Antonio Ambrosetti

On the Number of Solutions of Some Semilinear Elliptic Problems

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.3, p. 313–319.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_3_313_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 313-319

On the Number of Solutions of Some Semilinear Elliptic Problems

ANTONIO AMBROSETTI

Dedicated to the memory of Giovanni Prodi

Abstract. – *We show that a class of semilinear boundary value problems possess exactly one positive solution and one negative solution.*

1. – Introduction and main results

This paper is related to a topic marked by the pioneering researches by Giovanni Prodi on the geometric properties of nonlinear partial differential equations and sigularity theory. Precisely, we deal with semilinear elliptic Dirichlet boundary value problems like

$$(D_h)\qquad \begin{cases} -\Delta u &= \lambda u - f(u) + h(x) \quad x \in \Omega, \\ u(x) &= 0 \qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with $C^{0,\nu}$ boundary $\partial\Omega$, $h \in C^{0,\nu}(\overline{\Omega})$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(f1)\qquad f \in C^2(\mathbb{R}), f(0) = f'(0) = 0 \text{ and } f''(s)s > 0\ \forall\, s \neq 0$$

$$(f2)\qquad \lim_{s \to \pm\infty} \frac{f(s)}{s} = +\infty.$$

If $h = 0$ the problem (D_h) will be denoted by (D). By a solution of (D_h) we mean a $C_0^{2,\nu}(\Omega)$ classical solution.

In order to state our main results some notation is in order. If $m \in L^\infty(\Omega)$ is such that $meas\{x \in \Omega : m(x) > 0\} > 0$, the linear eigenvalue problem

$$(1)\qquad \begin{cases} -\Delta u &= \lambda m(x) u \quad x \in \Omega \\ u(x) &= 0 \qquad\qquad x \in \partial\Omega \end{cases}$$

has a sequence $0 < \lambda_1[m] < \lambda_2[m] \leq \ldots \leq \lambda_k[m] \leq \ldots$ of eigenvalues. If $m(x) \equiv 1$, we set $\lambda_k[m] = \lambda_k$. See e.g. [6, Chapter 1]. In particular, in the sequel we will use the monotonicity property of eigenvalues: let $\widehat{m}$ share the same properties of m;

if $m(x) \geq \widehat{m}(x)$ and $m(x) > \widehat{m}(x)$ in a subset of Ω with positive measure, then $\lambda_k[m] < \lambda_k[\widehat{m}]$, see [6, Prop. 1.12 A].

Using singularity theory jointly with topological degree and bifurcation theory, we will show:

THEOREM 1. – *Suppose that f satisfies $(f1-2)$ and let $\lambda > \lambda_1$. Then (D) has exactly one positive u_1 and one negative solution u_2. Moreover, if $\lambda \neq \lambda_k$, there exists $\varepsilon_\lambda > 0$ such that if $\|h\|_{L^2} \leq \varepsilon_\lambda$, then (D_h) has exactly one solution near each of $u = 0, u_1, u_2$.*

REMARKS 2. – (i) In [2] it has been proved that if $(f1-2)$ hold and $\lambda_1 < \lambda < \lambda_2$ then (D) has exactly one positive and one negative solution and (D) has no other solution. As a byproduct of our arguments we will give a simple proof of this fact, see Remark 10. Moreover, in [2] it is shown that if λ_2 is simple and $\lambda_2 < \lambda \leq \lambda_2 + \delta$, with $\delta > 0$ sufficiently small, (D) has exactly 4 non-trivial solutions. In addition, there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that (D_h) with $\|h\|_{L^2} \leq \varepsilon$ has precisely 3 solutions, resp. 5 solutions, provided $\lambda_1 < \lambda < \lambda_2$, resp. $\lambda_2 < \lambda \leq \lambda_2 + \delta$.

