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A Paradox in the Two Envelope Paradox?

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Dedicated to the memory of Giovanni Prodi

Abstract. – *We describe accurately the history of the two envelope paradox. We also formulate a new version of Schrödinger's paradox which reveals a close connection between the two sorts of paradoxes. Finally, we show that built into the most popular version of the two envelope paradox there is a logical paradox reminiscent of the unexpected hanging paradox.*

1. – Introduction

This paper concerns the well-known two envelope paradox, which can be formulated as follows.

PARADOX 1

Ann is given two envelopes and is informed that both contain a check, one twice the amount of the other. She is allowed to open one of the envelopes and then to make a choice: she may either keep the check or choose to open the second envelope. In the latter case she wins the amount written on the second check.

She may argue: "In my envelope I found X dollars, so in the other, with the same probability, there are either $2X$ *or* $X/2$ *dollars. Therefore if I switch my expected win is*

$$\frac{1}{2}\left(2X + \frac{X}{2}\right) = X + \frac{X}{4} > X\,,$$

so it is favorable to switch."

Here and elsewhere we may assume that she either plays against a bookmaker or against a second player, Bill, who may open the second envelope and is offered the same opportunity. Now both players are convinced that they improve their chances if they switch. But this is paradoxical because, as economists say, "There Ain't No Such Thing As A Free Lunch".

The paradox is attributed to Barry Nalebuff ([N], Puzzle 4), but he quotes a paper by Zabell [Z], who in turn ascribes it to Steve Budry.

Giovanni Prodi [P] published a paper concerning this paradox, carrying out an amusing idea: he proposed the puzzle to nine colleagues and collected their explanations of the paradox, adding his own.

He conjectured that the wide variety of approaches (much more varied than would be expected in other areas of mathematics) is due to the the fact that probability is a relatively young discipline compared to more traditional ones and he concluded that this makes probability more interesting.

We want to stress that, in spite of the fact that the problem is stated as a puzzle, it does not admit a simple solution (Martin Gardner presented a version of it in [G] saying that he had no clue how to solve it) and that a satisfactory solution required several decades. Moreover, it contributed to reach a deeper insight into fundamental problems like the ranking of random variables from one observation [SSS].

In Section 2 we present variants of Paradox 1, in particular the well-known one due to Kraïchik, which we call Paradox 2.

In Section 3 we present the Schrödinger paradox and a a new variant of it which relates to the original paradox as Paradox 2 relates to Paradox 1. We also formulate a winning strategy for the player against a bookmaker for this new game.

In Sections 2 and 3, in discussing the various versions of the paradox, we describe the state of the art and piece together an accurate history of the problem which often appears incomplete and sometimes even contradictory between the most important papers dedicated to it.

In the last section we show that Paradoxes 1 and 3 cannot be used to organize a game because of logical obstacles reminiscent of the "unexpected hanging paradox" and of a puzzle proposed by Littlewood [L].

In this paper we use the word "paradox" in two different senses: as a sentence which defies our intuition (in the first three sections), but also, in the last section, as a contradictory situation which does not yet have a universally accepted resolution.

2. – Other versions of the paradox

The oldest "paradox" related to Nalebuff's puzzle is due to the Belgian mathematician (born in Minsk) Maurice Kraïchik who proposed in [K] a puzzle equivalent to the following scenario.

PARADOX 2

Ann and Bill get from the referee one envelope each, containing checks with different amounts. Both participants can choose: either they decide to open their envelope and win the amount written on the check, or they allow the referee to *open both envelopes and then whoever has the least money in her or his envelope collects the money from both envelopes.*

Ann can reason: "I have the amount X in my envelope. That is the maximum that I could lose. If I ask the referee to open the envelopes, with probability $1/2$ *the amount I will have at the end of the game will be more than* $2X$*. Therefore opening both envelopes is favorable to me."*

But Bill can reason in exactly the same way, so they should both ask the referee to open the envelopes. In fact, by symmetry, the game is fair. Where is the mistake in the reasoning of Ann and Bill?

Bill's appearance on the scene highlights the contradiction: the game is at the same time fair and favorable to Ann (and to Bill), which is impossible.

There is a substantial variant of the game: assume that Ann plays against a bookmaker and she (alone) is given the opportunity to switch. This variant provides a different twist to the puzzle as there exists a strategy which gives her the edge over the bookmaker. We will discuss this matter at the end of the next section.

