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Some Remarks on Nonlinear Composition Operators in Spaces of Differentiable Functions

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To the Memory of Giovanni Prodi (1925-2010)

Abstract. – *In this note we study the nonlinear composition operator $f \mapsto g \circ f$ in various spaces of differentiable functions over an interval. It turns out that this operator is always bounded in the corresponding norm, whenever it maps such a space into itself, but continuous only in exceptional cases.*

Given a function $g : \mathbb{R} \to \mathbb{R}$, in what follows we shall be interested in studying the nonlinear composition operator T_g defined by

$$T_g f := g \circ f \tag{1}$$

in various spaces of functions $f : [a, b] \to \mathbb{R}$. This operator occurs in many places in nonlinear analysis and exhibits, inspite of its simple structure, several unexpected features. Some problems of this kind are treated in the monograph [1] and, more recently, in [2], with a particular emphasis on differential calculus in Banach spaces and applications to implicit function theorems and bifurcation phenomena.

A natural problem related to the operator (1) reads as follows:

- *Given a class X of functions $f : [a, b] \to \mathbb{R}$, find conditions on the function g, possibly both necessary and sufficient, under which the operator T_g generated by g maps the class X into itself.*

This problem is sometimes referred to as the *composition operator problem* (or COP, for short) in the literature, see e.g. [6,7]. Following [6] we introduce the notation

$$COP(X) := \{g : T_g(X) \subseteq X\}. \tag{2}$$

The explicit description of the set $COP(X)$ for given X is sometimes very easy, sometimes highly nontrivial. For example, it follows from the classical Tietze-Uryson extension lemma for continuous functions that $COP(C) = C$, where $C = C([a, b])$ denotes the set of continuous real functions on $[a, b]$.

When analyzing the set (2) for given X, one may establish, loosely speaking, the following "golden rule" which applies quite frequently:

- If X contains not only differentiable functions, then $COP(X) = Lip_{loc}(\mathbb{R})$.
- If X contains only differentiable functions, then $COP(X) = X_{loc}(\mathbb{R})$.

In other words, in the first case the operator (1) maps X into itself if and only if the corresponding function g is locally Lipschitz continuous on the real line, and so the COP has an *extrinsic and universal solution*. This was proved for $X = BV$ (functions of bounded variation) in [11], for $X = WBV_p$ (functions of bounded p-variation in Wiener's sense) in [3], for $X = RBV_p$ (functions of bounded p-variation in Riesz's sense) in [18], for $X = AC$ (absolutely continuous functions) in [16], and for $X = Lip$ (Lipschitz continuous functions) and $X = Lip_\alpha$ (Hölder continuous functions) in [9]. In all these spaces X it is rather easy to see that the local Lipschitz condition

$$|g(u) - g(v)| \leq k(r)|u - v| \qquad (|u|, |v| \leq r) \tag{3}$$

implies that T_g maps X into itself. To prove the (nontrivial) converse, it was shown recently in [4] that, if (3) *fails*, one may construct a function $f \in Lip([a, b])$ such that $g \circ f \notin BV([a, b])$. Since

$$Lip([a, b]) \subseteq RBV_p([a, b]) \subseteq AC([a, b]) \subseteq BV([a, b]),$$

the necessity of (3) for $T_g(X) \subseteq X$ follows.

On the other hand, in the second case the operator (1) maps X into itself if and only if the corresponding function g belongs (locally) to the same class, which is therefore an *algebra* with respect to composition, and so the COP has an *intrinsic and individual solution*. This is obvious for $X = C^1$ (continuously differentiable functions) and was shown for $X = WBV_p^1$ (primitives of functions belonging to WBV_p) in [7]. We remark that the space WBV_p^1 is particularly important in applications to differential equations, since it is closely related to the Besov spaces $\dot{B}_{p,1}^{1+1/p}$ and $\dot{B}_{p,\infty}^{1+1/p}$, see [6].

We point out that the above alternative is not exhaustive, since other possibilities may occur in certain spaces. For example, it was shown in [13,14] that

$$COP(W_p^1) = \{g \in Lip_{loc}(\mathbb{R}) : g' \text{ is bounded on } \mathbb{R}\},$$

where W_p^1 denotes as usual the first order Sobolev space with integrability index $p \in [1, \infty)$. A particularly drastic example of degeneracy [5] is the space $X = D^{-1}$ of all functions having a primitive (sometimes called "Darboux continuous functions" in the literature, since such functions share the intermediate value property with continuous functions): Here the set $COP(D^{-1})$ is extremely poor: it consists only of *affine* functions $g(y) := \alpha y + \beta$ $(\alpha, \beta \in \mathbb{R})$.

The aim of this paper is to describe the set $COP(X)$ for the spaces $X = C^1$, $X = RBV_p^1$, $X = AC^1$, and $X = Lip^1$ (for the precise definition see below). In particular, we will show that all these spaces fall into the second category, i.e., fulfill $COP(X) = X_{loc}(\mathbb{R})$. Moreover, we will be interested in conditions on g, possibly both necessary and sufficient, under which the operator (1) is continuous or bounded in X. (Recall that, in contrast to linear operators, a nonlinear operator may be continuous and unbounded, or bounded and discontinuous.) Such continuity and boundedness conditions are important in view of applications of existence principles of nonlinear analysis to specific problems like integral equations, functional differential equations, boundary value problems, or eigenvalue problems.