(ii) Problem (D_h), with $f(u) = u^3$ and homogeneous Neumann boundary conditions, has been also studied in [8]. Using the theory of singularities, it is evaluated the exact number of solutions of (D_h) (with no restriction on the norm of h) provided $\lambda_1 < \lambda < \lambda_1 + \Lambda$ for some $\Lambda > 0$. In general, the result cannot be extended to cover all $\lambda_1 < \lambda < \lambda_2$, see [9].

(iii) The first paper in which singularity theory is used to find geometric properties of a class of partial differential equations is [4]. As a consequence of a global inversion theorem in the presence of singularities, the precise number of multiple solutions of semilinear elliptic problems with *jumping nonlinearities* is established. See also [5] and references therein. □

2. – Preliminaries

We set $H = L^2(\Omega)$ with scalar product $(.\,|\,.)$ and norm $\|.\|$, and define $K \in L(H,H)$ by setting

$$Kv = u \iff -\Delta u = v, \quad v_{|\partial\Omega} = 0.$$

Let us consider $F_\lambda \in C^2(H,H)$,

$$F_\lambda(u) = u - \lambda K u - K f(u).$$

With this notation, $u \in H$ is a weak (and, by regularity, classical) solution of (D_h), resp. (D), whenever $F_\lambda(u) = Kh$, resp. $F_\lambda(u) = 0$.

We denote by Σ_λ the set of $u \in H$ such that $Ker[F'_\lambda(u)] \neq \{0\}$. The set Σ_λ is called the *singular set* of F_λ. See e.g. [3, Section 3].

REMARK 3. – If $u \in \Sigma_\lambda$, then there exists an integer $i \geq 1$ such that $\lambda_i[\lambda - f'(u)] = 1$. □

We denote by $\mathcal{S}_\lambda$ the set of nontrivial solutions of (D). It is well known that, if $\lambda > \lambda_1$ then (D) has at least 2 nontrivial solutions $u_1 > 0$ and $u_2 < 0$. This result can be proved in several manner: by variational methods, by sub- and super-solutions or by degree, cfr. e.g. [1]. See also Remark 11 later on.

Let us point out that $F_\lambda = Id - Compact$ and the solutions of (D) are bounded: there exists $C > 0$ such that

$$\|u\|_{L^2} \leq C, \qquad \forall\, u \in \mathcal{S}_\lambda. \tag{2}$$

As a consequence, for all $r > C$ there holds

$$deg(F_\lambda, B_r, 0) = 1, \tag{3}$$

where B_r denotes the ball of radius r in H and deg denotes the Leray-Schauder (LS for short) degree.

3. – Some lemmas

In this section we discuss some lemmas that will be used to prove Theorem 1. It is always understood that $(f1-2)$ hold.

Let $\mathcal{S}_{\lambda,1}$ be the set of $u \in \mathcal{S}_\lambda$ which do not change sign in Ω.

As remarked in Section [2], if $\lambda > \lambda_1$ then $\mathcal{S}_{\lambda,1} \neq \emptyset$.

LEMMA 4. – *Let $\lambda > \lambda_1$ and let $u \in \mathcal{S}_{\lambda,1}$. Then there exists a unique C^1 function $t_u = t_u(x)$ such that $0 < t_u(x) < 1$ in Ω, $t_u u \in H$ and $F'_\lambda(t_u u)[u] = 0$. Hence $t_u u \in \Sigma_\lambda$ and $\Sigma_\lambda \neq \emptyset$.*

PROOF. – Since $u(x) \neq 0$ for all $x \in \Omega$, the assumptions on f imply that there exists a unique $t_u = t_u(x)$ such that $0 < t_u(x) < 1$ in Ω, satisfying

$$f(u(x)) = f'(t_u(x)u(x))u(x). \tag{4}$$

Moreover, applying the elementary Implicit Function Theorem to $\gamma(t,x) := f'(tu(x))u(x) - f(u(x)) = 0$, it follows that $x \mapsto t_u(x)$ is C^1. In particular, $t_u u \in H$. We claim that $t_u u \in \Sigma_\lambda$. Since u is a solution of (D) one has

$$u = \lambda K u - K f(u). \tag{5}$$

Using (4) and (5), we get

$$F'_\lambda(t_u u)[u] = u - \lambda K u + K f'(t_u u) u = K f'(t_u u) u - K f(u) = 0.$$

