Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.3, p. 393–407.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_3_393_0>

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Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 393-407

A Peculiar Liapunov Functional for Ternary Reaction-Diffusion Dynamical Systems

SALVATORE RIONERO

To the memory of Giovanni Prodi.

Abstract. – *A Liapunov functional W, depending - together with the temporal derivative $\dot{W}$ along the solutions - on the eigenvalues via the system coefficients, is found. This functional is "peculiar" in the sense that W is positive definite and simultaneously $\dot{W}$ is negative definite, if and only if all the eigenvalues have negative real part. An application to a general type of ternary system often encountered in the literature, is furnished.*

1. – Introduction

Let $\Omega \subset \mathbb{R}^q$, $(q = 1, 2, 3)$, be a smooth bounded domain. This paper is concerned with the reaction-diffusion systems

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{L}\boldsymbol{u} + \boldsymbol{F}, \qquad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

with $\boldsymbol{u} = (u_1, u_2, u_3)^T$, $\boldsymbol{F} = (F_1, F_2, F_3)^T$,

$$L = \begin{pmatrix} a_{11} + \gamma_1 \Delta & a_{12} & a_{13} \\ a_{21} & a_{22} + \gamma_2 \Delta & a_{23} \\ a_{31} & a_{32} & a_{33} + \gamma_3 \Delta \end{pmatrix}; \tag{1.2}$$

$F_i = F_i(u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3)$, $(i = 1, 2, 3)$, being (generally) nonlinear and

$$\begin{cases} a_{ij} = \text{const.} \in \mathbb{R}, \quad \gamma_i = \text{const.} > 0, \quad i,j \in \{1,2,3\}, \\ u_i : (\boldsymbol{x}, t) \in \Omega \times \mathbb{R}^+ \to u_i(\boldsymbol{x}, t) \in \mathbb{R}, \quad \forall i \in \{1,2,3\}. \end{cases} \tag{1.3}$$

To (1.1) we append the Robin boundary conditions

$$\beta \boldsymbol{u} + (1 - \beta) \nabla \boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.4}$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$,

$$\begin{cases} \beta : \boldsymbol{x} \in \partial\Omega \to \beta(\boldsymbol{x}) \in \mathbb{R}, \\ 0 \le \beta \le 1, \quad \forall \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{1.5}$$

β being a sufficiently regular function not identically zero.

The nonlinear functions $F_i = F_i(u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3)$ are assumed to be sufficiently regular and such that

$$(F_i)_{u_1=u_2=u_3=0} = 0, \qquad \forall i \in \{1,2,3\}. \tag{1.6}$$

Therefore (1.1)-(1.6) admits the zero solution. To the L^2-stability of this solution is precisely devoted the present paper.

REMARK 1.1. – As it is well known, the stability of a non zero solution of a system S can be reduced to the stability of the zero solution of a system S^* easily linked to S.

We assume that Ω is of class C^p $(p > 2)$ and has the interior cone property. We denote by

- $\langle \cdot, \cdot \rangle$ the scalar product of $L^2(\Omega)$;
- $\langle \cdot, \cdot \rangle_{\partial\Omega}$ the scalar product of $L^2(\partial\Omega)$;
- $\| \cdot \|$ the norm of $L^2(\Omega)$;
- $\| \cdot \|_{\partial\Omega}$ the norm of $L^2(\partial\Omega)$;
- $W^{1,2}(\Omega, \beta)$ the functional space such that

$$W^{1,2}(\Omega, \beta) = \left\{ \varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial\Omega), \beta\varphi + (1-\beta)\nabla\varphi \cdot \boldsymbol{n} = 0, \text{ on } \partial\Omega \right\}.$$

For $\beta > 0$, $\beta \not\equiv 1$, it follows {cfr. [1], pp. 92-98 } that

$$\left\| \sqrt{\frac{\beta}{1-\beta}}\varphi \right\|^2_{\partial\Omega} + \|\nabla\varphi\|^2 \geq \bar{\alpha}\|\varphi\|^2, \tag{1.7}$$

where $\bar{\alpha} = \bar{\alpha}(\Omega, \beta) = \text{const.} > 0$, is the smallest eigenvalue of the spectral problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & \text{in } \Omega, \\ \beta\varphi + (1-\beta)\nabla\varphi \cdot \boldsymbol{n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

i.e. the principal eigenvalue of $-\Delta$ in $W^{1,2}(\Omega, \beta)$.

In the sequel *we assume that*

i) (1.1)-(1.5) *has the properties of a dynamical system* [2] *embedded in* $W^{1,2}(\Omega, \beta)$ *and hence*

$$u_i \in W^{1,2}(\Omega, \beta); \tag{1.9}$$

ii) *the functions F_i are such that*

$$\left\langle \sum_{i=1}^{3} |u_i|, \sum_{j=1}^{3} |F_j| \right\rangle \leq k_1(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2)^{1+\varepsilon_1} + \\ + k_2(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2)^{\varepsilon_2}(\|\nabla u_1\|^2 + \|\nabla u_2\|^2 + \|\nabla u_3\|^2), \tag{1.10}$$

with k_i, ε_i, $(i = 1, 2)$, non negative constants.

