Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.3, p. 445–468.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_3_445_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 445-468

Approximation by Multivariate Generalized Sampling Kantorovich Operators in the Setting of Orlicz Spaces

DANILO COSTARELLI - GIANLUCA VINTI

In memory of Professor Giovanni Prodi with deep respect and high esteem

Abstract. – *In this paper we study a linear version of the sampling Kantorovich type operators in a multivariate setting and we show applications to Image Processing. By means of the above operators, we are able to reconstruct continuous and uniformly continuous signals/images (functions). Moreover, we study the modular convergence of these operators in the setting of Orlicz spaces* $L^{\varphi}(\mathbb{R}^n)$ *that allows us to deal the case of not necessarily continuous signals/images. The convergence theorems in* $L^p(\mathbb{R}^n)$*-spaces,* $L^{\alpha}\log^{\beta}L(\mathbb{R}^n)$*-spaces and exponential spaces follow as particular cases. Several graphical representations, for the various examples and Image Processing applications are included.*

1. – Introduction

In [1] the problem of convergence for a family of linear generalized sampling operators in the Kantorovich sense has been studied in the setting of Orlicz spaces in one-dimensional case. Later these results have been considered in a more general context in [36].

The main purpose of this paper consists on the extension of the above results ([1]) in the multivariate setting.

The treatement of the theory in multivariate setting is important also from the point of view of the applications; indeed in signal theory, in order to deal with image processing, one has to work with multivariate signals. This explain the presence on this paper of concrete examples showing how the theory can be applied to image approximation. As concern the multivariate theory for the classical generalized sampling series, see [10].

The family of operators we take into consideration are of the form

$$\text{(I)} \qquad (S_w^{\chi} f)(\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u} \right] \qquad (\underline{x} \in \mathbb{R}^n),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$. Here $\chi : \mathbb{R}^n \to \mathbb{R}$ is a kernel function satisfying suitable properties, $t_{\underline{k}} = \left(t_{k_1}, \dots, t_{k_n}\right)$ is a vector where $(t_{k_i})_{k_i \in \mathbb{Z}}$, $i = 1, \dots, n$ is a sequence of real numbers with some properties and where

$$R_{\underline{k}}^{w} = \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times \dots \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right] \qquad (w > 0),$$

and $A_{\underline{k}} = \Delta_{k_1} \cdot \Delta_{k_2} \cdot \ldots \cdot \Delta_{k_n}$ with $\Delta_{k_i} = t_{k_i+1} - t_{k_i}$, $i = 1, \dots, n$.

The above family (I) represents a Kantorovich version of the generalized sampling operators (see e.g. [10]) where instead of the sampling values $f(\underline{k}/w)$ one has an average of f in a small pluri-rectangle around $\underline{k}/w$ (here instead of $\underline{k}$, we have a general sequence $t_{\underline{k}}$, obtaining a non uniform sampling).

This situation very often occurs in Signal Processing, when one cannot match exactly the "node" $t_{\underline{k}}/w$: this represents the so called "jitter-error".

Therefore our theory reduces jitter-errors calculating the information in a neighbourhood of a point rather than exactly at that point.

The generalized sampling series have been introduced by P.L. Butzer and his school at Aachen in the years '80 as an efficient approximation process to reconstruct signals (see e.g. [8, 13, 32, 9, 14, 15, 4, 11, 16, 17, 5, 34, 6, 27, 35]); for the theory of classical sampling operators, the reader can see e.g. [33, 23, 18, 20, 7, 21, 22].

The present paper deal with convergence for the family (I) to the function f both in case of pointwise and uniform convergence for bounded and continuous or uniformly continuous functions and in the more general setting of functions belonging to Orlicz spaces. In this case, the convergence considered here is the modular convergence of Orlicz spaces. This approach allow us to work even with discontinuous signals, treating in this way a problem not covered by the classical theory, which considers only continuous functions. The importance of this fact, release just in the multivariate setting, where the possibility of approximating images in case of discontinuous functions means to detail the countours of the image itself, since discontinuities represent jumps of grey levels that imply high contrast. Therefore our theory becomes very important when one deals with image enhancement, in particular in case of biomedical images where the shape of the countours can suggest some specific patology.

The paper is organized as follows: Section 2 is devoted to notations and preliminaries while in Section 3 we define our operators and give some basic results. In Section 4 we present convergence results both in classical setting, i.e. pointwise and uniform convergence and in the setting of Orlicz spaces. Section 5 shows the unitary approach given by the treatement in Orlicz spaces, giving us the possibility to consider particular cases as, $L^p(\mathbb{R}^n)$-spaces, $L^\alpha \log^\beta L(\mathbb{R}^n)$-spaces and exponential spaces. Finally, Section 6 is devoted to particular examples with special kernels and to graphical representations. Here we also consider how the theory can be applied to image approximation (Section 6.1).

2. – Notations and preliminaries

In the following, let $\mathbb{N}^n$ denote the set of all n-tuples $\underline{k} = (k_1, \ldots, k_n)$ of elements from $\mathbb{N}$; $\mathbb{Z}^n$ and $\mathbb{R}^n$ are definided analogously. In particular, $\mathbb{R}^n$ is the Euclidean n-space endowed with the norm $\|\underline{u}\|_2 = (u_1^2 + \ldots + u_n^2)^{1/2}$, where $\underline{u} = (u_1, \ldots, u_n)$, $u_i \in \mathbb{R}$, $i = 1, \ldots, n$. Further, $B(\underline{x}, r)$ is the closed ball of $\mathbb{R}^n$ of center $\underline{x}$ and radius $r > 0$ containing all the vector $\underline{u} \in \mathbb{R}^n$ such that $\|\underline{x} - \underline{u}\|_2 \leq r$. Finally, $\alpha\underline{u} = (\alpha u_1, \ldots, \alpha u_n)$ is the product of $\underline{u}$ with $\alpha \in \mathbb{R}$ and $\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i$ is the scalar product of $\underline{u}$, $\underline{v} \in \mathbb{R}^n$.

We denote by $C(\mathbb{R}^n)$ (resp. $C^0(\mathbb{R}^n)$) the space of all uniformly continuous and bounded (resp. continuous and bounded) functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ endowed with the usual sup-norm $\|f\|_{C(\mathbb{R}^n)} = \|f\|_\infty := \sup_{\underline{u} \in \mathbb{R}^n} |f(\underline{u})|$, and by $C_c(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ the subspace of the elements having compact support. Moreover, $M(\mathbb{R}^n)$ will denote the linear space of all (Lebesgue) measurable real functions defined on $\mathbb{R}^n$.

We now recall some basic facts concerning Orlicz spaces.
Let $\varphi : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ be a φ-function, i.e. φ satisfies the following assumptions:

1. $\varphi(0) = 0$, $\varphi(u) > 0$ for every $u > 0$;
2. φ is continuous and non decreasing on $\mathbb{R}_0^+$;
3. $\lim_{u \to \infty} \varphi(u) = +\infty$.

For a fixed φ-function φ, one can consider the functional $I^\varphi : M(\mathbb{R}^n) \to [0, +\infty]$ defined by

$$I^\varphi[f] := \int_{\mathbb{R}^n} \varphi(|f(\underline{x})|)\, d\underline{x} \qquad (f \in M(\mathbb{R}^n)).$$

As it is well-known I^φ is a modular functional on $M(\mathbb{R}^n)$, i.e. I^φ satisfies the following assumptions (see e.g. [28, 3]):

(a) $I^\varphi[f] = 0$ if and only if $f = 0$;
(b) $I^\varphi[f] = I^\varphi[-f]$;
(c) $I^\varphi[\alpha f + \beta g] \leq I^\varphi[f] + I^\varphi[g]$, for every α, $\beta > 0$ such that $\alpha + \beta = 1$.

The Orlicz space generated by φ is defined by

$$L^\varphi(\mathbb{R}^n) = \{f \in M(\mathbb{R}^n) : \ I^\varphi[\lambda f] < \infty, \text{ for some } \lambda > 0\}.$$

The Orlicz space $L^\varphi(\mathbb{R}^n)$ is a vector space and a norm on $L^\varphi(\mathbb{R}^n)$ is given by

$$\|f\|_\varphi \ := \ \inf\{\lambda > 0 : \ I^\varphi[f/\lambda] \leq \lambda\},$$

(in case of φ convex, $\|f\|_\varphi = \inf\{\lambda > 0 : \ I^\varphi[f/\lambda] \leq 1\}$).