Then $F'_\lambda(t_u u)[u] = 0$ has the nontrivial solution $u \neq 0$ and hence $t_u u \in \Sigma_\lambda$, as claimed. □

REMARK 5. – As a consequence of the results of [7] on the Unique Continuation Property, if $u \in \mathcal{S}_\lambda$ then the set $\Omega_u = \{x \in \Omega : u(x) = 0\}$ has zero measure. Actually, u satisfies $-\Delta u = V(x)u$ where $V \in L^\infty(\Omega)$ is defined by

$$V(x) = \begin{cases} \lambda - \dfrac{f(u(x))}{u(x)}, & \text{if } u(x) \neq 0; \\ \lambda, & \text{if } u(x) = 0. \end{cases}$$

Then we can repeat the proof of Lemma 4 in the case that u is any nontrivial solution of (D), showing that for all $x \in \Omega \setminus \Omega_u$, there exists a unique $t_u(x) \in (0,1)$ which is continuous a.e. in Ω, $t_u u \in H$ and $F'_\lambda(t_u u)[u] = 0$. □

EXAMPLE. – If $f(s) = |s|^{p-1}s$ then (4) yields $t_u(x) \equiv p^{-1/(p-1)}$. □

REMARK 6. – We do not exclude that $tu \in \Sigma_\lambda$ for some $t \neq t_u$. For example, if $u \in \mathcal{S}_{\lambda,1}$ and $\lambda > \lambda_2$, the continuous increasing function $\chi(t) := \lambda_2[\lambda - f'(tu)]$ is such that $\chi(0) = \lambda_2\lambda^{-1} < 1$ and $\chi(1) = \lambda_2[\lambda - f'(u)] > \lambda_1[\lambda - f'(u)] > 1$ (see Lemma 9 below). Hence $\exists\, t^* \in (0,1)$ such that $\lambda_2[\lambda - f'(t^* u)] = 1$. But, t_u is the only one such that $u \in Ker[F'_\lambda(t_u u)]$. □

If $\lambda \geq \lambda_1$ the fact that $\Sigma_\lambda \neq \emptyset$ can also be proved in the following way. If $\Sigma_\lambda = \emptyset$ one could apply the Global. Actually, if $\lambda > \lambda_1$ and $\Sigma_\lambda = \emptyset$ (if $\lambda = \lambda_1$ then $\Sigma_\lambda = \{0\}$), one could apply the Global Inversion Theorem (see [3, Theorem 3.1.8]) and (D) should have the trivial solution, only.

We set

$$\Sigma_{\lambda,1} = \{u \in \Sigma_\lambda \,:\, \lambda_1[\lambda - f'(u)] = 1\}.$$

We explicitly point out that the weight function $m(x) := \lambda - f'(u(x))$ $(\lambda > 0)$ satisfies the conditions stated in Section 1. In particular, for x near $\partial\Omega$ one has that $|u(x)| \ll 1$ and hence $m(x) = \lambda - f'(u(x)) > 0$ for these x.

LEMMA 7. – *If* $\lambda > \lambda_1$ *then* $\Sigma_{\lambda,1} \neq \emptyset$.

PROOF. – As remarked in Section 2, if $\lambda > \lambda_1$ then $\mathcal{S}_{\lambda,1} \neq \emptyset$. Let $u \in \mathcal{S}_{\lambda,1}$. By Lemma 4, $t_u u \in \Sigma_\lambda$ and a corresponding eigenfunction is u which does not change sign in Ω. Since the only eigenvalue with an eigenfunction that does not change sign in Ω is the first one, then $\lambda_1[\lambda - f'(t_u u)] = 1$. Hence $t_u u \in \Sigma_{\lambda,1}$, which is therefore not empty. □

REMARKS 8. – (i) If $\lambda > \lambda_1$ one has that $\Sigma_\lambda = \emptyset$. Moreover, if $\lambda < \lambda_2$ one has that $\lambda - f'(u) < \lambda_2$ and hence $\lambda_i[\lambda - f'(u)] > 1$ for all $i \geq 2$. Thus $\Sigma_\lambda = \Sigma_{\lambda,1}$.