Setting

$$b_{11} = a_{11} - \bar{\alpha}\gamma_1, \; b_{22} = a_{22} - \bar{\alpha}\gamma_2, \; b_{33} = a_{33} - \bar{\alpha}\gamma_3, \tag{1.11}$$

in [3] have been found conditions on a_{ij}, with $i \neq j$, able to reduce the stability of the zero solution of (1.1)-(1.6) to the stability of the zero solution of the linear system of O.D.Es

$$\frac{d\boldsymbol{u}}{dt} = \mathcal{L}\boldsymbol{u}, \tag{1.12}$$

with either

$$\mathcal{L} = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & a_{23} \\ 0 & a_{32} & b_{33} \end{pmatrix} \tag{1.13}$$

or - when $a_{ij}a_{ji} > 0$, $(i,j = 1,2,3)$ -

$$\mathcal{L} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{1.14}$$

b_{11}, b_{22}, b_{33} being given by (1.11) and

$$b_{ij} = b_{ji} = (\text{sign } \; a_{ij})\sqrt{a_{ij}a_{ji}}. \tag{1.15}$$

In the present paper, in the guideline of [3]-[6], we reconsider the problem aimed to show that:

i) *the local stability* ([1]) *of the zero solution of* (1.1)-(1.6) *can be reduced always to the stability of the zero solution of the linear system of O.D.Es.*

$$\frac{d\boldsymbol{x}}{dt} = \tilde{\mathcal{L}}\boldsymbol{x}, \tag{1.16}$$

with

$$\tilde{\mathcal{L}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix} \tag{1.17}$$

λ_1 *being a real eigenvalue of*

$$\tilde{L} = \begin{pmatrix} b_{11} & a_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & a_{32} & b_{33} \end{pmatrix} \tag{1.18}$$

([1]) In the context of the Navier-Stokes equations, the concept of local stability was developed extensively by G. Prodi [10] *and* $\tilde{a}_{ij}$ *real constants, linked in a suitable simple way to* a_{ij} {*cfr. Lemma* 2.3} *and such that*

$$I = \tilde{a}_{22} + \tilde{a}_{33} = I_1 - \lambda_1, \quad A = \tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{32} = \frac{I_3}{\lambda_1}, \tag{1.19}$$

where I_1 *and* I_3 *are the invariants of* $\tilde{L}$ *given by*

$$\begin{cases} I_1 = b_{11} + b_{22} + b_{33} = \lambda_1 + \lambda_2 + \lambda_3 \\ I_3 = \text{det of } \tilde{L} = \lambda_1\lambda_2\lambda_3, \end{cases} \tag{1.20}$$

λ_i, $(i = 1, 2, 3)$, *being the eigenvalues of* $\tilde{L}$;

ii) *the function*

$$W = \frac{1}{2}\left[x_1^2 + A\left(x_2^2 + x_3^2\right) + (\tilde{a}_{22}x_3 - \tilde{a}_{32}x_2)^2 + (\tilde{a}_{23}x_3 - \tilde{a}_{33}x_2)^2\right], \tag{1.21}$$

having the temporal derivative along (1.16) *given by*

$$\dot{W} = \lambda_1 x_1^2 + IA(x_2^2 + x_3^2), \tag{1.22}$$

is a "peculiar" Liapunov function for (1.16) *in the sense that – W is positive definite and simultaneously* $\dot{W}$ *is negative definite – if and only if the real part of the eigenvalues* λ_i *are negative,*

iii) *the functional*

$$W^* = \frac{1}{2}\left\{\left[||\tilde{u}_1||^2 + A(||\tilde{u}_2||^2 + ||\tilde{u}_3||^2) + ||\tilde{a}_{22}\tilde{u}_3 - \tilde{a}_{32}\tilde{u}_2||^2 + ||\tilde{a}_{23}\tilde{u}_3 - \tilde{a}_{33}\tilde{u}_2||^2\right]\right\} \tag{1.23}$$

with $\tilde{u}_i$ *linked to* u_i *in a suitable linear way, is a peculiar Liapunov function for* (1.1)-(1.6), *when* (1.10) *and some large conditions on* (γ_2, γ_3) *hold* {*cfr. Lemma* 3-2 *of* [3]}.

Section 2 is devoted to some preliminary Lemmas concerned with the stability of matrices of systems of O.D.Es. To the L^2-stability of the zero of (1.1)-(1.4) is addressed Section 3 while in Section 4 the results obtained in the previous Sections are applied to a class of systems modeling various phenomena. The paper ends with an appendix in which is recalled a remark concerned with the eigenvalues of (1.1)-(1.4).