This puzzle has been popularized by Martin Gardner in his famous book [G].

To make the different games more comparable, we have changed the original scenarios: Kraïchik imagined two friends comparing the (unknown) value of their neckties, presents of their respective wives, while Gardner puts on the stage a professor who asks two students to show her how much money they have in their wallets.

We want to stress at this point an important difference between Paradox 1 and Paradox 2: in the second there is no relation between the amounts in the two envelopes; they are "free" (they just have to be different, to avoid a draw), while in the first one there is a precise functional relation between the two amounts.

In fact we can modify slightly Paradox 1 by assuming that the amounts in the two envelopes are X and $f(X)$, where f is, say, continuous and strictly increasing for $x > 0$ (and hence invertible), such that $f(0) = 0$ and

$$\frac{1}{2}\left(f(x) + f^{-1}(x)\right) > x,$$

for $x > 0$. Ann's argument runs as before.

REMARK 2.1. – There are plenty of such functions, for instance $f(x) = \alpha x$, for any $\alpha > 1$. Another large family is provided by the next statement.

PROPOSITION 2.2. – *Suppose f is strictly convex for $x > 0$, such that $f(0) = 0$, and with right derivative 1 at 0. Then for every $x > 0$, we have*

$$\frac{1}{2}\left(f(x) + f^{-1}(x)\right) > x.$$

PROOF. – Let $x_0 > 0$. If we denote by x_1 and x_2 the x-coordinates of the intersections of the line $y = x_0$ with the line $y = f(x_0)x/x_0$ and with the graph of the function f, respectively, then since f is strictly convex we have that $x_1 < x_2 = f^{-1}(x_0)$ and therefore, as $f(x) > x$ for $x > 0$,

$$f(x_0) - x_0 > x_0 - x_1 > x_0 - x_2 = x_0 - f^{-1}(x_0)$$

or, in other words,

$$\frac{1}{2}\left(f(x_0) + f^{-1}(x_0)\right) > x_0 ,$$

as we wanted to prove.

3. – The Schrödinger-Székely paradox

Another paradox is often mentioned in connection with the two envelope paradox. It was proposed by Gabor Székely in his book ([S], p. 178) and is an elaboration of the following two puzzles presented by Littlewood [L] (p. 26 in [B]).

The first puzzle is formulated as follows.

(a) "There is an indefinite supply of cards marked 1 *and* 2 *on opposite sides, and of cards marked* 2 *and* 3, 3 *and* 4, *and so on. A card is drawn at random by a referee and held between the players A, B so that each sees one side only. Either player may veto the round, but if it is played the player seeing the highest number wins. The point now is that every round is vetoed. If A sees a* 1, *on the other side is* 2 *and he must veto. If he sees a* 2 *on the other side is* 1 *or* 3*; if* 1 *then B must veto; if he does not then A must. And so on by induction."*

The next puzzle (discussed in [E]) is a probabilistic game Littlewood attributed to the physicist Erwin Schrödinger (his paradox involving a cat is far more intriguing).

(b) "We have cards similar to those in (a) but this time there are 10^n *of the ones of type* $(n, n+1)$, *and the player seeing the* lower *number wins. A and B may now bet each with a bookie (or for that matter with each other) backing themselves at evens. The position now is that whatever A sees he 'should' bet, and the same is true for B, the odds in favour being* 9 *to* 1 *[since] ... whatever number A sees, it is* 10 *times more probable that the other side is* $n+1$ *than that it is* $n-1$*."*

We will reconsider puzzle (a) in the last section.

Székely (quoting [L]) combined some suggestions from the two puzzles and created a new one, calling it "Schrödinger's" paradox (see [S], p. 178), which will be presented below as Paradox 3. We believe that it should be called the "Schrödinger-Székely paradox", because it is a substantial modification of puzzle (b). However, the current literature consistently calls it "Schrödinger's paradox".

PARADOX 3

Suppose a referee takes a card marked n and $n+1$ on opposite sides and holds it in front of Ann so that she can see one side only. If the number she sees is the largest, she loses the amount of dollars written on the other side; otherwise she wins the amount of dollars shown to her. She has the right to veto if the number she sees appears to be too large.

Ann may think: I can see the number n and therefore on the other side there is either $n-1$ or $n+1$, with the same probability. If I lose, I pay $n-1$ dollars, otherwise I win n dollars and therefore it is not worth vetoing.

