in U_a$, to $(\varphi_a)_x^+ : E_x^+ \to S^k$. Then we can consider the map:

$$\hat{\varphi}_{a,x} := ((\varphi_a)_x^{+\,-1})^{*k} : \tilde{h}^k(E_x^+) \longrightarrow \tilde{h}^k(S^k) \; . \tag{5}$$

The proof of the following lemma and theorem can be found in [9], chap. V.

LEMMA 2.1. – *Let u be an h-orientation of a rank-n vector bundle $\pi : E \to B$, let (U_a, φ_a) be a contractible local chart for E and let $\hat{\varphi}_{a,x}$ be defined by* (5)*. Then $\hat{\varphi}_{a,x}(u|_{E_x^+})$ is constant in x with value γ^k or $-\gamma^k$.*

THEOREM 2.2. – *If a vector bundle $\pi : E \to B$ of rank k is h-orientable, then given trivializing contractible charts $\{U_a\}_{a \in I}$ it is always possible to choose trivializations $\varphi_a : \pi^{-1}(U_a) \to U_a \times \mathbb{R}^k$ such that $(\varphi_a^+)_x^{*k}(\gamma^k) = u|_{E_x^+}$. In particular, for $x \in U_{a\beta}$ the homeomorphism $(\varphi_\beta \varphi_a^{-1})_x^+ : (\mathbb{R}^k)^+ \simeq S^k \longrightarrow (\mathbb{R}^k)^+ \simeq S^k$ satisfies $((\varphi_\beta \varphi_a^{-1})_x^+)^*(\gamma^k) = \gamma^k$.*

Therefore, we can give the following definition:

DEFINITION 2.3. – *An atlas satisfying the conditions of Theorem* 2.2 *is called h-oriented atlas.*

LEMMA 2.3. – *Let $\pi : E \to B$ be an h^*-orientable vector bundle of rank k, for h^* a multiplicative cohomology theory. Then E is orientable also with respect to the singular cohomology with coefficients in $h^0\{*\}$. Therefore, if* $\mathrm{char}(h^0\{*\}) > 2$*, it is orientable in the usual sense. In particular, an atlas is h-oriented with respect to u or $-u$ if and only if it is oriented.*

PROOF. – We call $\{\varphi_{a\beta}\}$ the transition functions, and $\{\varphi_{a\beta}^+\}$ their extension to the compactified fibers. Since $\varphi_{a\beta}^+$ is a homeomorphism, it has degree 1 or -1, and the degree of a map is independent of the cohomology theory [3]. If $\mathrm{char}(h^0\{*\}) > 2$, an atlas is h-oriented, with respect to u or $-u$, if and only if the degree of each $\varphi_{a\beta}^+$ is 1 and not -1, since $\varphi_{a\beta}^+(\gamma^k) = \gamma^k$ (Theorem 2.2). The degree of $\varphi_{a\beta}^+$ is 1 if and only if the determinant of $\varphi_{a\beta}$ is positive, thus the thesis follows. If $\mathrm{char}(h^0\{*\}) = 2$ the thesis is trivial. □

Let X be a compact smooth n-manifold and $Y \subset X$ a compact embedded p-dimensional submanifold, such that the normal bundle $N(Y) = (TX|_Y)/TY$ is h-

orientable. Then, since Y is compact, there exists a tubular neighborhood U of Y in X [3], i.e. there exists a homeomorphism $\varphi_U : U \to N(Y)$. If $i : Y \to X$ is the embedding, from this data we can define the Gysin map:

$$i_! : h^*(Y) \longrightarrow \tilde{h}^{*+n-p}(X).$$

In fact, we first apply the Thom isomorphism ([5] page 7) $T : h^*(Y) \longrightarrow h^{*+n-p}_{\text{cpt}}(N(Y)) = \tilde{h}^{*+n-p}(N(Y)^+)$; then we naturally extend φ_U to $\varphi_U^+ : U^+ \to N(Y)^+$ and apply $(\varphi_U^+)^* : h^*_{\text{cpt}}(N(Y)) \to h^*_{\text{cpt}}(U)$; finally, considering the natural map $\psi : X \to U^+$, which sends $X \setminus U$ to the point at infinity, we apply $\psi^* : \tilde{h}^*(U^+) \longrightarrow \tilde{h}^*(X)$. Summarizing:

$$i_!(a) = \psi^* \circ \left(\varphi_U^+\right)^* \circ T(a). \tag{6}$$

REMARK. – One could try to use the immersion $i : U^+ \to X^+$ and the retraction $r : X^+ \to U^+$ to have a splitting $h(X) = h(U) \oplus h(X, U) = h(Y) \oplus h(X, U)$. But this is false, since the immersion $i : U^+ \to X^+$ is not continuous: *since X is compact*, $\{\infty\} \subset X^+$ is open, but $i^{-1}(\{\infty\}) = \{\infty\}$, and $\{\infty\}$ is not open in U^+ since U is non-compact.