2. – Preliminaries

We collect here some Lemmas useful for the sequel

LEMMA 2.1. – *The asymptotic stability of the null solution of*

(2.1)
$$\begin{cases} \dfrac{dx_1}{dt} = \lambda_1 x_1 + f_1(x_1, x_2, x_3) \\ \dfrac{dx_2}{dt} = \tilde{a}_{22}x_2 + \tilde{a}_{23}x_3 + f_2(x_1, x_2, x_3) \\ \dfrac{dx_3}{dt} = \tilde{a}_{32}x_2 + \tilde{a}_{33}x_3 + f_3(x_1, x_2, x_3) \end{cases}$$

with $f_i (i = 1, 2, 3)$, *nonlinear functions such that*

(2.2)
$$\left(\sum_{i=1}^{3} |x_i|\right)\left(\sum_{i=1}^{3} |f_i|\right) \leq k(x_1^2 + x_2^2 + x_3^2)^{1+\varepsilon}$$

with k and ε positive constants, is guaranteed iff

(2.3)
$$\lambda_1 < 0, \quad I < 0, \quad A > 0.$$

PROOF. – The proof can be obtained either by observing that (2.3) are equivalent to the Routh-Hurwitz necessary and sufficient conditions for all the eigenvalues of $\mathcal{L}$ have negative real part [7]-[8] {cfr. Remark 2.1} or by introducing the peculiar Liapunov function (1.21) which temporal derivative along the solution of (2.1) is given by [3]

(2.4)
$$\dot{W} = \lambda_1 x_1^2 + IA(x_2^2 + x_3^2) + \Psi$$

with

(2.5)
$$\begin{cases} \Psi = (\alpha_1 x_2 - \alpha_2 x_3)f_2 + (\alpha_2 x_3 - \alpha_3 x_2)f_3 + x_1 f_1 \\ \alpha_1 = A + \tilde{a}_{32}^2 + \tilde{a}_{33}^2,\ \alpha_2 = A + \tilde{a}_{22}^2 + \tilde{a}_{23}^2,\ \alpha_3 = \tilde{a}_{22}\tilde{a}_{32} + \tilde{a}_{23}\tilde{a}_{33}, \end{cases}$$

□

LEMMA 2.2. – *Let* (2.2) *hold. Then the asymptotic stability of the null solution of the ternary systems of O.D.Es.*

(2.6)
$$\begin{cases} \dfrac{dx_1}{dt} = \lambda_1 x_1 + \tilde{a}_{12}x_2 + \tilde{a}_{13}x_3 + f_1 \\ \dfrac{dx_2}{dt} = \tilde{a}_{22}x_2 + \tilde{a}_{23}x_3 + f_2 \\ \dfrac{dx_3}{dt} = \tilde{a}_{32}x_2 + \tilde{a}_{33}x_3 + f_3 \end{cases}$$

can be reduced to the stability of the null solution of (2.1).

PROOF. – Setting

(2.7)
$$x_i = \mu_i y_i,$$

with μ_i, $(i = 1, 2, 3)$, scalings to be chosen suitably later, (2.6) becomes

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda_1 y_1 + \tilde{f}_1 \\ \dfrac{dy_2}{dt} = \tilde{a}_{22} y_2 + \dfrac{\mu_3}{\mu_2} \tilde{a}_{23} y_3 + \tilde{f}_2 \\ \dfrac{dy_3}{dt} = \dfrac{\mu_2}{\mu_3} y_2 + \tilde{a}_{33} y_3 + \tilde{f}_3 \end{cases} \tag{2.8}$$

with

$$\begin{cases} \tilde{f}_1 = \dfrac{\mu_2}{\mu_1} \tilde{a}_{12} y_2 + \dfrac{\mu_3}{\mu_1} \tilde{a}_{13} y_3 + \dfrac{1}{\mu_1} f_1(\mu_1 y_1, \mu_2 y_2, \mu_3 y_3) \\ \tilde{f}_j = \dfrac{1}{\mu_j} f_j(\mu_1 y_1, \mu_2 y_2, \mu_3 y_3), \qquad j = 2, 3. \end{cases} \tag{2.9}$$

Introducing the functional $\tilde{W}$, analogous to (1.21)

$$\tilde{W} = \frac{1}{2}\left[y_1^2 + A(y_2^2 + y_3^2) + (\tilde{a}_{22} y_3 - \frac{\mu_2}{\mu_3} \tilde{a}_{32} y_2)^2 + \left(\frac{\mu_3}{\mu_2} \tilde{a}_{23} y_3 - \tilde{a}_{33} y_2 \right)^2 \right] \tag{2.10}$$

and taking into account (2.2), it turns out that

$$\frac{d\tilde{W}}{dt} \leq \lambda_1 y_1^2 + IA(y_2^2 + y_3^2) + \Phi + k_1 \tilde{W}^{1+\varepsilon_1} \tag{2.11}$$

with k_1 and ε_1 positive constants and Φ given by

$$\Phi = \frac{\mu_2}{\mu_1} \tilde{a}_{12} y_1 y_2 + \tilde{a}_{13} \frac{\mu_3}{\mu_1} y_1 y_3. \tag{2.12}$$