Now we start with the definition of the four spaces we are interested in. To this end, we first recall the definition of their "parent spaces" $X = C, X = RBV_p$, $X = AC$, and $X = Lip$. As usual, the space $C = C([a, b])$ is equipped with the norm

$$\|f\|_C := \max_{a \le t \le b} |f(t)|.$$

By $AC = AC([a, b])$ we denote the linear space of all absolutely continuous functions $f : [a, b] \to \mathbb{R}$. Since a function is absolutely continuous if and only if it is the primitive (a.e.) of some L_1 function (see, e.g., [12]) it seems natural to consider on $AC([a, b])$ the norm

$$\|f\|_{AC} := |f(a)| + \int_a^b |f'(t)|\, dt$$

which turns $AC([a, b])$ into a Banach space. Similarly, we consider the Banach space $RBV_p = RBV_p([a, b])$ $(1 \le p < \infty)$ of all functions $f : [a, b] \to \mathbb{R}$ which have bounded p-variation in Riesz's sense, equipped with the natural norm

$$\|f\|_{RBV_p} := |f(a)| + \mathrm{Var}_p(f, [a, b])^{1/p},$$

where

$$\mathrm{Var}_p(f, [a, b]) := \sup \left\{ \sum_{j=1}^{m} \frac{|f(t_j) - f(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}} : \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}([a, b]) \right\} \tag{4}$$

denotes the total p-variation of f in Riesz's sense on $[a, b]$, and the supremum in (4) is taken over the family $\mathcal{P}([a, b])$ of all partitions $\{t_0, t_1, \ldots, t_m\}$ of the interval $[a, b]$. In particular, the space RBV_1 coincides with the classical Banach space BV of functions of bounded variation.

Finally, by $Lip = Lip([a, b])$ we denote the Banach space of all Lipschitz continuous functions $f : [a, b] \to \mathbb{R}$, equipped with the natural norm

$$\|f\|_{Lip} := |f(a)| + lip(f, [a, b]), \tag{5}$$

where

$$lip(f) = lip(f,[a,b]) := \sup\left\{\frac{|f(s)-f(t)|}{|s-t|} : a \le s < t \le b\right\} \tag{6}$$

is the smallest Lipschitz constant of f on $[a,b]$. We point out that functions in RBV_p admit an interesting characterization by means of their derivatives: the classical Riesz lemma [19] states that $f \in RBV_p([a,b])$ in case $1<p<\infty$ if and only if f is absolutely continuous and f' (which exists a.e.) belongs to $L_p([a,b])$. An analogous result for $p = 1$ is not true, because a function $f \in RBV_1([a,b]) = BV([a,b])$ is in general not continuous, let alone absolutely continuous. However, if we *assume* that f is absolutely continuous on $[a,b]$ we have

$$\mathrm{Var}(f,[a,b]) = \int_a^b |f'(t)|\,dt. \tag{7}$$

This implies, in particular, that $\|f\|_{AC} = \|f\|_{BV}$ for all $f \in AC([a,b])$.

Finally, let us recall that $f \in Lip([a,b])$ if and only if f is absolutely continuous and f' (which exists a.e.) belongs to $L_\infty([a,b])$.

Now, to each space $X \in \{C, RBV_p, AC, Lip\}$ we associate the corresponding space $X^1 := \{f : f' \in X\}$, equipped with the norm

$$\|f\|_{X^1} := |f(a)| + \|f'\|_X. \tag{8}$$

In this way we get the four spaces C^1, RBV_p^1, AC^1, and Lip^1 in which we are going to study the composition operator (1). Note that the definition of the spaces AC^1 and RBV_p^1 carries over without any change to the case of unbounded intervals (in particular, on the whole real line). The spaces $Lip^1(\mathbb{R})$ and $Lip_{loc}^1(\mathbb{R})$, however, are very different, since the Lipschitz constant (6) essentially depends on the "size" of the underlying domain, and we may have $k(r) \to \infty$ in (3) as $r \to \infty$.

To begin with, let us state a simple result on imbeddings between these spaces. As usual, we write $X \hookrightarrow Y$ if $X \subseteq Y$ and there exists some $k \in \mathbb{R}$ such that $\|f\|_Y \le k\|f\|_X$ for all $x \in X$. In this case the norms (8) of the corresponding spaces X^1 and Y^1 satisfy $\|f\|_{Y^1} \le \max\{k,1\}\|f\|_{X^1}$, and so $X^1 \hookrightarrow Y^1$ as well.

PROPOSITION 1. – *The continuous imbeddings*

$$Lip^1([a,b]) \hookrightarrow RBV_p^1([a,b]) \hookrightarrow AC^1([a,b]) \hookrightarrow C^1([a,b]) \tag{9}$$

hold for $p > 1$.

