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Lewy-Stampacchia Inequality in Quasilinear Unilateral Problems and Application to the G-convergence

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a Italo (ardito, a che giammai ...) per i suoi 60 anni (+ε)

Abstract. – *In the paper* [5] *in collaboration with Italo Capuzzo Dolcetta, the use of the Lewy-Stampacchia inequality was the main tool for the study of the G-convergence in unilateral problems with linear differential operators. In this paper we prove a Lewy-Stampacchia inequality for unilateral problems with more general differential operators (quasilinear operators with lower order term having quadratic growth with respect to the gradient) in order to study the G-convergence in unilateral problems with such type of differential operators.*

1. – Introduction

We assume that Ω is bounded open set in $\mathbb{R}^N$, $N \geq 2$, and M is a measurable matrix such that

$$a|\xi|^2 \leq M(x)\xi\xi, \quad |M(x)| \leq \beta, \tag{1.1}$$

where a, $\beta > 0$. Moreover we assume that

$$f \in L^m(\Omega), \ m > \frac{N}{2}, \tag{1.2}$$

$$\mu \in \mathbb{R}^+, \tag{1.3}$$

$$\gamma \in \mathbb{R}. \tag{1.4}$$

In the paper [5] in collaboration with Italo Capuzzo Dolcetta, the use of the Lewy-Stampacchia inequality (see [11], [14]) was the main tool for the study of the G-convergence in unilateral problems with linear differential operators. In this paper we prove a Lewy-Stampacchia inequality for unilateral problems with more general differential operators (quasilinear operators with lower order term having quadratic growth with respect to the gradient) in order to study the G-convergence in unilateral problems with such type of differential operators.

To be more precise, in this paper, we will give a new (see [7], [1]) proof of the existence of a solution u for the following unilateral problem

$$(1.5)\qquad \begin{cases} u \geq 0,\ u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \int\limits_\Omega M(x)DuD[v-u] + \mu \int\limits_\Omega u[v-u] + \gamma \int\limits_\Omega |Du|^2[v-u] \geq \int\limits_\Omega f[v-u], \\ \text{for every } v \geq 0,\ \ v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

The proof is quite simple, thanks to the use of the homographic approximation, introduced in [10] and used in [6], [15], [16].

An additional advantage of this approach is that we easily deduce the following *Lewy-Stampacchia inequality*

$$(1.6)\qquad f \leq -\operatorname{div}(M(x)Du) + \mu u + \gamma|Du|^2 \leq f^+.$$

We point out that a similar result (with a different proof) can be found in [12].

Notice that our proof does not change if the principal part of the differential operator is nonlinear (even if the operator is defined in $W_0^{1,p}(\Omega)$, $p > 1$) or if the lower order term is more general, but with quadratic (or p) growth with respect to the gradient. On the contrary, if the obstacle is not zero, but a function $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, the scheme of the proof still holds, but more technical points are needed.

2. – Homographic approximation

We use the notation $\varepsilon = \dfrac{1}{n}$. Let u_ε be the solution of the following Dirichlet problem.

$$(2.1)\qquad \begin{cases} u_\varepsilon \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ -\operatorname{div}(M(x)Du_\varepsilon) + \mu u_\varepsilon + \dfrac{\gamma|Du_\varepsilon|^2}{1+\varepsilon|Du_\varepsilon|^2} + f^- \dfrac{u_\varepsilon}{\varepsilon + |u_\varepsilon|} = f^+. \end{cases}$$

LEMMA 2.1. – *Assume that hypotheses* (1.1), (1.2), (1.3) *and* (1.4) *are satisfied. Then,* $\forall\ \varepsilon > 0$, $u_\varepsilon \geq 0$ *almost everywhere in* Ω.