If Bill is looking at the other side of the card, he will come to the same conclusion. His expected win is

$$\frac{1}{2}(n-(n-1))=\frac{1}{2},$$

so that the game is favorable for him. Why should he veto in a game which assures him on average half a dollar each round?

In Paradoxes 1 and 2 there is a functional relation between the two numbers. Note that in Remark 2.1 it was necessary to require $\alpha > 1$. Here $\alpha = 1$, but with a positive additive constant ($n+k$ instead of $n+1$ is an inessential modification) and changing the winning conditions we get again a puzzling conclusion.

We now state a "free" version of the previous game, which is the missing tile in this mosaic of puzzles.

PARADOX 4

Suppose a card is marked by a referee on opposite sides with two distinct positive numbers. Ann can see one side only. The rules are as in the previous paradox: if the number she sees is the largest, she loses the amount written on the other side; otherwise she wins the amount she sees. If the latter appears to be too high she may veto and play another round.

Ann may argue: m is the number I see so if I win, I get m and if I lose, I have to pay an amount $n<m$. Therefore my expected win is $(m-n)/2>0$ and it is not worth vetoing.

We fixed the stakes in order to make Paradox 4 as similar as possible to the Schrödinger-Székely paradox, but the puzzle keeps its nature if we change them so that if Ann wins, she gets $m+n$, while if she loses, she remains empty-handed. Our intention is to get closer to Paradox 2.

In fact, the most interesting feature of this new paradox is that it is not quite new!

There is an obvious relation with Paradox 2, which is evident specially if we consider the modified stakes. Let us reconsider that scenario with a small change: suppose that this time the referee opens the envelope which remained in his hands, while Ann's envelope stays closed. Now Anna is given the possibility to accept the amount in referee's envelope or to open her own. The same arguments which there convinced Ann to switch, now suggest her not to switch. The closed envelope is always more attractive!

This shows that there is a very precise relation between the Schrödinger-Székely paradox and the two envelope paradox.

After having described so many different scenarios it is time to analyze the puzzles and to present the explanations offered by recent (and less recent) literature.

Paradoxes 1 and 3 (the "functional" ones) are dismissed by the fact that there exists no probability distribution over pairs of numbers X_0 and X_1 such that each is equally likely to be larger, and such that observing any one of them does not change this equality. This has been proved in full generality in [SSS] (last two pages). See also [Li].

On the other hand Paradoxes 2 and 4 are not paradoxical at all.

Let us investigate in particular the new one, Paradox 4. Imagine a variant in which Ann is playing against a bookmaker and she alone has the possibility (instead of vetoing) of turning the card.

In Paradoxes 2 and 4 (the "free" ones), if Ann is playing against a bookmaker there is actually a strategy which allows her to win with probability larger than 1/2. This strategy has been proposed and analyzed for Paradox 2 in [SV] for the discrete case and in [SSS] for the most general case. Both papers attribute the idea to Blackwell [Bl].

We will see that the same argument can also be used for Paradox 4 in the variant in which Ann is allowed to turn the card. This is not surprising at all, in view of its equivalence with Paradox 2, and it is nothing more than an appropriate reformulation of the arguments proposed in [SV] and [SSS].

A winning strategy

Ann fixes a threshold k (using common sense or making a random experiment). If we denote by $X_0 < X_1$ the two numbers on the card, there are three possibilities. Denote by W the event that she wins.

i) If $k < X_0$, then she does not turn the card and her probability of winning is $P(W|k < X_0) = \frac{1}{2}$.

ii) If $X_0 \leq k < X_1$, then Ann is sure to win, because if she sees X_1, she will turn the card and win X_1, while if she sees X_0 she will not turn the card and will again win X_1. Therefore $P(W|X_0 \leq k < X_1) = 1$

iii) If $k \geq X_1$, then she turns the card and her probability of winning is $P(W|k \geq X_1) = \frac{1}{2}$ as in case (i).

Thus the overall probability of winning is

$$P(W) = P(W|k < X_0)P(k < X_0) + P(W|X_0 \leq k < X_1)P(X_0 \leq k < X_1) + P(W|k \geq X_1)P(k \geq X_1) > \frac{1}{2}\,.$$

Of course there are plenty of probability distributions (discrete or not) which can be used by the bookmaker to produce two different positive numbers. The probability that Ann will win depends on how the bookmaker picks the two numbers, but is in any case larger than $1/2$.