PROOF. – By what we have just observed, it suffices to prove the continuous imbeddings

$$Lip([a,b]) \hookrightarrow RBV_p([a,b]) \hookrightarrow AC([a,b]) \hookrightarrow C([a,b])$$

for the derivatives. Let $f \in Lip([a,b])$ and $L > lip(f)$, and let $\{t_0, t_1, \ldots, t_m\}$ be any partition of $[a,b]$. Then we have

$$\sum_{j=1}^{m} \frac{|f(t_j) - f(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}} \leq L^p \sum_{j=1}^{m} |t_j - t_{j-1}| = L^p(b-a)$$

which shows that $f \in RBV_p([a,b])$ with $\|f\|_{RBV_p} \leq \max\{(b-a)^{1/p}, 1\}\|f\|_{Lip}$. Moreover, from Hölder's inequality and the Riesz lemma it follows that, for all $f \in RBV_p([a,b])$,

$$\left(\int_a^b |f'(t)|\,dt\right)^p \leq (b-a)^{p-1}\int_a^b |f'(t)|^p\,dt = (b-a)^{p-1}\mathrm{Var}_p(f,[a,b]),$$

and so $\|f\|_{AC} \leq \max\{(b-a)^{1-1/p}, 1\}\|f\|_{RBV_p}$. Finally, given $f \in AC([a,b])$ and considering the special partition $\{a, x, b\} \in \mathcal{P}([a,b])$ we get $|f(x) - f(a)| \leq \mathrm{Var}(f,[a,b])$. Consequently,

$$|f(x)| \leq |f(a)| + \mathrm{Var}(f,[a,b]) = \|f\|_{AC},$$

and so $\|f\|_C \leq \|f\|_{AC}$ as claimed. □

In the following Example 2 we show that all inclusions in (9) are strict.

EXAMPLE 2. – For $\alpha \geq 1$, consider the function $f_\alpha : [0,1] \to \mathbb{R}$ defined by $f_\alpha(x) := x^\alpha$. A straightforward calculation shows that

$$f_\alpha \in Lip^1([0,1]) \;\Leftrightarrow\; f_{\alpha-1} \in Lip([0,1]) \;\Leftrightarrow\; \alpha \geq 2,$$

$$f_\alpha \in RBV_p^1([0,1]) \;\Leftrightarrow\; f_{\alpha-1} \in RBV_p([0,1]) \;\Leftrightarrow\; \alpha \geq 2 - \frac{1}{p},$$

and

$$f_\alpha \in AC^1([0,1]) \;\Leftrightarrow\; f_{\alpha-1} \in AC([0,1]) \;\Leftrightarrow\; \alpha \geq 1.$$

So for $2 - \frac{1}{p} \leq \alpha < 2$ we have $f_\alpha \in RBV_p^1([0,1]) \setminus Lip^1([0,1])$, while for $1 \leq \alpha < 2 - \frac{1}{p}$ we have $f_\alpha \in AC^1([0,1]) \setminus RBV_p^1([0,1])$.

To show that the last inclusion in (9) is strict, let $\phi : [0,1] \to \mathbb{R}$ be the classical Cantor function. It is well known that ϕ is an increasing and continuous map of $[0,1]$ onto itself (see, e.g., [8]). Consequently, the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \int_0^x \phi(t)\,dt \qquad (0 \leq x \leq 1) \tag{10}$$

is a C^1 function. On the other hand, its derivative $f' = \phi$ does not map nullsets into nullsets, and so fails to be absolutely continuous, by the Vitali-Banach-Zaretskij theorem [12]. Consequently, $f \in C^1([0,1]) \setminus AC^1([0,1])$. □

Since we are going to deal with functions which have first derivatives everywhere and second derivatives almost everywhere, we will use the fact that, under reasonable conditions on f and g, we have

$$(g \circ f)' = (g' \circ f)f' \tag{11}$$

and

$$(g \circ f)'' = (g'' \circ f)f'^2 + (g' \circ f)f''. \tag{12}$$

Since we are also interested in conditions on g under which the corresponding operator T_g is bounded in some normed space X, we introduce the *growth function*

$$\mu_r(g,X) := \sup\{\|g \circ f\|_X : \|f\|_X \leq r\} \qquad (r > 0)$$

of $g \in COP(X)$. Thus, $T_g : X \to X$ is bounded if and only if $\mu_r(g,X) < \infty$ for all $r > 0$.

Now we start our analysis of the operator (1). For the sake of completeness, we recall first a rather obvious statement about the COP in the space C^1.

THEOREM 3. – *The operator* (1) *maps the space* $C^1([a,b])$ *into itself if and only if* $g \in C^1(\mathbb{R})$. *Moreover, in this case the operator* (1) *is automatically bounded and continuous in the norm*

$$\|f\|_{C^1} = |f(a)| + \|f'\|_C = |f(a)| + \max_{a \leq x \leq b} |f'(x)|. \tag{13}$$

PROOF. – The proof is almost trivial. The inclusion $COP(C^1) \subseteq C^1(\mathbb{R})$ follows from the fact that the identity $f(x) = x$ is C^1, while the inclusion $COP(C^1) \supseteq C^1(\mathbb{R})$ follows from the chain rule. To prove boundedness of T_g we introduce the notation

$$\gamma_0(r) := \sup_{|u| \leq r} |g(u)|, \quad \gamma_1(r) := \sup_{|u| \leq r} |g'(u)| \qquad (r > 0) \tag{14}$$

for a function $g \in C^1(\mathbb{R})$. From the chain rule it follows then that $\|f\|_{C^1} \leq r$ implies

$$|(g \circ f)(a)| \leq \gamma_0(r), \quad |(g \circ f)'(t)| \leq |g'(f(t))|\,|f'(t)| \leq r\gamma_1(r)$$

which shows that

$$\mu_r(g, C^1) \leq \gamma_0(r) + r\gamma_1(r) \qquad (r > 0),$$

and so T_g is bounded.