(ii) If $\lambda < \lambda_2$ then $\mathcal{S}_\lambda = \mathcal{S}_{\lambda,1}$. Otherwise, let z be a solution of (D) which changes sign. Then $t_z z \in \Sigma_\lambda$. Precisely, by (i) one has that $z \in \Sigma_{\lambda,1}$, namely

$\lambda_1[\lambda - f'(t_z z)] = 1$. Moreover $z \in Ker[\lambda - f'(t_z z)]$, a contradiction because z changes sign.

(*iii*) In the sequel we will show that if $\lambda > \lambda_2$ then (D) has a changing sign solution, see Proposition 12; moreover, $\Sigma_\lambda \setminus \Sigma_{\lambda,1} \neq \emptyset$, see Remark 14.

(*iv*) $\Sigma_{\lambda,1}$ is a smooth manifold of codimension 1 in H. To see this, let $v \in \Sigma_{\lambda,1}$, namely $\lambda_1[\lambda - f'(v)] = 1$. Obviously $\lambda_1[\lambda - f'(v)]$ is simple, $Ker[F'_\lambda(v)]$ is one dimensional and spanned by some $\phi_\lambda \in H$. By $f''(v)v > 0$ it follows

$$(F''_\lambda(v)[v, \phi_\lambda] \mid \phi_\lambda) = -\int_\Omega f''(v)v\phi_\lambda^2 < 0.$$

This suffices to apply [3, Lemma 3.2.1] and the result follows. □

We now focus our attention to $\mathcal{S}_{\lambda,1}$ and $\Sigma_{\lambda,1}$. If $u \in H$ is a non degenerate (i.e. non singular) solution of $F_\lambda(u) = 0$, we denote by $ind(F_\lambda, u)$ its LS index.

LEMMA 9. – *Every $u \in \mathcal{S}_{\lambda,1}$ is non degenerate and $ind(F_\lambda, u) = 1$.*

PROOF. – By the preceding arguments, if $u \in \mathcal{S}_{\lambda,1}$ then $\lambda_1[\lambda - f'(t_u u)] = 1$, with $0 < t_u(x) < 1$, $\forall x \in \Omega$. Then $f'(t_u(x)\, u(x)) < f'(u(x))$, $\forall x \in \Omega$ and the monotonicity property of eigenvalues implies $\lambda_1[\lambda - f'(u)] > 1$, proving the Lemma. □

REMARK 10. – Lemma 9 allows us to give a simple proof of the result of [2] for $\lambda \in]\lambda_1, \lambda_2[$, cited in Remark 2-(*i*). Actually, for such λ, $u = 0$ is non-degenerate with $ind(F_\lambda, 0) = -1$ while the total degree on a ball of radius $r \gg 1$ is 1, see (3). The $u \in \mathcal{S}_{\lambda,1}$ are also non-degenerate with index 1 and hence, F_λ being proper, their number is finite, say k. Moreover, if $\lambda \in]\lambda_1, \lambda_2[$, Remark 8-(*ii*) yields $\mathcal{S}_\lambda = \mathcal{S}_{\lambda,1}$. Then using the additivity property of the degree we get

$$1 = ind(F_\lambda, 0) + \sum_{u \in \mathcal{S}_{\lambda,1}} ind(F_\lambda, u) = -1 + k,$$

and thus $k = 2$. □

4. – Proof of Theorem 1

Theorem 1 cannot be proved by using the degree arguments outlined in Remark 10, because for $\lambda > \lambda_2$ problem (D) has changing sign solutions, see Proposition 12, and we do not know if they are degenerate or not.