$\|\cdot\|_\varphi$ is called Luxemburg norm and defines a strong notion of convergence in $L^\varphi(\mathbb{R}^n)$. It is easy to show that, a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is norm convergent to a function $f \in L^\varphi(\mathbb{R}^n)$, i.e. $\|f_w - f\|_\varphi \to 0$ for $w \to \infty$, if and only if $I^\varphi[\lambda(f_w - f)] \to 0$ for $w \to \infty$, for every $\lambda > 0$. We can also introduce in $L^\varphi(\mathbb{R}^n)$, a weak notion of convergence, called "modular convergence", which induces a topology (modular topology) on the space.

We will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is modularly convergent to a function $f \in L^\varphi(\mathbb{R}^n)$ if

$$\lim_{w\to\infty} I^\varphi[\lambda(f_w - f)] = 0$$

for some $\lambda > 0$. Obviously, the norm convergence implies the modular convergence, while the converse implication is true if and only if the φ-function φ satisfies the Δ_2-condition, i.e., if there exists a constant $M > 0$ such that

$$\varphi(2u) \leq M\varphi(u) \qquad (u \in \mathbb{R}_0^+). \tag{1}$$

The space

$$E^\varphi(\mathbb{R}^n) := \{f \in M(\mathbb{R}^n) : I^\varphi[\lambda f] < \infty, \text{ for every } \lambda > 0\},$$

is a vector subspace of $L^\varphi(\mathbb{R}^n)$ and it is called the space of all finite elements of $L^\varphi(\mathbb{R}^n)$. It is easy to show that $C_c(\mathbb{R}^n) \subset E^\varphi(\mathbb{R}^n)$. In general $E^\varphi(\mathbb{R}^n)$ is a proper subspace of $L^\varphi(\mathbb{R}^n)$ and these two spaces coincide if and only if φ satisfies the Δ_2-condition.

As a last basic property on Orlicz space, we recall that $C_c(\mathbb{R}^n)$ is modularly dense in $L^\varphi(\mathbb{R}^n)$.

We now give some interesting examples of Orlicz spaces. First, we consider the Orlicz spaces generated by the convex φ-functions $\varphi(u) = u^p$, $1 \leq p < \infty$ and $\varphi_{\alpha,\beta}(u) = u^\alpha \log^\beta(e + u)$ for $\alpha \geq 1$ and $\beta > 0$; they both satisfy the Δ_2-condition and so, in these cases, modular convergence and norm convergence are equivalent. Moreover, $E^\varphi(\mathbb{R}^n) = L^\varphi(\mathbb{R}^n)$ are respectively $L^p(\mathbb{R}^n)$ and $L^\alpha \log^\beta L(\mathbb{R}^n)$, both treated in Section 5, as particular cases.

On the other hand, the exponential spaces, generated by the convex φ-function $\varphi(u) = e^{u^\alpha} - 1$ for $\alpha > 0$, are examples for which $E^\varphi(\mathbb{R}^n)$ is included but not equal to $L^\varphi(\mathbb{R}^n)$ and norm convergence is stronger than modular convergence. Indeed, it is easy to see that this φ-function does not satisfy the Δ_2-condition.

For further details concerning Orlicz spaces, see the following monographs [25, 28, 24, 26, 30, 31, 3] and [29].

3. – The multivariate generalized sampling Kantorovich operators

In this section we introduce the class of operators we will discuss in this paper. Let $\Pi^n = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ be a sequence defined by $t_{\underline{k}} = (t_{k_1}, \ldots, t_{k_n})$, where each $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, \ldots, n$ is a sequence of real numbers with $-\infty < t_{k_i} < t_{k_{i+1}} < +\infty$,

$\lim_{k_i \to \pm\infty} t_{k_i} = \pm\infty$, for every $i = 1, \ldots, n$ and such that there exist Δ, $\delta > 0$ for which $\delta \leq \Delta_{k_i} = t_{k_i+1} - t_{k_i} \leq \Delta$, for every $i = 1, \ldots, n$.

In what follows, a function $\chi : \mathbb{R}^n \longrightarrow \mathbb{R}$ will be called a kernel if it satisfies the following properties:

- (K1) $\chi \in L^1(\mathbb{R}^n)$ and is bounded in a neighbourhood of $\underline{0} \in \mathbb{R}^n$;
- (K2) For every $\underline{u} \in \mathbb{R}^n$, $\sum_{\underline{k} \in \mathbb{Z}^n} \chi(\underline{u} - t_{\underline{k}}) = 1$;
- (K3) For some $\beta > 0$,

$$m_{\beta,\Pi^n}(\chi) = \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} |\chi(\underline{u} - t_{\underline{k}})| \cdot \|\underline{u} - t_{\underline{k}}\|_2^{\beta} < +\infty.$$

We define the set

$$R_{\underline{k}}^w = \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times \ldots \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right] \quad (w > 0).$$

For every $\underline{k} \in \mathbb{Z}^n$ and $w > 0$, $R_{\underline{k}}^w$ is a subset of $\mathbb{R}^n$; if we denote by $A_{\underline{k}} = \Delta_{k_1} \cdot \Delta_{k_2} \cdot \ldots \cdot \Delta_{k_n}$, the Lebesgue measure of $R_{\underline{k}}^w$ is given by $\dfrac{A_{\underline{k}}}{w^n}$ and satisfies the following inequality

$$\frac{\delta^n}{w^n} \leq \frac{A_{\underline{k}}}{w^n} \leq \frac{\Delta^n}{w^n}.$$

We can now introduce the definition of our operators.

DEFINITION 3.1. – *Let χ be a kernel. We define a family of operators $(S_w^{\chi})_{w>0}$ by*

$$(S_w^{\chi} f)(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}) \cdot \left[\frac{w^n}{A_{\underline{k}}} \cdot \int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u}\right] \quad (\underline{x} \in \mathbb{R}^n),$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a locally integrable function such that the series is convergent for each $\underline{x} \in \mathbb{R}^n$.

$S_w^{\chi} f$ $(w > 0)$, will be called the multivariate generalized sampling Kantorovich operator.

The term $\dfrac{w^n}{A_{\underline{k}}} \cdot \displaystyle\int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u}$ represent a mean of the function f on a pluri-rectangle of $\mathbb{R}^n$.

We begin giving the proof of the following lemma.

LEMMA 3.2. – *Let χ be a kernel. We have*

(i) $m_{0,\Pi^n}(\chi) := \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} |\chi(\underline{u} - t_{\underline{k}})| < +\infty$;

(ii) *For every* $\gamma > 0$

$$\lim_{w\to\infty} \sum_{\|w\underline{u}-t_{\underline{k}}\|_2>\gamma w} |\chi(w\underline{u}-t_{\underline{k}})| \;=\; 0,$$

uniformly with respect to $\underline{u} \in \mathbb{R}^n$.

(iii) *For every* $\gamma > 0$ *and* $\varepsilon > 0$ *there exists a constant* $M > 0$ *such that*

$$\int_{\|\underline{u}\|_2>M} w^n |\chi(w\underline{u}-t_{\underline{k}})| \, d\underline{u} \;<\; \varepsilon,$$

for sufficiently large $w > 0$ *and* $t_{\underline{k}}$ *such that* $\|t_{\underline{k}}\|_2 \leq \gamma w$.

PROOF. – (i) By (K1), there exists a closed ball $B(\underline{0}, a)$ and a positive constant T, such that $|\chi(\underline{u})| \leq T$, for every $\underline{u} \in B(\underline{0}, a)$. We can assume $a \leq \delta$. We write

$$\sum_{\underline{k}\in\mathbb{Z}^n} |\chi(\underline{u}-t_{\underline{k}})| \;=\; \left(\sum_{\|\underline{u}-t_{\underline{k}}\|_2\leq a} + \sum_{\|\underline{u}-t_{\underline{k}}\|_2> a} \right) |\chi(\underline{u}-t_{\underline{k}})| \;=\; I_1 + I_2.$$

Since $a \leq \delta$, the sum in I_1 contains at most 2^n terms and so $I_1 \leq 2^n T < +\infty$. For I_2 we have

$$I_2 \;\leq\; \frac{1}{a^\beta} \sum_{\|\underline{u}-t_{\underline{k}}\|_2> a} |\chi(\underline{u}-t_{\underline{k}})| \|\underline{u}-t_{\underline{k}}\|^\beta \;\leq\; \frac{1}{a^\beta} m_{\beta,\Pi^n}(\chi) < +\infty.$$

Since $I_1 + I_2 < +\infty$ for every $\underline{u} \in \mathbb{R}^n$, (i) is proved.