Choosing $\mu_2 = \mu_3$ and setting

$$\mu = \frac{\mu_2}{\mu_1} = \frac{\mu_3}{\mu_1} \tag{2.13}$$

it follows that

$$\Phi \leq m\mu |y_1| (|y_2| + |y_3|) \leq \frac{m^2 \mu^2 y_1^2}{|IA|} + \frac{1}{2} |IA| (y_2^2 + y_3^2) \tag{2.14}$$

with

$$m = max(|\tilde{a}_{12}|, |\tilde{a}_{13}|) \tag{2.15}$$

and hence

$$\mu^2 = \frac{|\lambda_1 IA|}{2m^2} \tag{2.16}$$

implies

$$\Phi \leq \frac{1}{2}\left[|\lambda_1| y_1^2 + IA(y_2^2 + y_3^2)\right]. \tag{2.17}$$

Then by virtue of (2.11) and (2.17), one obtains

$$\frac{d\tilde{W}}{dt} \leq -(k_* - k_1 \tilde{W}^{\varepsilon_1})\tilde{W} \tag{2.18}$$

with k_* positive constant and hence

$$\tilde{W}^{\varepsilon_1}(0) < \frac{k_*}{k_1} \Rightarrow \tilde{W} \leq \tilde{W}(0) \exp\left[-(k_* - k_1 \tilde{W}^{\varepsilon_1}(0))\right] t. \tag{2.19}$$

LEMMA 2.3. – *Let λ_1 be a (real) eigenvalue of $\tilde{L}$ and $\boldsymbol{U} = (U_1, U_2, U_3)$ to the associated eigenvector with $U_1 \neq 0$. Then the transformation*

$$\boldsymbol{X} = \tilde{L}_1 \boldsymbol{Z} \tag{2.20}$$

with $\boldsymbol{X} = (X_1, X_2, X_3)^T$, $\boldsymbol{Z} = (Z_1, Z_2, Z_3)^T$ and

$$\tilde{L}_1 = \begin{pmatrix} U_1 & 0 & 0 \\ U_2 & 1 & 0 \\ U_3 & 0 & 1 \end{pmatrix} \tag{2.21}$$

reduces the ternary system

$$\begin{cases} \dfrac{dX_1}{dt} = b_{11}X_1 + a_{12}X_2 + a_{13}X_3 \\ \dfrac{dX_2}{dt} = a_{21}X_1 + b_{22}X_2 + b_{23}X_3 \\ \dfrac{dX_3}{dt} = a_{31}X_1 + a_{32}X_2 + b_{33}X_3 \end{cases} \tag{2.22}$$

$$\begin{cases} \dfrac{dZ_1}{dt} = \lambda_1 Z_1 + \tilde{a}_{12}Z_2 + \tilde{a}_{13}Z_3 \\ \dfrac{dZ_2}{dt} = \tilde{a}_{22}Z_2 + \tilde{a}_{23}Z_3 \\ \dfrac{dZ_3}{dt} = \tilde{a}_{32}Z_2 + \tilde{a}_{33}Z_3 \end{cases} \tag{2.23}$$

with

$$\begin{cases} \tilde{a}_{12} = \dfrac{a_{12}}{U_1}, & \tilde{a}_{13} = \dfrac{a_{13}}{U_1}, \\ \tilde{a}_{22} = b_{22} - U_2\tilde{a}_{12}, & \tilde{a}_{23} = a_{23} - U_2\tilde{a}_{13}, \\ \tilde{a}_{32} = a_{32} - U_3\tilde{a}_{12}, & \tilde{a}_{33} = b_{33} - U_3\tilde{a}_{13}. \end{cases} \tag{2.24}$$

PROOF. – We apply, in the case (2.20)-(2.21), the procedure given in {[11], pp. 196-197} and begin by observing that

$$\left\{\begin{array}{l} b_{11}U_1+a_{12}U_2+a_{13}U_3=\lambda_1 U_1 \\ a_{22}U_1+b_{22}U_2+a_{23}U_3=\lambda_1 U_2 \\ a_{31}U_1+a_{32}U_2+b_{33}U_3=\lambda_1 U_3 \end{array}\right. \tag{2.25}$$

and

$$X_1=U_1Z_1, \quad X_2=U_2Z_1+Z_2, \quad X_3=U_3Z_1+Z_3 \tag{2.26}$$

$$Z_1=\frac{1}{U_1}X_1, \quad Z_2=X_2-\frac{U_2}{U_1}X_1, \quad Z_3=X_3-\frac{U_3}{U_1}X_1 \tag{2.27}$$

hold. Then, by virtue of (2.25)-(2.27), it turns out that

$$\begin{aligned} \frac{dZ_1}{dt}&=\frac{1}{U_1}\frac{dX_1}{dt}=\frac{1}{U_1}(b_{11}X_1+a_{12}X_2+a_{13}X_3)\\ &=\frac{1}{U_1}[(b_{11}U_1+a_{12}U_2+a_{13}U_3)Z_1+a_{12}U_2Z_2+a_{13}U_3Z_3]. \end{aligned}$$

i.e.