PROOF. – Let (see [7], [8], [9])

$$(2.2)\qquad \phi(t) = (e^{2\lambda|t|} - 1)\operatorname{sgn}(t), \qquad \lambda > \frac{|\gamma|}{2a}.$$

Note that, for every ε, u_ε is a bounded function (see [18]), thus it is possible to take

$\phi[(u_\varepsilon)^-]$ as test function in the weak formulation of (2.1). We get

$$-\int_\Omega M(x)D(u_\varepsilon)^-\,D(u_\varepsilon)^-\,\phi'[(u_\varepsilon)^-]-\mu\int_\Omega (u_\varepsilon)^-\phi[(u_\varepsilon)^-]$$

$$-\int_\Omega f^-\frac{(u_\varepsilon)^-}{\varepsilon+|u_\varepsilon|}\phi[(u_\varepsilon)^-]+\int_\Omega\frac{\gamma|Du_\varepsilon|^2}{1+\varepsilon|Du_\varepsilon|^2}\phi[(u_\varepsilon)^-]=\int_\Omega f^+\phi[(u_\varepsilon)^-],$$

which implies

$$\int_\Omega M(x)D(u_\varepsilon)^-\,D(u_\varepsilon)^-\,\phi'[(u_\varepsilon)^-]+\mu\int_\Omega (u_\varepsilon)^-\phi[(u_\varepsilon)^-]$$

$$+\int_\Omega f^-\frac{(u_\varepsilon)^-}{\varepsilon+|u_\varepsilon|}\phi[(u_\varepsilon)^-]=\int_\Omega\frac{\gamma|Du_\varepsilon|^2}{1+\varepsilon|Du_\varepsilon|^2}\phi[(u_\varepsilon)^-]-\int_\Omega f^+\phi[(u_\varepsilon)^-]$$

$$\leq|\gamma|\int_\Omega |Du_\varepsilon|^2\phi[(u_\varepsilon)^-]-\int_\Omega f^+\phi[(u_\varepsilon)^-],$$

which gives

$$\int_\Omega |D(u_\varepsilon)^-|^2\{a\phi'[(u_\varepsilon)^-]-|\gamma||\phi[(u_\varepsilon)^-]|\}\leq 0.$$

Since $a\phi'(s)-|\gamma||\phi(s)|\geq\frac{a}{2}$, by the choce of λ, the above inequality implies that $u_\varepsilon\geq 0$. Thus u_ε is solution of

$$(2.3)\qquad\begin{cases} u_\varepsilon\geq 0,\quad u_\varepsilon\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega):\\[2mm] -\mathrm{div}\,(M(x)Du_\varepsilon)+\mu u_\varepsilon+\dfrac{\gamma|Du_\varepsilon|^2}{1+\varepsilon|Du_\varepsilon|^2}+f^-\dfrac{u_\varepsilon}{\varepsilon+u_\varepsilon}=f^+.\end{cases}$$

Moreover we have, in the sense of distributions,

$$(2.4)\qquad f\leq-\mathrm{div}\,(M(x)Du_\varepsilon)+\mu u_\varepsilon+\frac{\gamma|Du_\varepsilon|^2}{1+\varepsilon|Du_\varepsilon|^2}\leq f^+,$$

since

$$0\leq f^-\frac{u_\varepsilon}{\varepsilon+u_\varepsilon}\leq f^-.$$

□

LEMMA 2.2. – *The sequence $\{u_\varepsilon\}$ is bounded in $L^\infty(\Omega)$.*

PROOF. – Following [9], we use $\phi(G_k(u_\varepsilon^+))$ as test function in (2.1), where $\phi(s)$ is defined in (2.2). Since $u_\varepsilon \geq 0$ and $f^- \geq 0$, we have

$$\int_\Omega f^- \frac{u_\varepsilon}{\varepsilon + u_\varepsilon} \phi(G_k(u_\varepsilon^+)) \geq 0. \tag{2.5}$$

Thus it is possible to repeat the proof of [9] in order to prove that the sequence $\{u_\varepsilon^+\}$ is bounded in $L^\infty(\Omega)$. □

LEMMA 2.3. – *The sequence* $\{u_\varepsilon\}$ *is bounded in* $W_0^{1,2}(\Omega)$.

PROOF. – Now we use $\phi(u_\varepsilon)$ as test function in (2.1). We repeat the remark (2.5), with $k = 0$. Thus it is possible to repeat the proof of [7], [8] in order to prove that the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,2}(\Omega)$. □

COROLLARY 2.4. – *There exist* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *and a subsequence, still denoted by* $\{u_\varepsilon\}$ *such that* u_ε *converges weakly to* u *in* $W_0^{1,2}(\Omega)$.