3. – Gysin map and Atiyah-Hirzebruch spectral sequence

In this section we follow the same line of [6], generalizing the discussion to any cohomology theory. We call X an orientable compact smooth n-dimensional manifold, and Y a compact embedded p-dimensional submanifold. We choose a finite oriented triangulation of X which restricts to a triangulation of Y [8]. We use the following notation:

- we denote the triangulation of X by $\Delta = \{\Delta_i^m\}$, where m is the dimension of the simplex and i enumerates the m-simplices;
- we denote by X_Δ^p the p-skeleton of X with respect to Δ.

The same notation is used for other triangulations or simplicial decompositions of X and Y. We now need the definition of "dual cell decomposition" with respect to a triangulation: we refer to [7] pp. 53-54. The following theorem coincides with Theorem 5.1 of [6], therefore we remand there for the proof.

THEOREM 3.1. – *Let X be an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold. Let:*

- *$\Delta = \{\Delta_i^m\}$ be a triangulation of X which restricts to a triangulation $\Delta' = \{\Delta_{i'}^m\}$ of Y;*
- *$D = \{D_i^{n-m}\}$ be the dual decomposition of X with respect to Δ;*
- *$\tilde{D} \subset D$ be subset of D made by the duals of the simplices in Δ'.*

Then, calling $|\tilde{D}|$ the support of $\tilde{D}$:

- *the interior of $|\tilde{D}|$ is a tubular neighborhood of Y in X;*
- *the interior of $|\tilde{D}|$ does not intersect X_D^{n-p-1}, i.e.:*

$$|\tilde{D}| \cap X_D^{n-p-1} \subset \partial|\tilde{D}| \, .$$

We now consider quintuples $(X, Y, \varDelta, D, \tilde{D})$ satisfying the following condition:

(#) X is an n-dimensional compact manifold and $Y \subset X$ a p-dimensional embedded compact submanifold such that $N(Y)$ is h-orientable. Moreover, $\varDelta$, D and $\tilde{D}$ are defined as in Theorem 3.1.

The following lemma coincides with Lemma 5.2 of [6], where the reader can find the proof.

LEMMA 3.2. – *Let $(X, Y, \varDelta, D, \tilde{D})$ be a quintuple satisfying (#), $U = \mathrm{Int}|\tilde{D}|$ and $a \in h^*(Y)$. Then:*

- *there exists a neighborhood V of $X \setminus U$ such that $i_!(a)|_V = 0$;*
- *in particular, $i_!(a)\,|_{X_D^{n-p-1}} = 0$.*

3.1 – *Trivial classes*

We start by considering the case of the unit class $1 \in h^0(Y)$ (see def. 2.2). Before we have assumed X orientable for simplicity. We denote by H the singular cohomology with coefficeints in $h^0\{*\}$: then the correct hypothesis is that X must by H-orientable, since we need the Poincaré duality with respect to H. Therefore, the orientability of X is necessary only if char $h^0\{*\} > 2$. If the normal bundle $N_Y X$ of Y in X is h-orientable, as in our hypotheses, then it is also H-orientable, thanks to Lemma 2.3. Actually, it also follows from the following argument. Y is an H-orientable manifold: for char $h^0\{*\} = 2$ any bundle is orientable (thus also the tangent bundle TY), otherwise, being Y a simplicial complex, in order to be a cycle in $C_p(X, h^0\{*\})$ it must be oriented as a simplicial complex, thus also as a manifold. Since also X is H-orientable, it follows that both $TX|_Y$ and TY are H-orientable, hence also $N_Y X$ is. Moreover, the atlas arising in the proof of Theorem 3.1 is naturally H-oriented, as follows from the construction of the dual cell decomposition.