$$\frac{dZ_1}{dt}=\lambda_1 Z_1+\frac{a_{12}}{U_1}U_2Z_2+\frac{a_{13}}{U_1}U_3Z_3 \tag{2.28}$$

$$\begin{aligned} \frac{dZ_2}{dt}&=\frac{dX_2}{dt}-U_2\frac{dZ_1}{dt}=(a_{21}X_1+b_{22}X_2+a_{23}X_3)-U_2\left(\lambda_1 Z_1+\frac{a_{12}}{U_1}Z_2+\frac{a_{13}}{U_1}Z_3\right)\\ &=a_{21}U_1Z_1+b_{22}(U_2Z_1+Z_3)+a_{23}(U_3Z_1+Z_3)-U_2\left(\lambda_1 Z_1+\frac{a_{12}}{U_1}Z_2+\frac{a_{13}}{U_1}Z_3\right)\\ &=[(a_{21}U_1+b_{22}U_2+a_{23}U_3)-\lambda_1 U_2]Z_1+\left(b_{22}-\frac{U_2}{U_1}a_{12}\right)Z_2+\left(a_{23}-\frac{U_2}{U_1}a_{13}\right)Z_3; \end{aligned}$$

i.e.

$$\frac{dZ_2}{dt}=+\left(b_{22}-\frac{U_2}{U_1}a_{12}\right)Z_2+\left(a_{23}-\frac{U_2}{U_1}a_{13}\right)Z_3 \tag{2.29}$$

$$\begin{aligned} \frac{dZ_3}{dt}&=\frac{dX_3}{dt}-U_3\frac{dZ_1}{dt}=(a_{31}X_1+a_{32}X_2+b_{33}X_3)-U_3\left(\lambda_1 Z_1+\frac{a_{12}}{U_1}Z_2+\frac{a_{13}}{U_1}Z_3\right)\\ &=a_{31}U_1Z_1+a_{32}(U_2Z_1+Z_2)+b_{33}(U_3Z_1+Z_3)-U_3\left(\lambda_1 Z_1+\frac{a_{12}}{U_1}Z_2+\frac{a_{13}}{U_1}Z_3\right)\\ &=[(a_{31}U_1+a_{32}U_2+b_{33}U_3)-\lambda_1 U_3]Z_1+\left(a_{32}-\frac{U_3}{U_1}a_{12}\right)Z_2+\left(b_{33}-\frac{U_3}{U_1}a_{13}\right)Z_3; \end{aligned}$$

i.e.

$$\frac{dZ_3}{dt} = \left(a_{32} - \frac{U_3}{U_1}a_{12}\right)Z_2 + \left(b_{33} - \frac{U_3}{U_1}a_{13}\right)Z_3 \tag{2.30}$$

By virtue of (2.28)-(2.30), (2.23) with the $\tilde{a}_{ij}$ given by (2.24) immediately follow.

REMARK 2.1. – Denoting by I_2 the invariant

$$I_2 = \lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 = \begin{vmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{13} \\ a_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & a_{23} \\ a_{32} & b_{33} \end{vmatrix} \tag{2.31}$$

of $\tilde{L}$, the characteristic (eigenvalues) equation of $\tilde{L}$ is easily found to be

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0. \tag{2.32}$$

Since λ_1 is supposed to be a root of (2.32) as expected, it turns out that

$$\lambda^2 + (I_1 - \lambda_1)\lambda + \frac{I_3}{\lambda_1} = \lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3 = 0. \tag{2.33}$$

On the other hand λ_2 and λ_3 have to be eigenvalues of $\tilde{\mathcal{L}}$ {cfr. appendix}, hence it follows that (1.19) hold.

REMARK 2.2. – By virtue of Lemma 2.1 with $f_1 = f_2 = f_3 = 0$, it follows that

$$W = \frac{1}{2}\left[Z_1^2 + A(Z_2^2 + Z_3^2) + (\tilde{a}_{22}Z_3 - \tilde{a}_{32}Z_2)^2 + (\tilde{a}_{23}Z_3 - a_{33}Z_2\right], \tag{2.34}$$

having the temporal derivative along (2.23) given by

$$\dot{W} = \lambda_1 Z_1^2 + IA(Z_2^2 + Z_3^2) \tag{2.35}$$

is a " peculiar" Liapunov function for (2.23) since the conditions (2.3)-equivalent to the Routh-Hurwitz conditions for all the eigenvalues of $\mathcal{L}$ have negative real part - guarantee that W is positive definite and $\dot{W}$ is negative definite.