To prove that T_g is continuous in the norm (13), let $(f_n)_n$ be a sequence of C^1 functions which converges in the norm (13) to some function $f \in C^1$. Then $(f_n)_n$ is bounded, say $\|f_n\|_{C^1} \leq r$. Putting

$$h_n := T_g f_n - T_g f = (g \circ f_n) - (g \circ f) \tag{15}$$

we get

$$h_n' = (g' \circ f_n) f_n' - (g' \circ f) f',$$

and we have to show that $\|h_n'\|_C \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since $g \in C^1(\mathbb{R})$ and $\|f_n - f\|_C \to 0$, by the mean value theorem, we may find $n_0 \in \mathbb{N}$ such that $|g'(f_n(t)) - g'(f(t))| \leq \varepsilon$ for $n \geq n_0$. But this implies that

$$\begin{aligned} |h_n'(t)| &\leq |g'(f_n(t))|\, |f_n'(t) - f'(t)| + |g'(f_n(t)) - g'(f(t))|\, |f'(t)| \\ &\leq \gamma_1(r) \|f_n' - f'\|_C + \varepsilon \|f'\|_C, \end{aligned}$$

which shows that $\|h_n'\|_C \to 0$ as $n \to \infty$. The relation $|h_n(a)| \to 0$ is obvious. □

In contrast to Theorem 3, the solution of the COP in the space AC^1 is not completely trivial.

THEOREM 4. – *The operator* (1) *maps the space* $AC^1([a,b])$ *into itself if and only if* $g \in AC^1(\mathbb{R})$. *Moreover, in this case the operator* (1) *is automatically bounded in the norm*

$$\|f\|_{AC^1} = |f(a)| + \|f'\|_{AC} = |f(a)| + |f'(a)| + \int_a^b |f''(x)|\, dx. \tag{16}$$

PROOF. – Without loss of generality we may assume that $[a,b] = [0,1]$. Suppose first that $g \in AC^1(\mathbb{R})$, and let $f \in AC^1([0,1])$. Then $g' \in AC(\mathbb{R})$ and $f' \in AC([0,1])$, and so $g' \circ f \in AC([0,1])$ and $(g \circ f)' \in AC([0,1])$, by (11). Similarly, from $g'' \in L_{1,loc}(\mathbb{R})$ and $f' \in AC([0,1]) \subseteq L_\infty([0,1])$ it follows that $(g'' \circ f) f'^2 \in L_1([0,1])$, while from $g' \in AC(\mathbb{R})$ and $f'' \in L_1([0,1])$ it follows that $(g' \circ f) f'' \in L_1([0,1])$. We conclude that $(g \circ f)'' \in L_1([0,1])$, by (12), and therefore $g \circ f \in AC^1([0,1])$ as claimed.

Now we suppose that $T_g(AC^1([0,1])) \subseteq AC^1([0,1])$ and show that $g \in AC^1([c,d])$ for every interval $[c,d] \subset \mathbb{R}$. The function $\ell_{c,d}(t) := c(1-t) + dt$ defines a strictly monotone (in fact, affine) C^∞ diffeomorphism from $[0,1]$ onto $[c,d]$, and so clearly belongs to $AC^1([0,1])$. By assumption, the function $h_{c,d} = T_g \ell_{c,d}$ belongs then to $AC^1([0,1])$ as well; consequently, the function $g = h_{c,d} \circ \ell_{c,d}^{-1}$ belongs to $AC^1([c,d])$ as claimed.

Now we show that the operator T_g is bounded in the norm (16) whenever it maps AC^1 into itself. Given $f \in AC^1([0,1])$ with $\|f\|_{AC^1} \leq r$, hence $|f(t)| \leq r$ and

$|f'(t)| \leq r$ for $0 \leq t \leq 1$, we get the estimates

$$|(g \circ f)(0)| \leq \gamma_0(r), \quad |(g \circ f)'(0)| \leq |g'(f(0))|\,|f'(0)| \leq r\gamma_1(r)$$

and

$$\|(g'' \circ f)f'^2\|_{L_1} \leq r^2 \|g'' \circ f\|_{L_1} \leq r^2 c(r),$$

where we use the notation (14) and the constant $c(r)$ is finite, since $g'' \circ f \in L_{1,loc}(\mathbb{R})$. Moreover,

$$\|(g' \circ f)f''\|_{L_1} \leq \gamma_1(r)\|f''\|_{L_1} \leq r\gamma_1(r).$$

So from (11) and (12) we conclude that

$$\mu_r(g, AC^1) \leq \gamma_0(r) + 2r\gamma_1(r) + r^2 c(r),$$

which shows that T_g is bounded. □

We do not know whether or not the operator (1) is also continuous in the norm (16) whenever it maps the space $AC^1([a,b])$ into itself.