(ii) Let $\gamma > 0$ be fixed. For every $w > 0$ we obtain as above,

$$\sum_{\|w\underline{u}-t_{\underline{k}}\|_2>\gamma w} |\chi(w\underline{u}-t_{\underline{k}})| \;\leq\; \frac{1}{(\gamma w)^\beta} m_{\beta,\Pi^n}(\chi),$$

and so the assertion follows.

(iii) Let $\varepsilon > 0$ be fixed. Since $\chi \in L^1(\mathbb{R}^n)$, there exists a constant $\widetilde{M} > 0$ such that

$$\int_{\|\underline{x}\|_2>\widetilde{M}} |\chi(\underline{x})| \, d\underline{x} \;<\; \varepsilon.$$

Let now $\gamma > 0$ and for every $w \geq 1$ consider all the elements $t_{\underline{k}}$ such that $\|t_{\underline{k}}\|_2 \leq w\gamma$. Further, let $M > 0$ such that $M - \gamma > \widetilde{M}$. We have

$$\begin{aligned}\int_{\|\underline{u}\|_2>M} w^n |\chi(w\underline{u}-t_{\underline{k}})| d\underline{u} &= \int_{\|\underline{x}+t_{\underline{k}}\|_2>Mw} |\chi(\underline{x})| d\underline{x} \\ &\leq \int_{\|\underline{x}\|_2>(M-\gamma)w} |\chi(\underline{x})| d\underline{x} \leq \int_{\|\underline{x}\|_2>\widetilde{M}} |\chi(\underline{x})| d\underline{x} < \varepsilon,\end{aligned}$$

and so (iii) is proved. □

REMARK 3.3. – (a) We note that, if $f \in L^\infty(\mathbb{R}^n)$, using Lemma 3.2 (i), the operators $S_w^\chi f$ are well-defined. Indeed,

$$|(S_w^\chi f)(\underline{x})| \;\leq\; m_{0,\Pi^n}(\chi)\|f\|_\infty \;<+\infty,$$

for every $\underline{x} \in \mathbb{R}^n$, i.e. $S_w^\chi : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

(b) Instead of assuming that the function χ is bounded in a neighbourhood of $\underline{0} \in \mathbb{R}^n$ and satisfies (K3), one can directly assume that for χ hold the properties (i) and (ii) of Lemma 3.2.

(c) If we consider the equally spaced sequence $t_{\underline{k}} = \underline{k} = (k_1, \ldots, k_n)$, $\underline{k} \in \mathbb{Z}^n$, the boundedness assumption on χ and the hypothesis (K3) can be replaced by $\sup\limits_{\underline{u}\in\mathbb{R}^n} \sum\limits_{\underline{k}\in\mathbb{Z}^n} |\chi(\underline{u} - t_{\underline{k}})| < +\infty$, where the convergence of the series is uniform on compact sets. In this case, (ii) of Lemma 3.2 becomes an easy consequence of the property

$$\lim_{R\to\infty} \sum_{\|\underline{u}-\underline{k}\|_2 > R} |\chi(\underline{u} - \underline{k})| \;=\; 0,$$

uniformly with respect to $\underline{u} \in \mathbb{R}^n$.

4. – Convergence results

We are now ready to prove the following convergence theorem.

THEOREM 4.1. – *Let $f \in C^0(\mathbb{R}^n)$. Then, for every $\underline{x} \in \mathbb{R}^n$,*

$$\lim_{w\to\infty} (S_w^\chi f)(\underline{x}) \;=\; f(\underline{x}).$$

In particular, if $f \in C(\mathbb{R}^n)$, then

$$\lim_{w\to\infty} \|S_w^\chi f - f\|_\infty \;=\; 0.$$

PROOF. – Let $\underline{x} \in \mathbb{R}^n$ be a point of continuity of f. We first note that $S_w^\chi f$ is well-defined by boundedness of f and let $\varepsilon > 0$ be fixed. Then there exists $\gamma > 0$ such that $|f(\underline{u}) - f(\underline{x})| < \varepsilon$ whenever $\|\underline{u} - \underline{x}\|_2 \leq \gamma$ ($\underline{u} \in \mathbb{R}^n$). By (K2) we obtain

$$|(S_w^\chi f)(\underline{x}) - f(\underline{x})| \;\leq\; \sum_{\underline{k}\in\mathbb{Z}^n} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} |f(\underline{u}) - f(\underline{x})|\, d\underline{u}$$

$$= \left(\sum_{\|w\underline{x}-t_{\underline{k}}\|_2 \leq w\gamma/2} + \sum_{\|w\underline{x}-t_{\underline{k}}\|_2 > w\gamma/2} \right) |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} |f(\underline{u}) - f(\underline{x})|\, d\underline{u}$$

$$= I_1 + I_2.$$

For every $\underline{u} \in \mathbb{R}^n$ with $\underline{u} \in R^w_{\underline{k}}$, if $\|w\underline{x} - t_{\underline{k}}\|_2 \leq w\gamma/2$, we have

$$\|\underline{u} - \underline{x}\|_2 \;\leq\; \left\|\underline{u} - \frac{1}{w} t_{\underline{k}}\right\|_2 + \left\|\frac{1}{w} t_{\underline{k}} - \underline{x}\right\|_2 \;\leq\; \sqrt{n} \cdot \frac{\Delta}{w} + \frac{\gamma}{2}.$$

Since there exists $\overline{w_1} > 0$ such that $\sqrt{n} \cdot \dfrac{\Delta}{w} < \dfrac{\gamma}{2}$ for every $w \geq \overline{w_1}$, we obtain

$$I_1 < \sum_{\|w\underline{x} - t_{\underline{k}}\|_2 \leq w\gamma/2} |\chi(w\underline{x} - t_{\underline{k}})| \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \varepsilon \, d\underline{u} \;\leq\; m_{0,\Pi^n}(\chi)\varepsilon.$$

Moreover,

$$I_2 \;\leq\; 2\|f\|_\infty \sum_{\|w\underline{x} - t_{\underline{k}}\|_2 > w\gamma/2} |\chi(w\underline{x} - t_{\underline{k}})|$$

and for the property (ii) of Lemma 3.2, there exists $\overline{w_2} > 0$ such that

$$\sum_{\|w\underline{x} - t_{\underline{k}}\|_2 > w\gamma/2} |\chi(w\underline{x} - t_{\underline{k}})| \;<\; \varepsilon$$

for every $w \geq \overline{w_2}$ and so the assertion follows taking $\overline{w} = max\{\overline{w_1}, \overline{w_2}\}$ and being $\varepsilon > 0$ arbitrary.

In particular, if $f \in C(\mathbb{R}^n)$, the inequality

$$|(S^\chi_w f)(\underline{x}) - f(\underline{x})| \;\leq\; I_1 + I_2 \;<\; \varepsilon,$$

holds for every $\underline{x} \in \mathbb{R}^n$ and for $w \geq \overline{w}$, for some $\overline{w} > 0$ that does not depend on $\underline{x}$. □