REMARK 2.3. – In view of (2.27), (2.32) and

$$\lambda_1 = \sqrt[[3]]{-\frac{q}{2} + \sqrt{\Delta^*}} + \sqrt[[3]]{-\frac{q}{2} - \sqrt{\Delta^*}} \tag{2.36}$$

with

$$\Delta^* = \frac{q^2}{4} + \frac{p^3}{27}, \quad p = I_2 + \frac{1}{3}I_1, \quad q = -I_3 + \frac{1}{3}I_1I_2 + \frac{2}{27}I_1^2 \tag{2.37}$$

it follows that the function

$$
\begin{aligned}
W = \frac{1}{2U_1^2}\Big\{X_1^2 + \frac{I_3}{\lambda_1}\Big[(U_1X_2 - U_2X_1)^2 + (U_1X_3 - U_3X_1)^2\Big] \\
+[\tilde{a}_{22}(U_1X_3 - U_3X_1) - \tilde{a}_{32}(U_1X_2 - U_2X_1)]^2 \\
+[\tilde{a}_{23}(U_1X_3 - U_3X_1) - \tilde{a}_{33}(U_1X_2 - U_2X_1)]^2\Big\}
\end{aligned}
\tag{2.38}
$$

with $\tilde{a}_{22}, \tilde{a}_{33}, \tilde{a}_{23}, \tilde{a}_{32}$ given by (2.24), has the temporal derivative along (2.22) given by

$$
\dot{W} = \frac{1}{U_1^2}\Big\{\lambda_1 X_1^2 + (I_1 - \lambda_1)\frac{I_3}{\lambda_1}\Big[(U_1X_2 - U_2X_1)^2 + (U_1X_3 - U_3X_1)^2\Big]\Big\}
\tag{2.39}
$$

and is a " peculiar" Liapunov function for (2.22).

REMARK 2.4. – The system

$$
\frac{d\boldsymbol{X}}{dt} = \tilde{L}\boldsymbol{X} + \boldsymbol{f}
\tag{2.40}
$$

with $\tilde{L}$ given by (1.20) and $\boldsymbol{f} = (f_1, f_2, f_3)^T$, by virtue of (2.26), is reduced to

$$
\begin{cases}
\dfrac{dZ_1}{dt} = \lambda_1 Z_1 + \tilde{a}_{12}Z_2 + \tilde{a}_{13}Z_3 + f_1^* \\
\dfrac{dZ_2}{dt} = \tilde{a}_{22}Z_2 + \tilde{a}_{23}Z_3 + f_2^* \\
\dfrac{dZ_3}{dt} = \tilde{a}_{32}Z_2 + \tilde{a}_{33}Z_3 + f_3^*
\end{cases}
\tag{2.41}
$$

with

$$
f_1^* = \frac{1}{U_1}f_1, \quad f_2^* = f_2 - \frac{U_2}{U_1}f_1, \quad f_3^* = f_3 - \frac{U_3}{U_1}f_1.
\tag{2.42}
$$

Further, when (2.2) holds, it is easily verified that exist two positive constants $\tilde{\varepsilon}, \tilde{k}$ such that

$$
\left(\sum_{i=1}^{3}|Z_i|\right)\left(\sum_{i=1}^{3}|f_i^*|\right) \le \tilde{k}\left(z_1^2 + z_2^2 + z_3^2\right)^{1+\tilde{\varepsilon}}.
\tag{2.43}
$$

Hence, by virtue of Lemma 2.2, it follows that (2.3) guarantee the (local) asymptotic stability of the null solution of (2.41) and that (2.35) is a " peculiar" Liapunov function for (2.41) and hence (2.39) for (2.40).

REMARK 2.5. – For the sake of completeness we end by recalling that the Routh-Hurwitz conditions for all the eigenvalues of (2.32) have negative real part are [7]-[8]

$$I_1 < 0, \;\; I_2 > 0, \;\; I_3 < 0, \;\; I_1 I_2 - I_3 < 0. \tag{2.44}$$

Therefore - in the case of $\tilde{\mathcal{L}}$- being

$$\begin{cases} I_1 = \lambda_1 + I, \quad I_2 = \lambda_1 I + A, \\ I_3 = \lambda_1 A, \qquad I_1 I_2 - I_3 = \lambda_1 I \left(I + \dfrac{\lambda_1^2 + A}{\lambda_1} \right) \end{cases} \tag{2.45}$$

it immediately follows that (2.44) are implied by (2.3). Viceversa let (2.44) hold. Then, in view of $(2.45)_3$ it immediately follows that $I_3 < 0$ with $\lambda_1 > 0$ and $A < 0$ is not admissible. In fact, by virtue of $\{I_1 < 0, \lambda_1 > 0\}$ and $(2.45)_1$ it follows that $I < -\lambda_1 < 0$ and hence $\lambda_1 I < 0$ which, together with $A < 0$, implies $I_2 < 0$. Therefore (2.44)-(2.45) imply $\{\lambda_1 < 0, A > 0\}$. It remains to obtain $I < 0$. Since by virtue of $(2.45)_4$, (2.44) is equivalent to

$$(-\lambda_1) I \left(I + \frac{\lambda_1^2 + A}{\lambda_1} \right) > 0$$

it follows that $I \notin \left[0, -\dfrac{\lambda_1^2 + A}{\lambda_1} \right]$. By virtue of $-\dfrac{\lambda_1^2 + A}{\lambda_1} > -\lambda_1$, $(2.44)_1$ does not allow, in view of $(2.45)_1$, $I > -\lambda_1$ hence $I < 0$.