Before solving the COP for the other spaces in (9) we make a remark on functions with so-called bounded second variation. Given a function $f : [a,b] \to \mathbb{R}$ and a partition $P = \{t_0, t_1, \ldots, t_m\}$ of $[a,b]$, the *second variation* of f on $[a,b]$ w.r.t. P is defined by

$$\mathrm{Var}^2(f, P; [a,b]) := \sum_{j=1}^{m-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|.$$

Following [17] we say that f has *bounded second variation* on $[a,b]$ and write $f \in BV^2([a,b])$ if

$$\mathrm{Var}^2(f; [a,b]) := \sup\left\{ \mathrm{Var}^2(f, P; [a,b]) : P \in \mathcal{P}([a,b]) \right\} < \infty.$$

There is an interesting relation between the spaces $BV^2([a,b])$ and $AC^1([a,b])$ due to Russell [20] which reads as follows: *$f \in AC^1([a,b])$ implies $f \in BV^2([a,b])$ with*

$$\mathrm{Var}^2(f; [a,b]) = \int_a^b |f''(t)|\, dt.$$

This may be viewed as a "higher order analogue" to the classical formula (7). It is also shown in [20] that a function f belongs to $BV^2([a,b])$ if and only if f may be represented as difference of two convex functions. This may be in turn viewed as a "higher order analogue" to the classical Jordan decomposition [10] of a function of bounded variation as difference of two increasing functions.

One could ask whether or not the requirement that $f \in AC^1([a,b])$ is essential in Russell's result. The following example shows that there exist in fact functions with bounded second variation whose derivatives are not absolutely continuous.

EXAMPLE 5. – Let $\phi : [0,1] \to \mathbb{R}$ be the Cantor function, and let $f : [0,1] \to \mathbb{R}$ be defined as in (10). We have already seen in Example 2 that $f \notin AC^1([0,1])$. On the other hand, since ϕ is monotonically increasing, f is convex, and so belongs to $BV^2([0,1])$. □

Example 5 shows that we cannot use Theorem 4 to solve the COP for the space $BV^2([a,b])$. As far as we know, the problem of describing $COP(BV^2)$ is still open.

We pass now to the space $RBV_p^1([a,b])$ for $1<p<\infty$. The restriction $1<p<\infty$ is important inasmuch as the space $BV([a,b])$ plays an exceptional role: as was shown in [11], we have $T_g(BV([a,b])) \subseteq BV([a,b])$ if and only if g satisfies (3). The following theorem is in contrast to this.

THEOREM 6. – *Let* $1<p<\infty$. *Then the operator* (1) *maps the space* $RBV_p^1([a,b])$ *into itself if and only if* $g \in RBV_p^1(\mathbb{R})$. *Moreover, in this case the operator* (1) *is automatically bounded in the norm*

$$\|f\|_{RBV_p^1} = |f(a)| + \|f'\|_{RBV_p} = |f(a)| + |f'(a)| + \mathrm{Var}_p(f',[a,b])^{1/p}. \tag{17}$$

PROOF. – Without loss of generality we assume again that $[a,b]=[0,1]$. Suppose first that $g \in RBV_p^1(\mathbb{R})$, and let $f \in RBV_p^1([0,1])$, and so $g'' \in L_{p,loc}(\mathbb{R})$ and $f'' \in L_p([0,1])$, by the Riesz lemma. Since $g' \in RBV_p(\mathbb{R}) \subset AC(\mathbb{R})$ and $f \in C([0,1])$, we have $g' \circ f \in C([0,1])$ and so $(g' \circ f)f'' \in L_p([0,1])$. Similarly, from $g'' \in L_{p,loc}(\mathbb{R})$, $f \in AC^1([0,1])$ and $f' \in AC([0,1])$ it follows that $(g'' \circ f)f'^2 \in L_p([0,1])$. So we have proved that $(g \circ f)'' \in L_p([0,1])$, by (12). The fact that $(g \circ f)' \in AC([0,1])$ is proved in the same way as in Theorem 4.

Now we suppose that $T_g(RBV_p^1([0,1])) \subseteq RBV_p^1([0,1])$ and show that $g \in RBV_p^1([c,d])$ for every interval $[c,d] \subset \mathbb{R}$. Let $\ell_{c,d}$ be defined as in the proof of Theorem 4; then $\ell_{c,d} \in RBV_p^1([0,1])$ with $\|\ell_{c,d}\|_{RBV_p^1} = |c| + |d-c|$. By assumption, the function $h_{c,d} = T_g \ell_{c,d}$ belongs then also to $RBV_p^1([0,1])$; consequently, the function $g = h_{c,d} \circ \ell_{c,d}^{-1}$ belongs to $RBV_p^1([c,d])$ as claimed.

Now we show that the operator T_g is bounded in the norm (17) whenever it maps RBV_p^1 into itself. Given $f \in RBV_p^1([0,1])$ with $\|f\|_{RBV_p^1} \le r$, hence $|f(t)| \le r$ and $|f'(t)| \le r$ for $0 \le t \le 1$, we get the estimates

$$|(g \circ f)(0)| \le \gamma_0(r), \quad |(g \circ f)'(0)| \le |g'(f(0))|\,|f'(0)| \le r\gamma_1(r)$$

and

$$\int_0^1 |g''(f(t))|^p |f'(t)|^{2p}\, dt \le r^{2p} \int_0^1 |g''(f(t))|^p\, dt \le r^{2p} c_p(r),$$

where we use the notation (14) and the constant $c_p(r)$ is finite, since $g'' \circ f \in L_{p,loc}(\mathbb{R})$. Moreover,

$$\int_0^1 |g'(f(t))|^p |f''(t)|^p \, dt \leq \gamma_1(r)^p \int_0^1 |f''(t)|^p \, dt = \gamma_1(r)^p \|f''\|_{L_p}^p \leq r^p \gamma_1(r)^p.$$