REMARK 4.2. – We note that, if $f \in C_c(\mathbb{R}^n)$ and $supp\, f \subset B(\underline{0}, \gamma)$, we have that $R^w_{\underline{k}} \cap B(\underline{0}, \gamma) = \emptyset$ for every $t_{\underline{k}} \notin B(\underline{0}, w\gamma)$ and therefore

$$\int_{R^w_{\underline{k}}} f(\underline{u}) \, d\underline{u} = 0.$$

It follows that $(S^\chi_w f)(\underline{x}) = \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} |\chi(w\underline{x} - t_{\underline{k}})| \left[\dfrac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) \, d\underline{u}\right]$ is well-defined for every $\underline{x} \in \mathbb{R}^n$ and $S^\chi_w f \in E^\varphi(\mathbb{R}^n) \subset L^\varphi(\mathbb{R}^n)$ where φ is a convex φ-function, for every $w > 0$. Indeed, for $\lambda > 0$, by Jensen's inequality, we have

$$\begin{aligned} I^{\varphi}[\lambda S_w^{\chi} f] &\leq \int_{\mathbb{R}^n} \varphi(\lambda \sum_{\|\frac{1}{w}\underline{t_k}\|_2 \leq \gamma} |\chi(w\underline{x} - t_{\underline{k}})| \|f\|_{\infty}) \, d\underline{x} \\ &\leq \sum_{\|\frac{1}{w}\underline{t_k}\|_2 \leq \gamma} \frac{\varphi(\lambda m_{0,\Pi^n}(\chi)\|f\|_{\infty})}{m_{0,\Pi^n}(\chi)} \int_{\mathbb{R}^n} |\chi(w\underline{x} - t_{\underline{k}})| \, d\underline{x} \\ &\leq \sum_{\|\frac{1}{w}\underline{t_k}\|_2 \leq \gamma} \frac{\varphi(\lambda m_{0,\Pi^n}(\chi)\|f\|_{\infty})}{w^n m_{0,\Pi^n}(\chi)} \|\chi\|_1 < +\infty. \end{aligned}$$

As noted in Remark 4.2, we can now prove the following theorem of norm-convergence for the family of multivariate sampling Kantorovich operators in the setting of Orlicz space generated by a convex φ-function.

THEOREM 4.3. – *Let φ be a convex φ-function. For every $f \in C_c(\mathbb{R}^n)$ we have*

$$\lim_{w \to \infty} \|S_w^{\chi} f - f\|_{\varphi} = 0.$$

PROOF. – We will prove that

$$\lim_{w \to \infty} I^{\varphi}[\lambda(S_w^{\chi} f - f)] = \lim_{w \to \infty} \int_{\mathbb{R}^n} \varphi(\lambda |\left(S_w^{\chi} f\right)(\underline{x}) - f(\underline{x})|) \, d\underline{x} = 0,$$

for every $\lambda > 0$, that is equivalent to show that the family $(\varphi(\lambda|S_w^{\chi} f - f|))_{w>0}$ converges to zero in $L^1(\mathbb{R}^n)$, for every $\lambda > 0$. We will use the Vitali convergence theorem in $L^1(\mathbb{R}^n)$. Let $\lambda > 0$ be fixed. Using Theorem 4.1 and for the continuity of φ, we have

$$\lim_{w \to \infty} \varphi(\lambda \|S_w^{\chi} f - f\|_{\infty}) = 0.$$

Now, let $\varepsilon > 0$ be fixed. Moreover let γ and $\overline{\gamma}$ be two positive constants, such that $supp\, f \subset B(\underline{0}, \overline{\gamma})$ and $\gamma > \overline{\gamma} + \Delta$. Then, if $t_{\underline{k}} \notin B(\underline{0}, w\gamma)$ we have that $R_{\underline{k}}^w \cap B(\underline{0}, \overline{\gamma}) = \emptyset$ and therefore

$$\int_{R_{\underline{k}}^w} |f(\underline{u})| \, d\underline{u} = 0.$$

By Lemma 3.2 (iii), for such γ, $\varepsilon > 0$, there exists a constant $M > 0$ (we can assume $\overline{\gamma} < M$), such that

$$\int_{\|\underline{x}\|_2 > M} w^n |\chi(w\underline{x} - t_{\underline{k}})| \, d\underline{x} < \varepsilon,$$

for every sufficiently large $w > 0$ and $t_{\underline{k}}$ such that $t_{\underline{k}} \in B(\underline{0}, w\gamma)$. Then, by Jensen's inequality and Fubini-Tonelli theorem we have

$$\int_{\|\underline{x}\|_2 > M} \varphi(\lambda |(S_w^\chi f)(\underline{x})|)\, d\underline{x}$$

$$\leq \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} \frac{\varphi(\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty)}{m_{0,\Pi^n}(\chi) w^n} \int_{\|\underline{x}\|_2 > M} w^n |\chi(w\underline{x} - t_{\underline{k}})| d\underline{x}$$

$$< \varepsilon \cdot \frac{\varphi(\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty)}{m_{0,\Pi^n}(\chi) w^n} \cdot L,$$

where $L > 0$ represents the number of terms of the above series in fact corresponding to the number of sets $R_{\underline{k}}^w$ having non-empty intersection with $B(\underline{0}, \gamma)$. We note that, for every $w \geq 1$ we obtain

$$L \leq \left[2\left(\left[\frac{\gamma w}{\delta}\right] + 1\right)\right]^n = 2^n \sum_{i=0}^{n} \binom{n}{i} \left(\left[\frac{\gamma w}{\delta}\right]\right)^{n-i} = 2^n \left[\left(\left[\frac{\gamma w}{\delta}\right]\right)^n \right.$$

$$\left. + n\left(\left[\frac{\gamma w}{\delta}\right]\right)^{n-1} + \ldots + 1\right] = 2^n w^n \cdot \left[\left(\left[\frac{\gamma}{\delta}\right]\right)^n + n\left(\left[\frac{\gamma}{\delta}\right]\right)^{n-1} \cdot \frac{1}{w} + \ldots + \frac{1}{w^n}\right]$$

$$\leq w^n \cdot \left\{2^n \left[\left(\left[\frac{\gamma}{\delta}\right]\right)^n + n\left(\left[\frac{\gamma}{\delta}\right]\right)^{n-1} + \ldots + 1\right]\right\} =: w^n \cdot K;$$

(here $[\cdot]$ denotes the integer part).

Thus,

$$\int_{\|\underline{x}\|_2 > M} \varphi(\lambda |(S_w^\chi f)(\underline{x})|)\, d\underline{x} < \varepsilon \cdot \frac{\varphi(\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty)}{m_{0,\Pi^n}(\chi)} \cdot K =: \varepsilon \cdot C,$$

for every $w \geq 1$.

Therefore, for $\varepsilon > 0$ there exists a set $E_\varepsilon = B(\underline{0}, M)$ such that for every measurable set F, with $F \cap E_\varepsilon = \emptyset$, we have

$$\int_F \varphi(\lambda |(S_w^\chi f)(\underline{x}) - f(\underline{x})|)\, d\underline{x} = \int_F \varphi(\lambda |(S_w^\chi f)(\underline{x})|)\, d\underline{x}$$

$$\leq \int_{\|\underline{x}\|_2 > M} \varphi(\lambda |(S_w^\chi f)(\underline{x})|)\, d\underline{x} < \varepsilon \cdot C.$$

Finally, for $\|f\|_\infty > 0$, let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ be a measurable set with

$$|B| < \frac{\varepsilon}{\varphi(2\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty)}.$$

Then, by Remark 3.3 (a) and since $m_{0,\Pi^n}(\chi) \geq 1$, in corrispondence to $\varepsilon > 0$ and for every $w > 0$,

$$\int_B \varphi(\lambda|(S_w^\chi f)(\underline{x}) - f(\underline{x})|) d\underline{x}$$
$$\leq \frac{1}{2}\int_B \varphi(2\lambda|(S_w^\chi f)(\underline{x})|)\, d\underline{x} + \frac{1}{2}\int_B \varphi(2\lambda|f(\underline{x})|)\, d\underline{x}$$
$$\leq \frac{1}{2}\int_B \varphi(2\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty)\, d\underline{x} + \frac{1}{2}\int_B \varphi(2\lambda\|f\|_\infty)\, d\underline{x}$$
$$\leq \int_B \varphi(2\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty)\, d\underline{x} = |B|\varphi(2\lambda m_{0,\Pi^n}(\chi)\|f\|_\infty) < \varepsilon.$$

Thus, the integrals

$$\int_{(\cdot)} \varphi(\lambda|(S_w^\chi f)(\underline{x}) - f(\underline{x})|)\, d\underline{x}$$

are equi-absolutely continuous. Since $\lambda > 0$ is arbitrary, we obtain the assertion. □

In order to obtain a modular convergence result in $L^\varphi(\mathbb{R}^n)$, we need a modular continuity property for the operators S_w^χ. We have the following.