REMARK 2.6. – We remark that the method introduced holds even when the characteristic equation of $\tilde{L}$ has multiple roots.

3. – L^2-stability of the zero solution of (1.1)-(1.4)

In view of (1.11), (1.1) can be written

$$\frac{\partial \boldsymbol{u}}{\partial t} = \tilde{L} \boldsymbol{u} + \boldsymbol{F} + \boldsymbol{F}^* \tag{3.1}$$

with

$$\boldsymbol{F}^* = (F_1^*, F_2^*, F_3^*)^T, \quad F_i^* = \gamma_i (\Delta u_i + \bar{\alpha} u_i), \quad (i = 1, 2, 3). \tag{3.2}$$

By virtue of Lemma 2.2 the transformation

$$\boldsymbol{u} = \tilde{L}_1 \boldsymbol{v} \tag{3.3}$$

i.e.

$$u_1 = U_1 v_1, \;\; u_2 = U_2 v_1 + v_2, \;\; u_3 = U_3 v_1 + v_3 \tag{3.4}$$

(3.1) becomes

$$\text{(3.5)}\qquad \begin{cases} \dfrac{\partial v_1}{\partial t} = \lambda_1 v_1 + \tilde{a}_{12} v_2 + \tilde{a}_{13} v_3 + \gamma_1(\Delta v_1 + \bar{\alpha} v_1) + P_1 \\ \dfrac{\partial v_2}{\partial t} = \tilde{a}_{22} v_2 + \tilde{a}_{23} v_3 + \gamma_2(\Delta v_2 + \bar{\alpha} v_2) + U_2(\gamma_2 - \gamma_1)(\Delta v_1 + \bar{\alpha} v_1) + P_2 \\ \dfrac{\partial v_3}{\partial t} = \tilde{a}_{32} v_2 + \tilde{a}_{33} v_3 + \gamma_3(\Delta v_3 + \bar{\alpha} v_3) + U_3(\gamma_3 - \gamma_1)(\Delta v_1 + \bar{\alpha} v_1) + P_3 \end{cases}$$

with the $\tilde{a}_{ij}$ given by (2.24) and

$$\text{(3.6)}\qquad \begin{cases} P_1 = \dfrac{1}{U_1} F_1(\tilde{L}_1 \boldsymbol{v}), \qquad P_2 = F_2(\tilde{L}_1 \boldsymbol{v}) - U_2 F_1(\tilde{L}_1 \boldsymbol{v}), \\ P_3 = F_3(\tilde{L}_1 \boldsymbol{v}) - U_3 F_1(\tilde{L}_1 \boldsymbol{v}). \end{cases}$$

By virtue of the linearity of (3.4)-(3.6), the boundary conditions

$$\text{(3.7)}\qquad \beta \boldsymbol{v} + (1-\beta)\nabla \boldsymbol{v} \cdot \boldsymbol{n} = 0 \qquad \text{on} \qquad \partial\Omega \times R^+$$

have to be appended to (3.5).

We remark that, by virtue of the linear transformation (3.4), a cross-diffusion term appears in $(3.5)_2$-$(3.5)_3$. To (3.5)-(3.7) the methodologies introduced in [11]-[12] can be applied. We here confine ourselves to the case $\gamma_1 = \gamma_2 = \gamma_3$. In this case to (3.5)-(3.7) - as it is easily verified - can be applied Theorem 3.1 of [3] which h guarantees that the (local) asymptotic stability hold iff

$$\text{(3.8)}\qquad \lambda_1 < 0, \quad I = I_1 - \lambda_1 < 0, \quad A = \frac{I_3}{\lambda_1} > 0$$

and

$$\text{(3.9)}\quad W = \frac{1}{2}\left[||v_1||^2 + A(||v_2||2 + ||v_3||^2) + ||\tilde{a}_{22} v_3 - \tilde{a}_{32} v_2||^2 + ||\tilde{a}_{23} v_3 - \tilde{a}_{33} v_2||^2 \right]$$

is a peculiar Liapunov function.