To overcome this difficulty we consider the bifurcation problem $F_\lambda(u) = 0$. It is well known that from λ_1 emanates a continuum $\mathcal{C}$ of solutions of $F_\lambda(u) = 0$. Moreover, near $(0, \lambda_1)$, $\mathcal{C}$ is a uniquely determined curve and if $(\lambda, u) \in \mathcal{C}$, u does not change sign in Ω. For the bifurcation from a simple eigenvalue we refer, e.g.

to [3, Sec. 5.4]. Let $\mathcal{C}^+ = \{(\lambda, u) : \lambda \in \mathbb{R},\ u > 0,\ F_\lambda(u) = 0\}$. By the previous remark, near $(\lambda_1, 0)$ one has that $\mathcal{C}^+ \subset \mathcal{C}$. If $(\lambda, u_\lambda) \in \mathcal{C}^+$ then by Lemma 9 u_λ is non-degenerate and hence $\mathcal{C}^+$ is a C^1 curve. In particular, there are no secondary bifurcations on $\mathcal{C}^+$. From (2) it follows that $\mathcal{C}^+$ is bounded in $[0, \Lambda] \times H$ for each $\Lambda > 0$. Since $F_\lambda(u) = 0$ has only the trivial solution provided $\lambda \leq \lambda_1$, then $\{u \in H : (\lambda, u) \in \mathcal{C}^+\} = \emptyset$ for these λ. Moreover $\overline{\mathcal{C}}^+$ (the closure of $\mathcal{C}^+$) cannot contain $(\lambda_k, 0)$, with $k \geq 2$, because λ_1 is the unique eigenvalue from which bifurcate positive solutions. Next, suppose that there exists a solution $z_\lambda > 0$ of $F_\lambda(u) = 0$ such that $(\lambda, z_\lambda) \notin \mathcal{C}^+$. Lemma 9 implies that z_λ is non-degenerate. By the continuation property of the topological degree, there is a branch (actually a C^1 curve) $\mathcal{C}^*$ containing (λ, z_λ). Repeating the preceding arguments, we deduce that $\mathcal{C}^*$ shares the same properties of $\mathcal{C}^+$. In particular, $\mathcal{C}^* \cap \mathcal{C}^+ = \emptyset$ because otherwise $\mathcal{C}^+$ (or $\mathcal{C}^*$) would have a secondary bifurcation. In addition $(\lambda_1, 0) \in \overline{\mathcal{C}}^*$. Since the branch $\mathcal{C}$ bifurcating from $(\lambda_1, 0)$ is (locally) unique, we find a contradiction, proving that the only positive solution of $F_\lambda(u) = 0$ belong to $\mathcal{C}^+$. In a quite similar way one shows that $\mathcal{C}^- = \{(\lambda, u) : \lambda \in \mathbb{R},\ u < 0,\ F_\lambda(u) = 0\}$ is a curve bifurcating from $(\lambda_1, 0)$ which contains all the negative solutions of $F_\lambda = 0$. This proves that (D) has precisely one solutions $u_1 > 0$ and one solution $u_2 < 0$. Both u_1, u_2 are non-degenerate. Moreover, if $\lambda \neq \lambda_k$, also $u = 0$ is non-degenerate. Thus the Local Inversion Theorem applies yielding a unique solution of (D_h) near $0, u_1, u_2$ provided $\|h\|_H \ll 1$. This completes the proof of Theorem 1. □

REMARK 11. – It is known that u_1, u_2 are local minima of the energy functional

$$J_\lambda(u) = \frac{1}{2} \int_\Omega \left[|\nabla u|^2 - \lambda u^2 \right] dx + \int_\Omega \left[\int_0^{u(x)} \tilde{f}(s) ds \right] dx,$$

where

$$\tilde{f}(s) = \begin{cases} \lambda s - f(s) & \text{if } -s^* \geq s \leq s^*, \\ -\lambda s^* + f(-s^*), & \text{if } s < -s^*, \\ \lambda s^* - f(s^*), & \text{if } s > s^*, \end{cases}$$

and $s^* > 0$ is such that $\lambda s^* - f(s^*) < 0$ and $-\lambda s^* - f(-s^*) > 0$. □