THEOREM 4.4. – *Let φ be a convex φ-function. For every $f \in L^\varphi(\mathbb{R}^n)$ there holds*

$$I^\varphi[\lambda S_w^\chi f] \leq \frac{\|\chi\|_1}{\delta^n \cdot m_{0,\Pi^n}(\chi)} I^\varphi[\lambda m_{0,\Pi^n}(\chi) f],$$

for some $\lambda > 0$.

In particular, S_w^χ maps $L^\varphi(\mathbb{R}^n)$ in $L^\varphi(\mathbb{R}^n)$.

PROOF. – Since $f \in L^\varphi(\mathbb{R}^n)$, there exists $\bar{\lambda} > 0$ such that $I^\varphi[\bar{\lambda} f] < +\infty$. Let $\lambda > 0$ such that $\lambda m_{0,\Pi^n}(\chi) \leq \bar{\lambda}$, then applying twice Jensen's inequality we obtain

$$I^\varphi[\lambda S_w^\chi f] = \int_{\mathbb{R}^n} \varphi(\lambda|(S_w^\chi f)(\underline{x})|)\, d\underline{x}$$
$$\leq \frac{1}{m_{0,\Pi^n}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \varphi\left(\frac{w^n}{A_{\underline{k}}} \lambda m_{0,\Pi^n}(\chi) \int_{R_{\underline{k}}^w} |f(\underline{u})|\, d\underline{u}\right) \int_{\mathbb{R}^n} |\chi(w\underline{x} - t_{\underline{k}})|\, d\underline{x}$$
$$\leq \frac{1}{m_{0,\Pi^n}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \left\{\frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} \varphi(\lambda m_{0,\Pi^n}(\chi)|f(\underline{u})|) d\underline{u} \cdot \int_{\mathbb{R}^n} |\chi(w\underline{x} - t_{\underline{k}})| d\underline{x}\right\}.$$

With the change of variable $w\underline{x} - t_{\underline{k}} = \underline{v}$ in the last integral, we have

$$\begin{aligned} I^{\varphi}[\lambda S_w^{\chi} f] &= \int_{\mathbb{R}^n} \varphi(\lambda |(S_w^{\chi} f)(\underline{x})|)\, d\underline{x} \\ &\leq \frac{1}{m_{0,\Pi^n}(\chi)} \sum_{\underline{k} \in \mathbb{Z}^n} \left\{ \frac{1}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} \varphi(\lambda m_{0,\Pi^n}(\chi)|f(\underline{u})|)\, d\underline{u} \right\} \|\chi\|_1 \\ &\leq \frac{1}{m_{0,\Pi^n}(\chi)} \frac{1}{\delta^n} \|\chi\|_1 I^{\varphi}[\lambda m_{0,\Pi^n}(\chi) f] < +\infty, \end{aligned}$$

since $I^{\varphi}[\lambda m_{0,\Pi^n}(\chi) f] \leq I^{\varphi}[\overline{\lambda} f] < +\infty$. □

Theorem 4.4 says that the family of operators $S_w^{\chi} : L^{\varphi}(\mathbb{R}^n) \longrightarrow L^{\varphi}(\mathbb{R}^n)$ is modularly continuous, for every $w > 0$.

We can finally prove the main theorem of this section, which provides a result of modular convergence.

THEOREM 4.5. – *Let φ be a convex φ-function. For every $f \in L^{\varphi}(\mathbb{R}^n)$, there exists $\lambda > 0$ such that*

$$\lim_{w \to \infty} I^{\varphi}[\lambda(S_w^{\chi} f - f)] = 0.$$

PROOF. – Let $f \in L^{\varphi}(\mathbb{R}^n)$ and $\varepsilon > 0$ be fixed. Since $C_c(\mathbb{R}^n)$ is modularly dense in $L^{\varphi}(\mathbb{R}^n)$ (see e.g. [2]), there exists $\overline{\lambda} > 0$ and a function $g \in C_c(\mathbb{R}^n)$ such that $I^{\varphi}[\overline{\lambda}(f - g)] < \varepsilon$. Let now $\lambda > 0$ such that $3\lambda(1 + m_{0,\Pi^n}(\chi)) \leq \overline{\lambda}$. By the properties of φ and Theorem 4.4, we have

$$\begin{aligned} & I^{\varphi}[\lambda(S_w^{\chi} f - f)] \\ &\quad \leq I^{\varphi}[3\lambda(S_w^{\chi} f - S_w^{\chi} g)] + I^{\varphi}[3\lambda(S_w^{\chi} g - g)] + I^{\varphi}[3\lambda(f - g)] \\ &\quad \leq \frac{1}{m_{0,\Pi^n}(\chi) \cdot \delta^n} \|\chi\|_1 I^{\varphi}[\overline{\lambda}(f - g)] + I^{\varphi}[3\lambda(S_w^{\chi} g - g)] + I^{\varphi}[\overline{\lambda}(f - g)] \\ &\quad < \left(\frac{1}{m_{0,\Pi^n}(\chi) \cdot \delta^n} \|\chi\|_1 + 1 \right) \varepsilon + I^{\varphi}[3\lambda(S_w^{\chi} g - g)]. \end{aligned}$$

The assertion follows from Theorem 4.3. □

5. – Approximation in $L^p(\mathbb{R}^n)$ and some other particular cases

We will now apply our convergence theorems to some special spaces. First we study the important case of $L^{\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ that occurs when $\varphi(u) = u^p$, $u \in \mathbb{R}_0^+$ and $1 \leq p < \infty$.

In this case $I^{\varphi}[f] = \|f\|_p^p$, and it is well-known that φ satisfies the Δ_2-condition and so $L^p(\mathbb{R}^n)$ coincides with its space of all finite elements and the modular convergence and usual norm-convergence are equivalent. Then we obtain the following corollary.

COROLLARY 5.1. – *For every $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, we have*

$$\lim_{w\to\infty} \|S_w^{\chi} f - f\|_p = 0.$$

Moreover, the following inequality holds

$$\|S_w^{\chi} f\|_p \leq \delta^{-n/p}(m_{0,\Pi^n}(\chi))^{(p-1)/p}\|\chi\|_1^{1/p}\|f\|_p \quad (f \in L^p(\mathbb{R}^n)).$$

As another particular case, we can apply our result developed in the previous section to the space $L^{\alpha}\log^{\beta}L(\mathbb{R}^n)$, generated by the φ-function $\varphi_{\alpha,\beta}(u) = u^{\alpha}\log^{\beta}(e+u)$, $u \geq 0$ for $\alpha \geq 1$ and $\beta > 0$. The modular corresponding to $\varphi_{\alpha,\beta}$ is

$$I^{\varphi_{\alpha,\beta}}[f] = \int_{\mathbb{R}^n} |f(\underline{x})|^{\alpha}\log^{\beta}(e+|f(\underline{x})|)\, d\underline{x}.$$

Note that the function $\varphi_{\alpha,\beta}$ satisfies the Δ_2-condition, which means that, as in the case of $L^p(\mathbb{R}^n)$, $L^{\alpha}\log^{\beta}L(\mathbb{R}^n) = L^{\varphi_{\alpha,\beta}}(\mathbb{R}^n) = E^{\varphi_{\alpha,\beta}}(\mathbb{R}^n)$ and the norm convergence is equivalent to the modular convergence.

We obtain the following, for the most interesting case $\alpha = \beta = 1$.

COROLLARY 5.2. – *For every $f \in L\log L(\mathbb{R}^n)$ and for every $\lambda > 0$ we have*

$$\lim_{w\to\infty} \int_{\mathbb{R}^n} |(S_w^{\chi} f)(\underline{x}) - f(\underline{x})| \log(e + \lambda|(S_w^{\chi} f)(\underline{x}) - f(\underline{x})|)\, d\underline{x} = 0$$

or equivalently,

$$\lim_{w\to\infty} \|S_w^{\chi} f - f\|_{L\log L} = 0,$$

where $\|\cdot\|_{L\log L}$ is the Luxemburg norm associated to $I^{\varphi_{1,1}}$.

Moreover, we have

$$\int_{\mathbb{R}^n} |(S_w^{\chi} f)(\underline{x})| \log(e + \lambda|(S_w^{\chi} f)(\underline{x})|)\, d\underline{x} \leq \frac{\|\chi\|_1}{\delta^n} \int_{\mathbb{R}^n} |f(\underline{x})| \log(e + \lambda m_{0,\Pi^n}(\chi)|f(\underline{x})|)\, d\underline{x},$$

for every $\lambda > 0$, i.e. $S_w^{\chi} : L\log L(\mathbb{R}^n) \to L\log L(\mathbb{R}^n)$.