THEOREM 3.3. – *Let $(X, Y, \varDelta, D, \tilde{D})$ be a quintuple satisfying (#), with X H-orientable, and $\varPhi_D^{n-p} : C^{n-p}(X, h^0(\{*\})) \to h^{n-p+q}(X_D^{n-p}, X_D^{n-p-1})$ be the standard canonical isomorphism. Let us define the natural projection and immersion:*

$$\pi^{n-p,\,n-p-1} : X_D^{n-p} \longrightarrow X_D^{n-p}/X_D^{n-p-1} \qquad\qquad i^{n-p} : X_D^{n-p} \longrightarrow X$$

and let $\mathrm{PD}_\varDelta(Y)$ be the representative of the Poincaré dual (with respect to H) $\mathrm{PD}_X[Y]$ given by the sum of the cells dual to the p-cells of $\varDelta$ covering Y. Then:

$$(i^{n-p})^*(i_!(1)) = (\pi^{n-p,\,n-p-1})^*(\varPhi_D^{n-p}(\mathrm{PD}_\varDelta(Y))) \, .$$

PROOF. – Let U be the tubular neighborhood of Y in X stated in Theorem 3.1. We define the space $(U^{+})^{n-p}_{D}$ obtained considering the interior of the $(n-p)$-cells intersecting Y transversally and compactifying this space to one point. The interiors of such cells forms exactly the intersection between the $(n-p)$-skeleton of D and U, i.e. $X^{n-p}_{D}\,|_{U}$, since the only $(n-p)$-cells intersecting U are the ones intersecting Y, and their interior is complitely contained in U, as stated in Theorem 3.1. If we close this space in X we obtain the closed cells intersecting Y transversally, whose boundary lies entirely in X^{n-p-1}_{D}. Thus the one-point compatification of the interior is:

$$(U^{+})^{n-p}_{D} = \frac{\overline{X^{n-p}_{D}\,|_{U}}^{X}}{X^{n-p-1}_{D}\,|_{\partial U}}$$

so that there is a natural inclusion $(U^{+})^{n-p}_{D} \subset U^{+}$ sending the denominator to ∞ (the numerator is exactly $X^{n-p}_{\tilde{D}}$ of Theorem 3.1). We also define:

$$\psi^{n-p} = \psi\,|_{X^{n-p}_{D}} : X^{n-p}_{D} \longrightarrow (U^{+})^{n-p}_{D}\ .$$

The latter is well-defined since the $(n-p)$-simplices outside U and all the $(n-p-1)$-simplices are sent to ∞ by ψ. Calling I the set of indices of the $(n-p)$-simplices in D, calling S^{k} the k-dimensional sphere and denoting by $\dot{\bigcup}$ the one-point union of topological spaces, there are the following canonical homeomorphisms:

$$\xi^{n-p}_{X} : \pi^{n-p}(X^{n-p}_{D}) \xrightarrow{\simeq} \dot{\bigcup_{i\in I}} S^{n-p}_{i}$$

$$\xi^{n-p}_{U^{+}} : \psi^{n-p}(X^{n-p}_{D}) \xrightarrow{\simeq} \dot{\bigcup_{j\in J}} S^{n-p}_{j}$$

where $\{S^{n-p}_{j}\}_{j\in J}$, with $J \subset I$, is the set of $(n-p)$-spheres corresponding to the $(n-p)$-simplices with interior contanined in U, i.e. corresponding to $\pi^{n-p}\big(\overline{X^{n-p}_{D}\,|_{U}}\big)$. The homeomorphism $\xi^{n-p}_{U^{+}}$ is due to the fact that the boundary of the $(n-p)$-cells intersecting U is contained in ∂U, hence it is sent to ∞ by ψ^{n-p}, while all the $(n-p)$-cells outside U are sent to ∞: hence, the image of ψ^{n-p} is homeomorphic to $\dot{\bigcup_{j\in J}} S^{n-p}_{j}$ sending ∞ to the attachment point. We define:

$$\rho : \dot{\bigcup_{i\in I}} S^{n-p}_{i} \longrightarrow \dot{\bigcup_{j\in J}} S^{n-p}_{j}$$

as the natural projection, i.e. ρ is the identity of S^{n-p}_{j} for every $j \in J$ and sends all the spheres in $\{S^{n-p}_{i}\}_{i\in I\setminus J}$ to the attachment point. We have that:

$$\xi^{n-p}_{U^{+}} \circ \psi^{n-p} = \rho \circ \xi^{n-p}_{X} \circ \pi^{n-p,\,n-p-1}$$

hence:

$$(\psi^{n-p})^* \circ (\xi^{n-p}_{U^+})^* = (\pi^{n-p,\,n-p-1})^* \circ (\xi^{n-p}_X)^* \circ \rho^* \; . \tag{7}$$