So from (11) and (12) we conclude that

$$\mu_r(g, RBV_p^1) \leq \gamma_0(r) + 2r\gamma_1(r) + r^2 c_p(r)^{1/p},$$

and so T_g is bounded. □

Again, we do not know whether or not the operator (1) is also continuous in the norm (17) whenever it maps the space $RBV_p^1([a,b])$ into itself. We remark that the continuity problem for the operator (1) in the Wiener space $WBV_p^1([a,b])$ was completely solved in [7]. Indeed, it was shown there that $T_g(WBV_p^1([a,b])) \subseteq WBV_p^1([a,b])$ if and only if $g \in WBV_{p,loc}^1(\mathbb{R})$; however, T_g is continuous in the corresponding norm

$$\|f\|_{WBV_p^1} := |f(a)| + |f'(a)| + \sup\left\{ \sum_{j=1}^m |f(t_j) - f(t_{j-1})|^p : \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}([a,b]) \right\}$$

(if and) only if g belongs to the closure of $WBV_{p,loc}^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ which is a proper subspace of $WBV_{p,loc}^1(\mathbb{R})$. We believe that a similar restriction is needed in the Riesz space $RBV_p^1([a,b])$, but we were unable to prove it.

THEOREM 7. – *The operator* (1) *maps the space* $Lip^1([a,b])$ *into itself if and only if* $g \in Lip_{loc}^1(\mathbb{R})$. *Moreover, in this case the operator* (1) *is automatically bounded in the norm*

$$\|f\|_{Lip^1} = |f(a)| + \|f'\|_{Lip} = |f(a)| + |f'(a)| + lip(f', [a,b]). \tag{18}$$

PROOF. – Again, without loss of generality we may assume that $[a,b] = [0,1]$. Suppose first that $g \in Lip_{loc}^1(\mathbb{R})$, and let $f \in Lip^1([0,1])$ and $h := T_g f$, hence $h' = (g' \circ f)f'$. From $g' \in Lip_{loc}(\mathbb{R})$ it follows that $T_{g'}(Lip([0,1])) \subseteq Lip([0,1])$. Combining this with $f' \in Lip([0,1])$ we conclude that $h' \in Lip([0,1])$, since $Lip([0,1])$ is an algebra. So we have shown that $h \in Lip^1([0,1])$, i.e., $T_g(Lip^1([0,1])) \subseteq Lip^1([0,1])$.

The proof of the converse implication is similar as in Theorem 4. If we define $\ell_{c,d} : [0,1] \to [c,d]$ as there and suppose that $T_g(Lip^1([0,1])) \subseteq Lip^1([0,1])$, we have $h_{c,d} := T_g \ell_{c,d} \in Lip^1([0,1])$ with $h'_{c,d} = (d-c)(g' \circ \ell_{c,d})$. But this implies that $g' \in Lip([c,d])$, and so $g \in Lip_{loc}^1(\mathbb{R})$, since $[c,d] \subset \mathbb{R}$ is arbitrary.

Now we show that the operator T_g is bounded in the norm (18) under the assumption $g \in Lip_{loc}^1(\mathbb{R})$. Given $f \in Lip^1([0,1])$ with $\|f\|_{Lip^1} \leq r$ and using again

(14) we have

$$\|f'\|_C \leq |f'(0)| + lip(f') = \|f'\|_{Lip} \leq r, \qquad \|g' \circ f\|_C \leq \gamma_1(r).$$

Moreover, since the operator $T_{g'}$ is bounded in the space $Lip([0,1])$, we have

$$lip(g' \circ f) \leq \|T_{g'} f\|_{Lip} \leq \mu_r(g', Lip),$$

and so

$$lip((g' \circ f)f') \leq lip(g' \circ f)\|f'\|_C + lip(f')\|g' \circ f\|_C \leq \mu_r(g', Lip)r + \gamma_1(r)r.$$

This shows that

$$\mu_r(g, Lip^1) \leq r[\mu_r(g', Lip) + \gamma_1(r)]$$

and so T_g is bounded as claimed. □

Observe that again we did not claim automatic continuity of T_g in Theorem 7. However, here we are able to present a counterexample which shows that the operator (1) may map the space $Lip^1([a,b])$ into itself without being continuous in the norm (18).