Finally, we consider the case of the exponential space generated by the φ-

function $\varphi_\alpha(u) = e^{u^\alpha} - 1$, $u \geq 0$ for some $\alpha > 0$. The Orlicz space $L^{\varphi_\alpha}(\mathbb{R}^n)$ consists of those function $f \in M(\mathbb{R}^n)$ for which

$$I^{\varphi_\alpha}[\lambda f] = \int_{\mathbb{R}^n} (e^{(\lambda|f(\underline{x})|)^\alpha} - 1)\, d\underline{x} < +\infty,$$

for some $\lambda > 0$. Since φ_α does not satisfy the Δ_2-condition, the space $L^{\varphi_\alpha}(\mathbb{R}^n)$ does not coincide with the space of its finite elements $E^{\varphi_\alpha}(\mathbb{R}^n)$. Moreover, modular convergence does not imply norm convergence. We obtain the following.

COROLLARY 5.3. – *For $f \in L^{\varphi_\alpha}(\mathbb{R}^n)$, there exists $\lambda > 0$ such that*

$$\lim_{w\to\infty} \int_{\mathbb{R}^n} (e^{(\lambda|(S_w^\chi f)(\underline{x}) - f(\underline{x})|)^\alpha} - 1)\, d\underline{x} \;=\; 0.$$

Moreover,

$$\int_{\mathbb{R}^n} (e^{(\lambda|(S_w^\chi f)(\underline{x})|)^\alpha} - 1)\, d\underline{x} \;\leq\; \frac{\|\chi\|_1}{\delta^n m_{0,\Pi^n}(\chi)} \int_{\mathbb{R}^n} (e^{(\lambda m_{0,\Pi^n}(\chi)|f(\underline{x})|)^\alpha} - 1)\, d\underline{x},$$

for some $\lambda > 0$.

6. – Examples with special kernels and graphical representations

In our theory, the choice of the kernels becomes important; indeed, in general it is not so easy to verify if a multivariate function satisfies the conditions (K1), (K2) and (K3). Therefore may be useful to use the well-known examples of univariate kernels to construct those in more variables (see e.g. [10]).

To this aim, consider the following procedure: let for simplicity $\Pi^n = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$, $t_{k_i} = k_i$, for $i = 1, \ldots, n$, the equispaced sequence, and let $\chi_1, \ldots, \chi_n \in L^1(\mathbb{R})$ such that

$$m_{0,\Pi_i^n}(\chi_i) \;:=\; \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_i(u-k)| < +\infty,$$

where the convergence of the series is uniform on compact sets of $\mathbb{R}$, and assume that $\sum_{k\in\mathbb{Z}} \chi_i(u-k) = 1$, for every $u \in \mathbb{R}$, for $i = 1, \ldots, n$. Setting $\chi(\underline{u}) := \prod_{i=1}^n \chi_i(u_i)$, we obtain that $\chi \in L^1(\mathbb{R}^n)$ since

$$\int_{\mathbb{R}^n} |\chi(\underline{u})|\, d\underline{u} = \int_{\mathbb{R}^n} \prod_{i=1}^n |\chi_i(u_i)|\, du_1 \ldots du_n = \prod_{i=1}^n \int_{\mathbb{R}} |\chi_i(u_i)|\, du_i < +\infty,$$

and

$$m_{0,\Pi^n}(\chi) = \sup_{\underline{u}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} |\chi(\underline{u}-\underline{k})| = \prod_{i=1}^{n} m_{0,\Pi_i^n}(\chi_i) < +\infty,$$

and the convergence is uniform on compact subsets of $\mathbb{R}^n$. Further,

$$\sum_{\underline{k}\in\mathbb{Z}^n} \chi(\underline{u}-\underline{k}) = \prod_{i=1}^{n} \sum_{k_i\in\mathbb{Z}} \chi_i(u_i - k_i) = 1,$$

for every $\underline{u} \in \mathbb{R}^n$. Thus, by Remark 3.3 (b) and (c), we can say that χ is a kernel.

For example, the univariate Fejér's kernel (see Figure 1) defined by

$$F(x) := \frac{1}{2} sinc^2\left(\frac{x}{2}\right) \qquad (x \in \mathbb{R}),$$

where the *sinc*-function is given by

$$sinc\,(x) := \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \in \mathbb{R}\setminus\{0\}, \\ 1, & x = 0, \end{cases}$$

satisfies the condition above mentioned, as shown in [1].

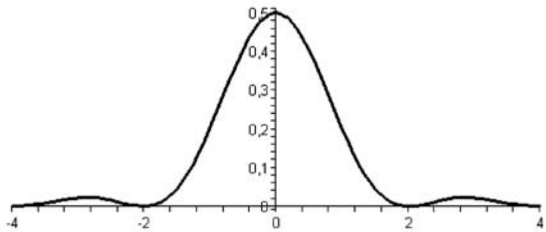

Fig. 1. – Univariate Fejér's kernel F.

Then, $\mathcal{F}_n(\underline{x}) = \prod_{i=1}^{n} F(x_i)$ is the multivariate Fejér's kernel (see Figure 2) and satisfies the condition upon a generalized kernel.

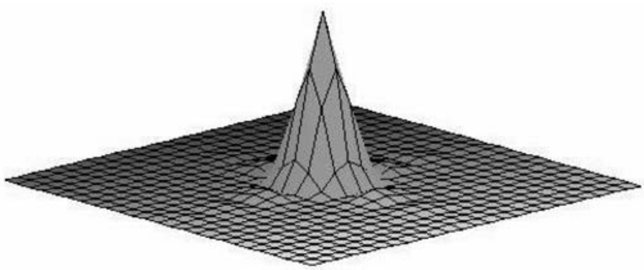

Fig. 2. – Bivariate Fejér's kernel $\mathcal{F}_2$.

The multivariate sampling Kantorovich operator of $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, takes now the form

$$(S_w^{\mathcal{F}_n} f)(\underline{x}) \;=\; \sum_{\underline{k} \in \mathbb{Z}^n} \left[w^n \int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u} \right] \cdot \mathcal{F}_n(w\underline{x} - \underline{k}) \qquad (\underline{x} \in \mathbb{R}^n).$$

From Corollaries 5.1, 5.2 and 5.3 we now obtain the following.

COROLLARY 6.1. – *(a) For every* $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, *we have*

$$\lim_{w \to \infty} \| S_w^{\mathcal{F}_n} f - f \|_p \;=\; 0.$$

(b) There holds for $f \in L \log L(\mathbb{R}^n)$ *and every* $\lambda > 0$,

$$\lim_{w \to \infty} \int_{\mathbb{R}^n} |\left(S_w^{\mathcal{F}_n} f\right)(\underline{x}) - f(\underline{x})| \log\left(e + \lambda |\left(S_w^{\mathcal{F}_n} f\right)(\underline{x}) - f(\underline{x})|\right) d\underline{x} \;=\; 0,$$

or, equivalently,

$$\lim_{w \to \infty} \| S_w^{\mathcal{F}_n} f - f \|_{L \log L} \;=\; 0.$$

(c) For $f \in L^{\varphi_\alpha}(\mathbb{R}^n)$, *there exists* $\lambda > 0$ *such that*

$$\lim_{w \to \infty} \int_{\mathbb{R}^n} \left(e^{(\lambda |\left(S_w^{\mathcal{F}_n} f\right)(\underline{x}) - f(\underline{x})|)^\alpha} - 1\right) d\underline{x} \;=\; 0.$$

For example, we apply now the sampling operator $S_w^{\mathcal{F}_2}$ to the particular discontinuous function $f \in L^p(\mathbb{R}^2)$, for every $1 \le p < \infty$, defined by (Figure 3)

$$f(x, y) \;=\; \begin{cases} 3, & -1 \le \; x \; \le 1 \; \textit{and} \; -1 \le \; y \; \le 1, \\ \dfrac{6}{x^2 + y^2}, & \textit{otherwise}. \end{cases} \tag{2}$$

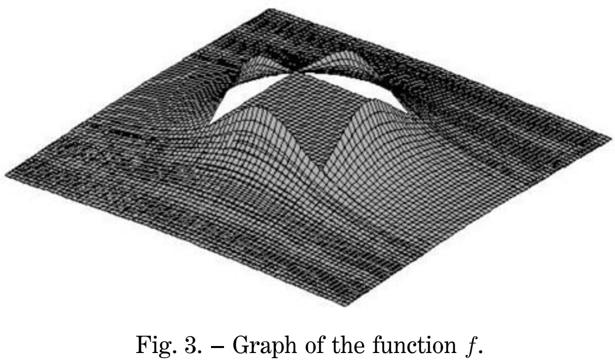

Fig. 3. – Graph of the function f.