The existence of a winning strategy confirms that Ann can take advantage of the information given by the card she sees and that it is not convenient to assume that the conditional expectations are equal to $1/2$. Proposition 1 of [SSS] describes the principle on which this observation is based.

4. – A paradox in the paradox?

We learned in the previous section that there is a flaw in the arguments used by Ann and Bill in Paradoxes 1 and 3. But this fact alone is not an impediment for a gambler to play the game against a bookmaker.

There is another obstacle, of a logical type, which makes the game related to Paradox 1 questionable, at least if we assume that the stakes, in dollars, say, are expressed by positive integers (or by multiples of a minimal unit, for instance 1 cent).

For simplicity, let \$1 be the smallest unit. Then it is not reasonable for the dealer to put in any envelope an odd amount of $2n+1$ dollars, because if Ann opens that envelope, knowing that $\frac{2n+1}{2}$ dollars is not an admissible stake, she will switch and win. Given that, he cannot put in any envelope a sum of $2(2n+1)$ dollars because if Ann finds it, she will switch and win knowing that no envelope can contain an odd number. But then also stakes of the kind $4(2n+1)$ dollars are not acceptable, and by induction no positive number of dollars is admissible.

This obstacle is eliminated if we allow real or at least rational stakes, but while this decision may make a mathematician happy, the host of a TV program or a contestant may be less so!

This argument is reminiscent of the "surprise exam" paradox, which can be stated in the following way:

The teacher wanted to punish the students, saying: "You shall have an examination one day next week, but you will not know in advance which day. It will be a surprise."

The students reasoned that if they are not be examined before Friday, an examination on Friday would not be a surprise. Therefore the teacher cannot possibly defer it until Friday.

But the argument can be repeated: since the examination cannot be on Friday, if held on Thursday it would not be a surprise. The same argument applies now to Wednesday, and one by one the students excluded all the days of the week concluding that there will be no exam.

But when the teacher handed out exam papers on Tuesday, the students were surprised!

This paradox has been proposed in various equivalent scenarios. Gardner [G] presented it as the unexpected tiger puzzle, quoting the paper by Quine [Q] who calls his version "the unexpected hanging paradox". The latter is probably the most popular version. T. Y. Chow refers to it, saying [C] that despite significant academic interest, "no consensus on its correct resolution has yet been established".

The argument we used before seems to be an "infinite" version of the unexpected hanging paradox. We began excluding the odd numbers as possible stakes and proceeded by induction eliminating step by step all positive integers. But, just as the teacher can examine on Tuesday without breaking the rules, the bookmaker can put in the envelopes 1,024 and 2,048 dollars and if Ann opens the one with 2,048 dollars, she has no argument to divine the content of the other envelope.

Littlewood's puzzle (b) can also be seen as an infinite version of the unexpected hanging paradox: the card $(1,2)$ is eliminated first and then, one by one, by induction all the other cards are eliminated. But if a player sees the number 3,748 on one side of the card, again she or he has no good reason to veto.

The Schrödinger-Székely game suffers from a similar hitch if we assume that the highest win, 10,000 dollars, say, is known.

Now if Ann sees 10,000, she will veto. If she sees 9,999 and Bill does not veto, she must. And so on backwards, by induction all the cards have to be eliminated.

However, if Ann sees the number 3,748 there is no obvious reason why she should veto.

But even if the highest win is not known, it is questionable if the Schrödinger-Székely puzzle can be used to arrange a reasonable game between two players. Littlewood's argument does not apply as it is, since the winning conditions are inverted, but it pops up in a different way.

The host wants to increase the audience and to keep it in suspense. Therefore he cannot show to Ann and Bill the card $(1,2)$, because whoever sees 1 knows that they have lost. Tossing a coin is simpler and would give the same excitement to the audience!

But now Littlewood's clockwork applies again: the host cannot use the card $(2,3)$ either, and by induction he cannot use any card.

On the other hand, if the card $(721,722)$ is shown to Ann and Bill, there is no evident reason for anybody to conclude that they have lost.

Conclusions

We believe that a paper devoted to the two envelope problem is appropriate to honor Giovanni Prodi's memory. It gives evidence of his broad interests which allowed him to contribute also to areas which are quite far from those which gave him a wide international reputation.

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