EXAMPLE 8. – Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(u) := \begin{cases} 0 & \text{for} \quad u \leq 0, \\ \frac{1}{2}u^2 & \text{for} \quad 0 < u < 1, \\ u - \frac{1}{2} & \text{for} \quad u \geq 1, \end{cases}$$

Since $g'(u) \equiv 0$ for $u \leq 0$ and $g'(u) = \min\{u, 1\}$ for $u > 0$, we certainly have $g \in Lip^1_{loc}(\mathbb{R})$ (even $g \in Lip^1(\mathbb{R})$), and so T_g maps $Lip^1([0,1])$ into itself, by Theorem 7. However, T_g is not continuous in the norm (18). To see this, consider the functions $f, f_n : [0,1] \to \mathbb{R}$ defined by

$$f(t) := t, \qquad f_n(t) := \frac{n+1}{n}t.$$

Clearly, $\|f_n - f\|_{Lip^1} = \frac{1}{n} \to 0$ as $n \to \infty$. On the other hand, the function h_n defined by (15) satisfies

$$h_n(t) = \begin{cases} \frac{2n+1}{2n^2}t^2 & \text{for} \quad 0 \leq t \leq \tau_n, \\ \frac{n+1}{n}t - \frac{1}{2}(1+t^2) & \text{for} \quad \tau_n < t \leq 1, \end{cases}$$

and

$$h_n'(t) = \begin{cases} \dfrac{2n+1}{n^2} t & \text{for} \quad 0 \le t \le \tau_n, \\ \dfrac{n+1}{n} - t & \text{for} \quad \tau_n < t \le 1, \end{cases}$$

where $\tau_n := \dfrac{n}{n+1}$. Consequently,

$$\|T_g f_n - T_g f\|_{Lip^1} \ge lip(h_n') \ge \frac{|h_n'(1) - h_n'(\tau_n)|}{1-\tau_n} = \frac{n+1}{n+1} \equiv 1,$$

which shows that $\|T_g f_n - T_g f\|_{Lip^1} \not\to 0$ as $n \to \infty$. □

Observe that the function g in Example 8 belongs to $Lip^1_{loc}(\mathbb{R})$, and so also to $C^1(\mathbb{R})$, but has no second derivative at 0 and 1. This is not accidental, as the following theorem shows which provides a necessary and sufficient continuity condition.

THEOREM 9. – *The operator* (1) *maps the space* $Lip^1([a,b])$ *into itself and is continuous in the norm* (18) *if and only if* $g \in C^2(\mathbb{R})$.

PROOF. – Suppose first that $g \in C^2(\mathbb{R})$. Let $(f_n)_n$ be a sequence of Lip^1 function and $f \in Lip^1([a,b])$ such that

$$\|f_n - f\|_{Lip^1} = |f_n(a) - f(a)| + |f_n'(a) - f'(a)| + lip(f_n' - f') \to 0 \qquad (n \to \infty).$$

Defining h_n as in (15), we have to show that

$$lip(h_n') = lip((g' \circ f_n) f_n' - (g' \circ f) f') \to 0 \qquad (n \to \infty).$$

Now, from $g' \in C^1(\mathbb{R})$ it follows that the operator $T_{g'}$ maps $Lip([a,b])$ into itself and is continuous in the norm (5). Consequently,

$$\|T_{g'} f_n - T_{g'} f\|_{Lip} = \|(g' \circ f_n) - (g' \circ f)\|_{Lip} \to 0 \qquad (n \to \infty).$$

Also, from $\|f_n - f\|_{Lip^1} \to 0$ it follows that $\|f_n' - f'\|_{Lip} \to 0$ as $n \to \infty$. Using again the fact that $Lip([a,b])$ is an algebra we obtain

$$\begin{aligned} lip(h_n') &\le lip((g' \circ f_n) f_n' - (g' \circ f_n) f') + lip((g' \circ f_n) f' - (g' \circ f) f') \\ &\le lip(g' \circ f_n) \|f_n' - f'\|_C + lip(f_n' - f') \|g' \circ f_n\|_C \\ &\quad + lip((g' \circ f_n) - (g' \circ f)) \|f'\|_C + lip(f') \|(g' \circ f_n) - (g' \circ f)\|_C, \end{aligned}$$

and all four terms in the last sum tend to zero as $n \to \infty$. The relations $|h_n(a)| \to 0$ and $|h_n'(a)| \to 0$ are obvious.

Now suppose that T_g maps $Lip^1([a,b])$ into itself and is continuous in the norm (18). In particular, we have then $g' \in Lip_{loc}(\mathbb{R})$, by Theorem 7, and we have to show that even $g' \in C^1(\mathbb{R})$.

Suppose that $g''(u_0)$ does not exist at some point $u_0 \in \mathbb{R}$. Then we find real sequences $(h_m)_m$ and $(k_m)_m$ such that $h_m \to 0$, $k_m \to 0$, and

$$\lim_{m\to\infty} \frac{g'(u_0+h_m)-g'(u_0)}{h_m} =: L_1 < L_2 := \lim_{m\to\infty} \frac{g'(u_0+k_m)-g'(u_0)}{k_m}.$$

Since g' is differentiable a.e., we may choose a real sequence $(u_n)_n$ such that $u_n \to u_0$ and $g''(u_n)$ exists for all n. The functions $f_n : [a,b] \to \mathbb{R}$ defined for fixed $\tau \in (a,b)$ by

$$f_n(t) := t - \tau + u_n \qquad (a \le t \le b)$$

belong to $Lip^1([a,b])$ and converge in the norm (18) to the function $f_0(t) := t - \tau + u_0$. By assumption, the functions $g \circ f_n$ then also belong to $Lip^1([a,b])$. We claim that T_g is discontinuous at f_0. Indeed, otherwise for $\varepsilon \in (0,(L_2-L_1)/2)$ we would find $n_0 \in \mathbb{N}$ such that $\|T_g f_n - T_g f_0\|_{Lip^1} \le \varepsilon$ for $n \ge n_0$. This would imply, in particular, that