The two-dimensional sampling Kantorovich operators for the function f defined in (2) in case of $w = 5$ and $w = 10$ are given in Figure 4 (in an octant of the plane).

Fig. 4. – The function f (black) with respectively the two-dimensional sampling Kantorovich operator $S_5^{\mathcal{F}_2} f$ (grey) and $S_{10}^{\mathcal{F}_2} f$ (grey).

The graphs show how the operator approximates the original function, and to give a better view of this fact, the function f and the operators $S_5^{\mathcal{F}_2} f$ and $S_{10}^{\mathcal{F}_2} f$ are plotted all together, as before, in an octant of the plane (Figure 5).

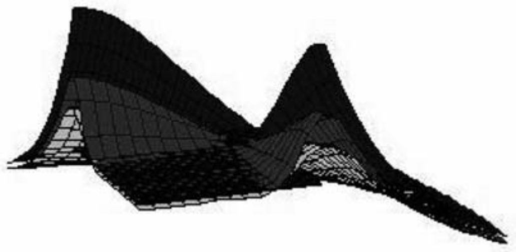

Fig. 5. – f (black), $S_5^{\mathcal{F}_2} f$ (grey), $S_{10}^{\mathcal{F}_2} f$ (dark grey).

Since Fejér's kernel $\mathcal{F}_n$ has unbounded support, one needs an infinite number of mean values $w^n \int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u}$ in order to evaluate the corresponding sampling series $(S_w^{\mathcal{F}_n} f)(\underline{x})$ at any given $\underline{x} \in \mathbb{R}^n$. If the function f has compact support, this problem does not arise, while, if the function has unbounded support (as in our example), one can only take a finite number of these mean value into account, so the infinite sampling series must be truncated to a finite one, which leads to the so-called truncation error.

In order to avoid the truncation error, one can take kernels χ with bounded support. The most convenient examples here are the multivariate kernel generated using the univariate B-splines of order $k \in \mathbb{N}$, defined by

$$M_k(x) := \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\frac{k}{2} + x - i \right)_+^{k-1} \quad (x \in \mathbb{R}),$$

(where the function $(x)_+ = \max\{x, 0\}$ is the positive part of x).

As shown in [1], M_k satisfies the conditions of univariate kernels, so

$$\mathcal{M}_k^n(\underline{x}) := \prod_{i=1}^{n} M_k(x_i),$$

is the multivariate B-spline kernel of order $k \in \mathbb{N}$.

Hence, we obtain the following, again from Corollaries 5.1, 5.2 and 5.3.

COROLLARY 6.2. – *(a) For the multivariate sampling Kantorovich operator*

$$(S_w^{\mathcal{M}_k^n} f)(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^n} \left[w^n \int_{R_{\underline{k}}^w} f(\underline{u}) d\underline{u} \right] \mathcal{M}_k^n(w\underline{x} - \underline{k}) \quad (\underline{x} \in \mathbb{R}^n),$$

of $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, there holds

$$\lim_{w \to \infty} \|S_w^{\mathcal{M}_k^n} f - f\|_p = 0.$$

(b) Likewise, one has for $f \in L \log L(\mathbb{R}^n)$ and every $\lambda > 0$,

$$\lim_{w \to \infty} \int_{\mathbb{R}^n} |\left(S_w^{\mathcal{M}_k^n} f \right)(\underline{x}) - f(\underline{x})| \log (e + \lambda |\left(S_w^{\mathcal{M}_k^n} f \right)(\underline{x}) - f(\underline{x})|) \, d\underline{x} = 0,$$

or, equivalently,

$$\lim_{w \to \infty} \|S_w^{\mathcal{M}_k^n} f - f\|_{L \log L} = 0.$$

(c) For $f \in L^{\varphi_\alpha}(\mathbb{R}^n)$, there exists $\lambda > 0$ such that

$$\lim_{w \to \infty} \int_{\mathbb{R}^n} (e^{\left(\lambda \left| \left(S_w^{\mathcal{M}_k^n} f \right)(\underline{x}) - f(\underline{x}) \right| \right)^\alpha} - 1) \, d\underline{x} = 0.$$

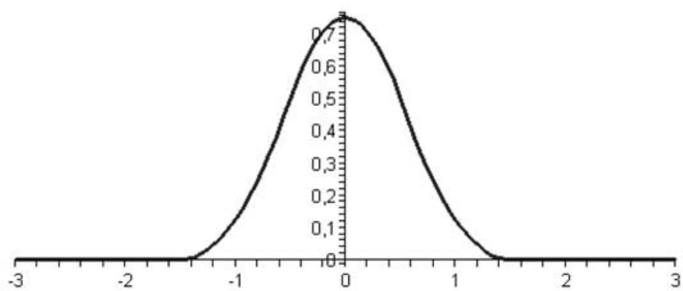

Fig. 6. – Univariate B-spline kernel M_3.

Let now consider the particular case $k = 3$ in detail. The B-spline M_3 (see

Figure 6) is given by

$$M_3(x) \ := \ \begin{cases} \dfrac{3}{4} - x^2, & |x| \leq \dfrac{1}{2}, \\ \dfrac{1}{2}\left(\dfrac{3}{2} - |x|\right)^2, & \dfrac{1}{2} < |x| \leq \dfrac{3}{2}, \\ 0, & |x| > \dfrac{3}{2}. \end{cases}$$

The two-dimensional sampling Kantorovich operators generated by $\mathcal{M}_3^2$ (see Figure 7), for the function f defined in (2) in case of $w = 5$ and $w = 10$, are plotted in Figure 8.

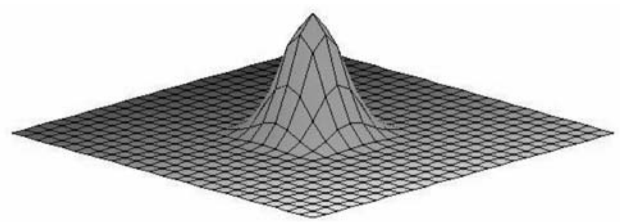

Fig. 7. – Bivariate B-spline kernel $\mathcal{M}_3^2$.

Fig. 8. – The function f (black) with respectively the two-dimensional sampling Kantorovich operator $S_5^{\mathcal{M}_3^2} f$ (grey) and $S_{10}^{\mathcal{M}_3^2} f$ (grey).

The graphs show how the operator approximates the original function, and to give a better view of this fact, the function f and the operators $S_5^{\mathcal{M}_3^2} f$ and $S_{10}^{\mathcal{M}_3^2} f$ are plotted all together in Figure 9.

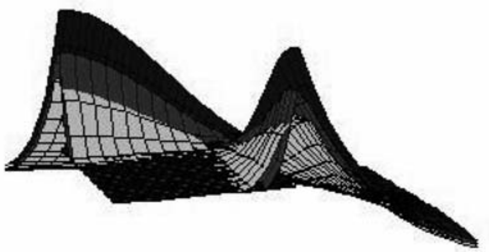

Fig. 9. – f (black), $S_5^{\mathcal{M}_3^2} f$ (grey), $S_{10}^{\mathcal{M}_3^2} f$ (dark grey).

Moreover, if we compare the graphs of the multivariate sampling Kantorovich operator generated by the spline kernel and the Fejér kernel, it is clear that the approximation by the series based on $\mathcal{M}_3^2$ is considerably better than the approximation by the series based on $\mathcal{F}_2$ kernel (see for example Figure 10). This means that applying the series $S_w^{\mathcal{M}_3^2} f$ a reasonably good approximation can be achieved by taking into account fewer mean values of f than for the series $S_w^{\mathcal{F}_2} f$.