4. – An useful application

As shown in {Lemma 4.1 of [3]}, if

$$\text{(4.1)}\qquad \begin{cases} a_{ij} a_{ji} > 0, \qquad i \neq j \\ a_{12} a_{23} a_{31} = a_{13} a_{21} a_{31}, \end{cases}$$

via a suitable scaling of the u_i, L can be symmetrized and the zero solution is stable iff the quadratic form associated to the symmetrized matrix is semi-negative definite. Therefore in order to put in evidence the easy applicability of the procedures introduced in the previous sections, we consider the (non-symmetrizable) system (1.1)-(1.4) with

$$\text{(4.2)}\qquad \begin{cases} a_{11} = a_{22} = -a_1, & a_{33} = -a, \qquad \gamma_1 = \gamma_2 = \gamma_3 = \gamma, \\ a_{12} = a_{21} = 0, & a_{13} = a_{31} = c, \quad a_{23} = -a_{32} = b, \end{cases}$$

(a, b, c, γ being positive constants) often encountered in literature [13]-[14]. Setting

$$\text{(4.3)}\qquad \begin{cases} R = a + \bar{\alpha}\gamma, \\ R_1 = a_1 + \bar{\alpha}\gamma, \end{cases}$$

it turns out that

$$\text{(4.4)}\qquad \begin{cases} \dfrac{\partial u_1}{\partial t} = -R_1 u_1 + c u_3 + \gamma(\varDelta u_1 + \bar{\alpha} u_1) + F_1 \\ \dfrac{\partial u_2}{\partial t} = -R_1 u_2 + b u_3 + \gamma(\varDelta u_2 + \bar{\alpha} u_2) + F_2 \\ \dfrac{\partial u_3}{\partial t} = c u_1 - b u_2 - R u_3 + \gamma(\varDelta u_3 + \bar{\alpha} u_3) + F_3 \end{cases}$$

As it is easily verified the matrix

$$\text{(4.5)}\qquad \tilde{L} = \begin{pmatrix} -R_1 & 0 & c \\ 0 & -R_1 & b \\ c & -b & -R \end{pmatrix}$$

admits the eigenvalues $\lambda_1 = -R_1$ with the associated eigenvector, $\boldsymbol{U} = (U_1 = b, U_2 = c, U_3 = 0)$ and hence

$$\text{(4.6)}\qquad \boldsymbol{u} = \tilde{L}_1 \boldsymbol{v}, \quad \tilde{L}_1 = \begin{pmatrix} b & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

give

$$\text{(4.7)}\qquad u_1 = b v_1, \qquad u_2 = c v_1 + v_2, \qquad u_3 = v_3.$$

Then it easily follows that

$$\text{(4.8)}\qquad \begin{cases} \dfrac{\partial v_1}{\partial t} = -R v_1 + \dfrac{c}{b} v_3 + \gamma(\varDelta v_1 + \bar{\alpha} v_1) + \dfrac{1}{b} F_1(\tilde{L}_1 \boldsymbol{u}) \\ \dfrac{\partial v_2}{\partial t} = -R v_2 + \dfrac{b^2 - c^2}{b} v_3 + \gamma(\varDelta v_2 + \bar{\alpha} v_2) + F_2(\tilde{L}_1 \boldsymbol{u}) - \dfrac{c}{b} F_1(\tilde{L}_1 \boldsymbol{u}) \\ \dfrac{\partial v_3}{\partial t} = -b v_2 - R v_3 + \gamma(\varDelta v_3 + \bar{\alpha} v_3) + F_3(\tilde{L}_1 \boldsymbol{u}). \end{cases}$$

Being

$$\lambda_1 = -R, \qquad I = -(R + R_1), \qquad A = RR_1 + b^2 - c^2 \tag{4.9}$$

with R and R_1 positive constants, it turns out that the zero solution is stable iff

$$C^2 < RR_1 + b^2 = (a + \bar{\alpha}\gamma)(a_1 + \bar{\alpha}\gamma) + b^2. \tag{4.10}$$

5. – Appendix

Systems (2.22) and (2.23) can be written respectively

$$\frac{d\boldsymbol{X}}{dt} = \tilde{L}\boldsymbol{X}, \tag{5.1}$$

$$\frac{d\boldsymbol{Z}}{dt} = \tilde{L}_1^{-1}\tilde{L}\tilde{L}_1\boldsymbol{Z}. \tag{5.2}$$

The characteristic equations of $\tilde{L}$ and $\tilde{L}_1^{-1}\tilde{L}\tilde{L}_1$ are coincident since [9]

$$\tilde{L}_1^{-1}(\tilde{L} - \lambda E)\tilde{L}_1 = \tilde{L}_1^{-1}\tilde{L}\tilde{L}_1 - \lambda E \tag{5.3}$$

with E unity matrix. Hence

$$\begin{aligned} \det(\tilde{L}_1^{-1}\tilde{L}\tilde{L}_1 - \lambda E) &= \det\tilde{L}_1^{-1}\det(\tilde{L} - \lambda E)\,\det\tilde{L}_1 \\ &= \det(\tilde{L} - \lambda E). \end{aligned} \tag{5.4}$$

Acknowledgments. This paper has been performed under the auspices of the G.N.F.M. of I.N.D.A.M. This work was supported in part from the Leverhulm Trust, "Tipping points: mathematics, metaphors and meanings".

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Received February 7, 2011 and in revised form April 19, 2011