PROPOSITION 2.5. – u_ε *converges strongly to* u *in* $W_0^{1,2}(\Omega)$.

PROOF. – Use $\phi(u_\varepsilon - u)$ as test function and note that the term

$$\int_\Omega f^- \frac{u_\varepsilon}{\varepsilon + u_\varepsilon} \phi(u_\varepsilon - u)$$

converges to zero. Recall the results of Lemma 2.2 and Lemma 2.3 Then the statement is a consequence of a result proved in [7], [8], [9]. □

COROLLARY 2.6. – *Proposition* 2.5 *and Vitali Theorem imply that*

$$\frac{\gamma |Du_\varepsilon|^2}{1 + \varepsilon |Du_\varepsilon|^2} \to \gamma |Du|^2, \qquad \text{in } L^1(\Omega).$$

THEOREM 2.7. – *There exists a solution* u *of the unilateral problem* (1.5).

PROOF. – Let v be a positive function in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and use $v - u_\varepsilon$ as test function in (2.1). Then

$$\int_\Omega M(x) Du_\varepsilon D[v - u_\varepsilon] + \mu \int_\Omega u_\varepsilon [v - u_\varepsilon] + \int_\Omega \frac{\gamma |Du_\varepsilon|^2}{1 + \varepsilon |Du_\varepsilon|^2} [v - u_\varepsilon]$$
$$= \int_\Omega f^+ [v - u_\varepsilon] - \int_\Omega f^- \frac{u_\varepsilon}{\varepsilon + u_\varepsilon} [v - u_\varepsilon].$$

Note that

$$\lim_{\varepsilon\to 0}\int_\Omega f^- \frac{u_\varepsilon}{\varepsilon+u_\varepsilon} u_\varepsilon \geq \lim_{\varepsilon\to 0}\int_{\{x:u(x)>0\}} f^- \frac{u_\varepsilon}{\varepsilon+u_\varepsilon} u_\varepsilon = \int_{\{x:u(x)>0\}} f^- u = \int_\Omega f^- u \tag{2.6}$$

and that, for every $v \geq 0$,

$$\lim_{\varepsilon\to 0}\int_\Omega f^- \frac{u_\varepsilon}{\varepsilon+u_\varepsilon} v \leq \lim_{\varepsilon\to 0}\int_\Omega f^- \frac{u_\varepsilon}{u_\varepsilon} v = \int_\Omega f^- v. \tag{2.7}$$

Then

$$\lim_{\varepsilon\to 0}\int_\Omega f^+[v-u_\varepsilon] - \lim_{\varepsilon\to 0}\int_\Omega f^- \frac{u_\varepsilon}{\varepsilon+u_\varepsilon}[v-u_\varepsilon] \geq \int_\Omega f[v-u],$$

so that we pass to the limit in (2.3), thanks to the above limits, Lemma 2.2, Proposition 2.5, Corollary 2.6. Thus we show that u is solution of the unilateral problem (1.5). Moreover, passing to the limit in (2.4), we show that u verifies (1.6).

3. – G-convergence

Let us recall the notion of G-convergence, which was introduced for symmetric matrices by S. Spagnolo (see [17]) and generalized by Murat and Tartar (see [13]).

We will denote by $\mathcal{M}(a,\beta;\Omega)$ the class of $(N\times N)$-matrices with components in $L^\infty(\Omega)$ such that

$$a|\xi|^2 \leq M(x)\,\xi\,\xi, \qquad |M(x)| \leq \beta,$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$.