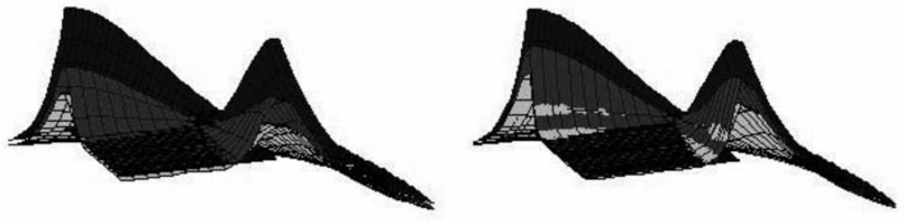

Fig. 10. – f (black) with respectively $S_5^{\mathcal{M}_3^2} f$ (dark grey), $S_5^{\mathcal{F}_2} f$ (grey) and $S_{10}^{\mathcal{M}_3^2} f$ (dark grey) and $S_{10}^{\mathcal{F}_2} f$ (grey).

One can also use linear combination of univariate B-spline of different degree, such as

$$\chi_1(x) \ := \ 4M_3(x) - 3M_4(x), \quad \chi_2(x) \ := \ 5M_4(x) - 4M_5(x) \quad (x \in \mathbb{R}),$$

or linear combinations of translates of B-splines, e.g.

$$\chi_3(x) \ := \ \frac{5}{4} M_3(x) - \frac{1}{8} \{M_3(x+1) + M_3(x-1)\} \quad (x \in \mathbb{R}),$$

to construct examples of multivariate kernels, in order to improve the rate of approximation.

6.1 – *Image Processing*

An interesting application of our theory concerns the image processing. A two-dimensional static image is represented by a function (signal) of two variables; similarly, a digital (static) image is a discrete signal and it is represented by a two-dimensional matrix. Every matrix can be modeled as a step function (with compact support) belonging to $L^p(\mathbb{R}^2)$, for every $1 \leq p < \infty$. Thus, one can use the two-dimensional sampling Kantorovich operator to obtain approximations of digital images. Then, we can build a function corresponding to the matrix (input image) in order to obtain a new matrix (image) that approximates the original one by using the bivariate sampling Kantorovich operator. If the sampling rate is chosen higher than the original sampling rate, one can get a new image that has an higher resolution than the original one, i.e. it is built with a larger number of pixel compared to the original image.

For example, in Figure 12 and 13 we have approximations of the image in

Figure 11 obtained by the bivariate sampling Kantorovich operators $S_w^{\mathcal{F}_2}$ and $S_w^{\mathcal{M}_3^2}$, for $w = 5$ and $w = 10$. The sampling rate is the same of the original image.

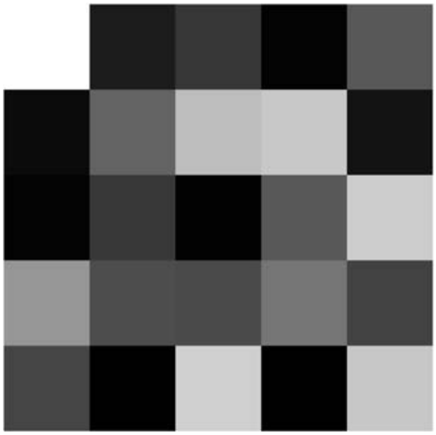

Fig. 11. – Original image.

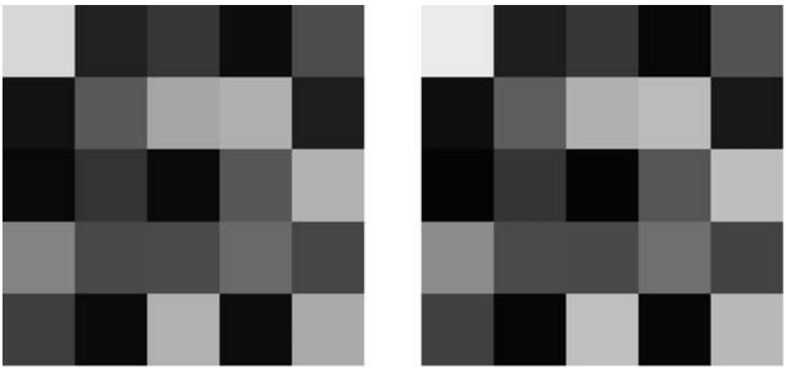

Fig. 12. – Approximation by $S_5^{\mathcal{F}_2}$ and $S_{10}^{\mathcal{F}_2}$.

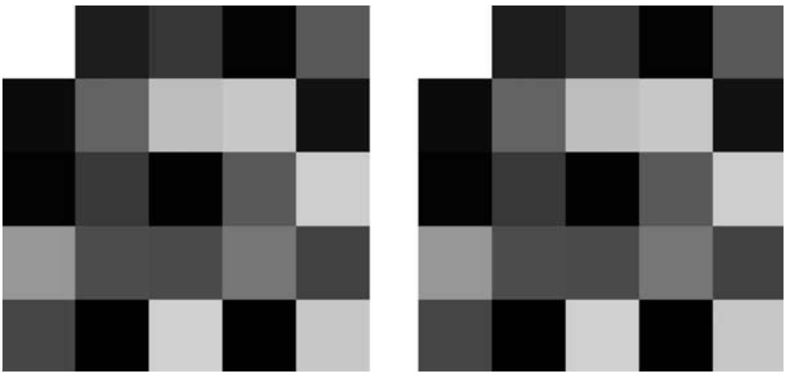

Fig. 13. – Approximation by $S_5^{\mathcal{M}_3^2}$ and $S_{10}^{\mathcal{M}_3^2}$.

In Figure 14 and 15 we have approximations of the image in Figure 11 obtained by the two-dimensional sampling Kantorovich operators $S_w^{\mathcal{F}_2}$ and $S_w^{\mathcal{M}_3^2}$, for $w = 5$ and $w = 10$ obtained taking into account a number of sample values bigger than the image in Figure 12 and 13 (twice the samples for each spatial variable).

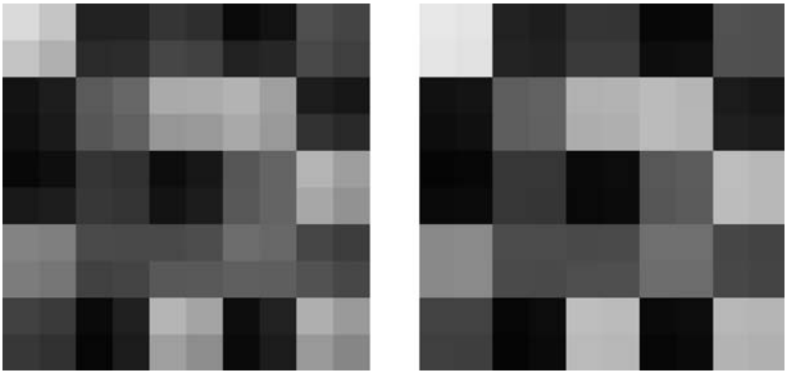

Fig. 14. – Approximation by $S_5^{\mathcal{F}_2}$ and $S_{10}^{\mathcal{F}_2}$ with increased resolution.

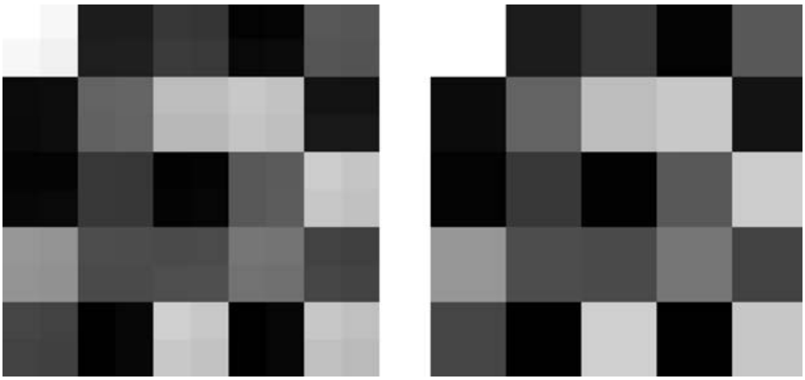

Fig. 15. – Approximation by $S_5^{\mathcal{M}_3^2}$ and $S_{10}^{\mathcal{M}_3^2}$ with increased resolution.

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Received February 15, 2011 and in revised form June 22, 2011