We put $N = N(Y)$ and $\tilde{u}_N = (\varphi^+_U)^*(u_N)$, where u_N is the Thom class of the normal bundle. Since (4) is unitary, from equation (6) we get $i_!(1) = \psi^* \circ (\varphi^+_U)^*(u_N)$. Then:

$$(i^{n-p})^*(i_!(1)) = (i^{n-p})^* \psi^*(\tilde{u}_N) = (\psi^{n-p})^* \left(\tilde{u}_N \,\big|_{(U^+)^{n-p}_D}\right)$$

and

$$(\xi^{n-p}_X)^* \circ \rho^* \circ ((\xi^{n-p}_{U^+})^{-1})^* \left(\tilde{u}_N \,\big|_{(U^+)^{n-p}_D}\right) = \Phi^{n-p}_D(\mathrm{PD}_\Delta Y)$$

since:

- $\mathrm{PD}_\Delta(Y)$ is the sum of the $(n-p)$-cells intersecting U, oriented as the normal bundle;
- hence $((\xi^{n-p}_X)^{-1})^* \circ \Phi^{n-p}_D(\mathrm{PD}_\Delta(Y))$ gives a γ^{n-p} factor to each sphere S^{n-p}_j for $j \in J$ and 0 otherwise, orienting the sphere orthogonally to Y;
- but this is exactly $\rho^* \circ ((\xi^{n-p}_{U^+})^{-1})^*(\tilde{u}_N \,|_{(U^+)^{n-p}_D})$ since, by definition of orientability, the restriction of $\tilde{\lambda}_N$ must be $\pm\,\gamma^n$ for each fiber of N^+. We must show that the sign ambiguity is fixed: this follows from the fact that the atlas arising from the tubular neighborhood in Theorem 3.1 is H-oriented, as we pointed out at the beginning of this section. For the spheres outside U, that ρ sends to ∞, we have that:

$$\begin{aligned}\rho^*\left(\tilde{u}_N \,\big|_{(U^+)^{n-p}_D}\right)\Big|_{\dot{\bigcup}_{i\in I\setminus J} S^{n-p}_i} &= \rho^*\left(\tilde{u}_N \,\big|_{\rho(\dot{\bigcup}_{i\in I\setminus J} S^{n-p}_i)}\right)\\ &= \rho^*\left(\tilde{u}_N \,\big|_{\{\infty\}}\right) = \rho^*(0) = 0 \; .\end{aligned}$$

Hence, from equation (7):

$$\begin{aligned} i_!(Y \times \mathbb{C}) \,\big|_{X^{n-p}_D} &= (\psi^{n-p})^* \left(\tilde{u}_N \,\big|_{(U^+)^{n-p}_D}\right)\\ &= (\pi^{n-p,\,n-p-1})^* \circ (\xi^{n-p}_X)^* \circ \rho^* \circ ((\xi^{n-p}_{U^+})^{-1})^* \left(\tilde{u}_N \,\big|_{(U^+)^{n-p}_D}\right)\\ &= (\pi^{n-p,\,n-p-1})^* \Phi^{n-p}_D(\mathrm{PD}_\Delta Y) \; . \end{aligned}$$

□

Let us now consider any trivial class $P^*\eta \in h^q(Y)$. Since (4) is unitary, we have that $P^*\eta \cdot u_N = \eta \cdot u_N$, hence Theorem 3.3 becomes:

$$(i^{n-p})^*(i_!(P^*\eta)) = (\pi^{n-p,\,n-p-1})^*(\Phi^{n-p}_D(\mathrm{PD}_\Delta(Y \otimes \eta))) \; .$$

In fact, the same proof applies considering that $\eta \cdot u_N$ provides a factor $\eta \cdot \gamma^{n-p}$ instead of γ^{n-p} for each spere of N^+, with $\eta \in h^q(\{*\}) \simeq \tilde{h}^q(S^q)$.

The following theorem encodes the link between Gysin map and AHSS.