Remark 11 allows us to prove:

PROPOSITION 12. – *If $\lambda > \lambda_2$ then (D) has a solution z that changes sign in Ω.*

PROOF. – As a consequence of Remark 11, if $\lambda > \lambda_2$ then (D) has at least a Mountain Pass solution $z \neq 0$. Such a z changes sign in Ω, otherwise by Theorem 1 either $z = u_1$ or $z = u_2$. □

REMARKS 13. – (*i*) To prove Proposition 12 one could also argue as follows. If $z \in \mathcal{S}_{\lambda,1}$ then Lemma 9 implies that z is non-degenerate and $ind(F_\lambda, z) = 1$, while any non-degenerate Mountain Pass solution has Morse index 1 and hence LS index -1, see e.g. [1, Theorem 12.31].

(*ii*) Unfortunately we are not able to estimate the precise number of changing sign solutions, the main difficulty being that we do not know if these solutions are non-degenerate. □

REMARK 14. – If $\lambda > \lambda_2$ then $\Sigma_\lambda \setminus \Sigma_{\lambda,1} \neq \emptyset$. To prove this claim, let $z \in \mathcal{S}_\lambda \setminus \mathcal{S}_{\lambda,1}$. By Remark 5, there exists $t_z(x) \in (0,1)$ such that $t_z z \in \Sigma_\lambda$ and a corresponding eigenfunction is z which changes sign in Ω. Then $\lambda_i[\lambda - f'(t_z z)] = 1$ for some integer $i > 1$, namely $t_z z \in \Sigma_\lambda \setminus \Sigma_{\lambda,1}$ as claimed. □

Acknowledgements. I like to thank Vittorio Coti Zelati for his comments and David Arcoya for useful discussions and for bringing my attention to reference [7].

REFERENCES

[1] A. AMBROSETTI - A. MALCHIODI, *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics No. 104, Cambridge Univ. Press, 2006

[2] A. AMBROSETTI - G. MANCINI, *Sharp nonuniqueness results for some nonlinear problems*, J. Nonlin. Anal. TMA, 3-5 (1979), 635-645.

[3] A. AMBROSETTI - G. PRODI, *A Primer of Nonlinear Analysis*, Cambridge Studies in Advanced Mathematics No. 34, Cambridge Univ. Press, 1993.

[4] A. AMBROSETTI - G. PRODI, *On the inversion of some differentiable appings with singularities between Banach spaces*. Ann. Mat. Pura Appl., **93** (1972), 231-246.

[5] D. G. DE FIGUEIREDO, *Lectures on Boundary Value Problems of Ambrosetti-Prodi Type,* 12th Brazilian Seminar of Analysis, Sao Paulo, Brasil, 1980.

[6] D. G. DE FIGUEIREDO, *Positive solutions of semilinear elliptic problems*, in *Differential equations*, Lecture Notes in Math. 957 Springer, 1982.

[7] D. G. DE FIGUEIREDO - J. P. GOSSEZ, *Strict monotonicity of eigenvalues and Unique Continuation*, Comm. P.D.E., **17**-1 (1993), 339-346.

[8] B. RUF, *Singularity theory and the geometry of a nonlinear elliptic equation*, Ann. Sc. Norm. Sup. Pisa, **17** (1990), 1-33.

[9] B. RUF, *Forced secondary bifurcation in an elliptic boundary value problem*, Diff. Int. Eq., **5**-4 (1992), 793-804.

S.I.S.S.A. via Bonomea 265, 34136 Trieste,
E-mail: ambr@sissa.it

Received October 12, 2010 and in revised form November 26, 2010