$$|g'(f_n(s)) - g'(f_n(t)) - g'(f_0(s)) + g'(f_0(t))| \le \varepsilon|s-t| \tag{19}$$

for $n \ge n_0$. Choosing $s = \tau + h_m$ and $t = \tau$ in (19) and taking into account the definition of f_n and f_0 yields

$$|g'(u_n+h_m) - g'(u_n) - g'(u_0+h_m) + g'(u_0)| \le \varepsilon|h_m|,$$

while putting $s = \tau + k_m$ and $t = \tau$ we obtain

$$|g'(u_n+k_m) - g'(u_n) - g'(u_0+k_m) + g'(u_0)| \le \varepsilon|k_m|.$$

But after letting $m \to \infty$ this gives both $|g''(u_n) - L_1| \le \varepsilon$ and $|g''(u_n) - L_2| \le \varepsilon$, hence

$$L_2 - L_1 = L_2 - g''(u_n) + g''(u_n) - L_1 \le 2\varepsilon < L_2 - L_1,$$

a contradiction. This completes the proof. □

Our previous discussion shows that in all spaces under consideration we get boundedness of the operator T_g for free, while continuity is a delicate problem: in the largest space $C^1([a,b])$ in (9) continuity holds, in the smallest space $Lip^1([a,b])$ in (9) continuity fails, and in the intermediate spaces $RBV_p^1([a,b])$ and $AC^1([a,b])$ in (9) we do not know the answer. For the reader's ease we summarize our results in the following synoptic table.

TABLE 1. – Behaviour of the operator $T_g f = g \circ f$ in $X \in \{C^1, AC^1, RBV_p^1, Lip^1\}$.

Banach space X	$T_g(X) \subseteq X$ *if and only if*	*automatic boundedness*	*automatic continuity*	T_g *continuous if and only if*
$C^1([a,b])$	$g \in C^1(\mathbb{R})$	yes	yes	$g \in C^1(\mathbb{R})$
$AC^1([a,b])$	$g \in AC^1(\mathbb{R})$	yes	???	???
$RBV_p^1([a,b])$	$g \in RBV_p^1(\mathbb{R})$	yes	???	???
$Lip^1([a,b])$	$g \in Lip_{loc}^1(\mathbb{R})$	yes	no	$g \in C^2(\mathbb{R})$

We conclude with a statement on stronger continuity properties. Interestingly, if we require *uniform continuity* of T_g we encounter a strong degeneracy phenomenon: only *affine* functions g generate uniformly continuous composition operators.

THEOREM 10. – *Suppose that the operator* (1) *generated by some function* $g : \mathbb{R} \to \mathbb{R}$ *maps one of the spaces* $X \in \{C^1, AC^1, RBV_p^1, Lip^1\}$ *into itself and is uniformly continuous in the respective norm. Then there exist constants* $\alpha, \beta \in \mathbb{R}$ *such that*

$$g(u) = \alpha + \beta u \qquad (u \in \mathbb{R}),$$

i.e., g is an affine function.

PROOF. – We use the fact that the largest space $C^1([a,b])$ in (9) is continuously imbedded in the space $Lip([a,b])$ with norm (5). From our assumptions it follows that we can find a $\delta > 0$ such that $\|T_g f - T_g \tilde{f}\|_X \leq 1$ for all $f, \tilde{f} \in X$ satisfying $\|f - \tilde{f}\|_{Lip} \leq \delta$.

Fix $\omega > 0$ and $v \in [-\delta, \delta]$, and define $f, \tilde{f} \in X$ by $f(t) := \omega t + v$ and $\tilde{f}(t) := \omega t$. Since $\|f - \tilde{f}\|_{Lip} = |v| \leq \delta$, we conclude that

$$lip(T_g f - T_g \tilde{f}) \leq \|T_g f - T_g \tilde{f}\|_{Lip} \leq \|T_g f - T_g \tilde{f}\|_X \leq 1,$$

hence

$$|g(\omega s + v) - g(\omega s) - g(\omega t + v) + g(\omega t)| \leq |s - t|.$$

Putting, in particular, $s = u/\omega$ and $t = 0$, we further conclude that

$$|g(u + v) - g(u) - g(v) + g(0)| \leq \left|\frac{u}{\omega}\right| \to 0 \qquad (\omega \to \infty).$$

Let us suppose for a moment that $g(0) = 0$. Then the last relation shows that

$$g(u + v) = g(u) + g(v) \qquad (u, v \in \mathbb{R},\ |v| \leq \delta)$$

which by standard arguments implies that $g(u) = \beta u$ with $\beta = g(1) \in \mathbb{R}$. Replacing in the general case g by the function $u \mapsto g(u) - g(0)$, the statement follows with $\alpha = g(0)$ and $\beta = g(1) - g(0)$. □

The hypotheses of Theorem 10 are satisfied, in particular, if the operator (1) satisfies a global Lipschitz condition

$$\|T_g f - T_g \tilde{f}\| \leq k\|f - \tilde{f}\| \qquad (f, \tilde{f} \in X)$$

in one of the spaces X covered by Theorem 10. This is a well-known degeneracy phenomenon which was proved first for the space C^1 by Matkowski in [15], and subsequently for many other function spaces as well, see Section 2.2 in [17] for a detailed discussion.

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