THEOREM 3.4. – *Let $(X, Y, \Delta, D, \tilde{D})$ be a quintuple satisfying (#), with X H-orientable, and $\Phi_D^{n-p} : C^{n-p}(X, h^q(\{*\})) \to h^{n-p+q}(X_D^{n-p}, X_D^{n-p-1})$ be the standard canonical isomorphism. Let us suppose that $\mathrm{PD}_\Delta Y$ is contained in the kernel of all the boundaries $d_r^{n-p,q}$ for $r \geq 1$. Then it defines a class:*

$$\{\Phi_D^{n-p}(\mathrm{PD}_\Delta(Y \otimes \eta))\}_{E_\infty^{n-p,q}} \in E_\infty^{n-p,q} \simeq \frac{\mathrm{Ker}(h^{n-p+q}(X) \longrightarrow h^{n-p+q}(X^{n-p-1}))}{\mathrm{Ker}(h^{n-p+q}(X) \longrightarrow h^{n-p+q}(X^{n-p}))} .$$

The following equality holds:

$$\{\Phi_D^{n-p}(\mathrm{PD}_\Delta(Y \otimes \eta))\}_{E_\infty^{n-p,q}} = [i_!(P^*\eta)] .$$

PROOF. – Considering the diagram:

$$(8) \qquad \begin{array}{ccccc} E_\infty^{n-p,q} = \mathrm{Im}\big(\tilde{h}^{n-p+q}(X/X_D^{n-p-1}) & & \xrightarrow{(f^{n-p})^*} & & \tilde{h}^{n-p+q}(X_D^{n-p})\big) \\ & \searrow^{(\pi^{n-p-1})^*} & & \nearrow^{(i^{n-p})^*} & \\ & & \tilde{h}^{n-p+q}(X) & & \end{array}$$

given a representative $a \in \mathrm{Im}\,(\pi_{n-r-1})^* = \mathrm{Ker}(h^{n-p+q}(X) \longrightarrow h^{n-p+q}(X_D^{n-p-1}))$, we have that $\{a\}_{E_\infty^{n-p,q}} = (i^{n-p})^*(a) = a\,|_{X_D^{n-p}}$. Moreover, we consider the diagram:

$$(9) \qquad \begin{array}{ccccc} E_\infty^{n-p,q} = \mathrm{Im}\big(\tilde{h}^{n-p+q}(X/X_D^{n-p-1}) & & \xrightarrow{(f^{n-p})^*} & & \tilde{h}^{n-p+q}(X_D^{n-p})\big) \\ & \searrow^{(i^{n-p,n-p-1})^*} & & \nearrow^{(\pi^{n-p,n-p-1})^*} & \\ & & \tilde{h}^{n-p+q}(X_D^{n-p}/X_D^{n-p-1}) . & & \end{array}$$

where $i^{n-p,n-p-1} : X_D^{n-p}/X_D^{n-p-1} \to X/X^{n-p-1}$ is the natural immersion. We have that:

- by formula (3) the class $\{\Phi_D^{n-p}(\mathrm{PD}_\Delta(Y \otimes \eta))\}_{E_\infty^{n-p,q}}$ is given in diagram 9 by $(\pi^{n-p,n-p-1})^*(\Phi_D^{n-p}(\mathrm{PD}_\Delta(Y \otimes \eta)))$;
- by Lemma 3.2 we have $i_!(1) \in \mathrm{Ker}(h^{n-p+q}(X) \to h^{n-p+q}(X_D^{n-p-1}))$, hence the class $[i_!(P^*\eta)]$ is well-defined in $E_\infty^{n-p,q}$, and, by exactness, $i_!(P^*\eta) \in \mathrm{Im}\,(\pi^{n-p-1})^*$;
- by Theorem 3.3 we have $(i^{n-p})^*(i_!(P^*\eta)) = (\pi^{n-p,n-p-1})^*(\Phi_D^{n-p}(\mathrm{PD}_\Delta(Y \otimes \eta)))$;
- hence $\{\Phi_D^{n-p}(\mathrm{PD}_\Delta(Y \otimes \eta))\}_{E_\infty^{n-p,q}} = [i_!(P^*\eta)]$. □

COROLLARY 3.5. – *Assuming the same data of the previous theorem, the fact that Y has orientable normal bundle with respect to h^* is a sufficient condition for $\mathrm{PD}_\Delta(Y)$ to survive until the last step of the spectral sequence. Thus, the Poincaré dual of any homology class $[Y] \in H_p(X, h^q\{*\})$ having a smooth representative with h-orientable normal bundle survives until the last step.*