Definition 3.1. – *Let M_n be a sequence of matrices in $\mathcal{M}(a,\beta;\Omega)$. We will say that M_n G-converges to a matrix $M_0\in\mathcal{M}(a,\beta';\Omega)$ if, for every sequence f_n such that*

$$f_n \to f_0 \textit{ in } W^{-1,2}(\Omega)\textit{-strong},$$

the solutions z_n and z_0 of the equations

$$z_n \in W_0^{1,2}(\Omega):\ -\operatorname{div}(M_n(x)\nabla z_n) = f_n,$$

$$z_0 \in W_0^{1,2}(\Omega):\ -\operatorname{div}(M_0(x)\nabla z_0) = f_0$$

satisfy

$$z_n \rightharpoonup z_0 \textit{ in } W_0^{1,2}(\Omega)\textit{-weak}, \quad M_n(x)\nabla z_n \rightharpoonup M_0(x)\nabla z_0 \textit{ in } (L^2(\Omega))^N\textit{-weak}.$$

Now we state the following theorem which is a generalization of the main result of the paper [5] in collaboration with Italo Capuzzo Dolcetta.

THEOREM 3.2. – *Let M_n be a sequence of matrices in $\mathcal{M}(\alpha,\beta;\Omega)$ which G-converges to a matrix $M_0 \in \mathcal{M}(\alpha,\beta';\Omega)$. Assume* (1.2), (1.3), (1.4) *and that u_n is a solution of the unilateral problem*

$$(3.1)\qquad \begin{cases} u_n \geq 0,\ u_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega): \\ \int_\Omega M_n(x)Du_nD[v-u_n] + \mu\int_\Omega u_n[v-u_n] + \gamma\int_\Omega |Du_n|^2[v-u_n] \\ \geq \int_\Omega f[v-u_n], \\ \textit{for every } v\geq 0,\ v\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega). \end{cases}$$

Then it is possible to construct a function $H(x,\xi)$, measurable with respect to x, with the property

$$(3.2)\qquad |H(x,\xi)-H(x,\tilde{\xi})| \leq (|\xi|+|\tilde{\xi}|)|\xi-\tilde{\xi}|, \quad \forall\, \xi,\, \tilde{\xi}\in\mathbb{R}^N,$$

such that (up to subequences) u_n converges weakly to u_0, where u_0 is a solution of the unilateral problem

$$(3.3)\qquad \begin{cases} u_0 \geq 0,\ u_0 \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega): \\ \int_\Omega M_0(x)Du_0D[v-u_0] + \mu\int_\Omega u_0[v-u_0] + \int_\Omega H(x,Du_0)[v-u_0] \\ \geq \int_\Omega f[v-u_0], \\ \textit{for every } v\geq 0,\ v\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega). \end{cases}$$

PROOF. – Thanks to the Lewy-Stampacchia inequality (1.6), we have that

$$-\operatorname{div}(M_n(x)Du_n) + \mu u_n + \gamma|Du_n|^2$$

is bounded in $L^m(\Omega)$, $m > \dfrac{N}{2}$. Thus we can say that

$$u_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega):\ -\operatorname{div}(M_n(x)Du_n) + \mu u_n + \gamma|Du_n|^2 = g_n,$$

with $f \leq g_n \leq f^+$ and $\{g_n\}$ bounded in $L^m(\Omega)$, $m > \dfrac{N}{2}$. With the use of the Rellich Theorem we improve the above statement: up to subsequences, we have

$$[-\operatorname{div}(M_n(x)Du_n) + \mu u_n + \gamma|Du_n|^2] \to g_0, \quad \text{in } W^{-1,m^*}(\Omega),$$

with $f \leq g_0 \leq f^+$. Since $m^* \geq 2$, the results of G-convergence for quasilinear elliptic equations (see [2], [3], [4]) say that it is possible to construct a function $H(x, \xi)$, measurable with respect to x, satisfying (3.2), such that (up to subequences) u_n converges weakly to $u^* \geq 0$, where u^* is a solution of

$$u^* \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : -\mathrm{div}\,(M_0(x)Du^*) + \mu u^* + H(x, Du^*) = g_0$$

and

$$\begin{cases} -\mathrm{div}\,(M_n(x)Du_n) + \mu u_n \to -\mathrm{div}\,(M_0(x)Du^*) + \mu u^*, & \text{in } W^{-1,2}(\Omega); \\ \gamma|Du_n|^2 \to H(x, Du^*), & \text{in } L^1(\Omega). \end{cases}$$

Thus it is possible to pass to the limit in (3.1) and to say that u^* is a solution of (3.3).

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