PROOF. – We put together the diagrams (8) and (9):

$$\begin{array}{ccc} \tilde{h}^{n-p}(X/X_D^{n-p-1}) & \xrightarrow{(\pi^{n-p-1})^*} & \tilde{h}^{n-p}(X) \\ \downarrow{\scriptstyle (i^{n-p,n-p-1})^*} & \searrow{\scriptstyle (f^{n-p})^*} & \downarrow{\scriptstyle (i^{n-p})^*} \\ \tilde{h}^{n-p}(X_D^{n-p}/X_D^{n-p-1}) & \xrightarrow{(\pi^{n-p,n-p-1})^*} & \tilde{h}^{n-p}(X_D^{n-p}) \end{array} \tag{10}$$

and the diagram commutes being $\pi^{n-p,n-p-1} \circ i^{n-p,n-p-1} = i^{n-p} \circ \pi^{n-p-1}$. Under the hypotheses stated, we have that $i_!(1) \in \mathrm{Im}(\pi^{n-p-1})^*$, so that $i_!(1) = (\pi^{n-p-1})^*(a)$. Then $(i^{n-p})^*(a) \in A^{n-p,0}$, so that it survives until the last step giving a class $(i^{n-p})^*(\pi^{n-p})^*(a)$ in the last step. □

One could inquire if the condition of having h-orientable normal bundle is homology invariant. This is not true: let us consider the example of K-theory, for which a bundle is orientable if and only if it is a spin^c bundle. In [2] the authors show that in general, for a manifold X, there exist homologous submanifolds Y and Y', such that the normal bundle of Y is spin^c, while the normal bundle of Y' is not. Since the second step of the Atiyah-Hirzebruch spectral sequence coincides with the cohomology of X, this means that both $\mathrm{PD}_\Delta Y$ and $\mathrm{PD}_{\Delta'} Y'$ (for suitable Δ and Δ') survive until the last step, even if the normal bundle of Y' is not orientable. Then, it is natural to inquire if it is true that a cohomology class survives until the last step if and only if it admits smooth representatives with orientable normal bundle, but we do not know the answer.

3.2 – *Generic cohomology class*

If we consider a generic class a over Y of rank $\mathrm{rk}(a)$, we can prove that $i_!(E)$ and $i_!(P^*\mathrm{rk}(a))$ have the same restriction to X_D^{n-p}: in fact, the Thom isomorphism gives $T(a) = a \cdot u_N$ and, if we restrict $a \cdot u_N$ to a *finite* family of fibers, which are transversal to Y, the contribution of a becomes trivial, so it has the same effect of the trivial class $P^*\mathrm{rk}(a)$. We now prove this.

LEMMA 3.6. – *Let* $(X, Y, \Delta, D, \tilde{D})$ *be a quintuple satisfying* $(\#)$ *and* $a \in h^*(Y)$ *a class of rank* $\mathrm{rk}(a)$. *Then:*

$$(i^{n-p})^*(i_! a) = (i^{n-p})^*(i_!(P^*\mathrm{rk}\, a)) .$$

PROOF. – Since X_D^{n-p} intersects the tubular neighborhood in a finite number of cells corresponding under φ_U^+ to a finite number of fibers of the normal bundle N attached to one point, it is sufficient to prove that, for any $y \in Y$, $(a \cdot u_N)\,|_{N_y^+} = P^*\mathrm{rk}(a) \cdot u_N\,|_{N_y^+}$. Let us consider the following diagram for $y \in B$:

$$\begin{array}{ccc} h^i(Y) \times h^n(N_y, N'_y) & \xrightarrow{\times} & h^{i+n}(Y \times N, Y \times N') \\ \downarrow{\scriptstyle (i^*)^i \times (i^*)^n} & & \downarrow{\scriptstyle (i\times i)^{*\, i+n}} \\ h^i\{y\} \times h^n(N_y, N'_y) & \xrightarrow{\times} & h^{i+n}(\{y\} \times N_y, \{*\} \times N'_y)\ . \end{array}$$

The diagram commutes by naturality of the product, thus $(a \cdot u_N)|_{N_y^+} = a|_{\{y\}} \cdot u_N|_{N_y^+}$. Thus, we just have to prove that $a|_{\{y\}} = (P^* \operatorname{rk}(a))|_{\{y\}}$, i.e. that $i^*a = i^*P^*p^*a = (p \circ P \circ i)^*a$. This immediately follows from the fact that $p \circ P \circ i = i$. □

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Received November 3, 2010 and in revised form February 7, 2011