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Abstract. – *Flows of the form $D_t u + \alpha(u) \ni h$, with α maximal monotone, are here formulated as null-minimization problems via Fitzpatrick's theory. By means of De Giorgi's notion of Γ-convergence, we study the compactness and the structural stability of these flows with respect to variations of the source h and of the operator α.*

1. – Introduction

1.1 – *The Fitzpatrick Theorem*

Let V be a real Banach space. For any proper operator $\alpha : V \to \mathcal{P}(V')$, in [6] Fitzpatrick defined the convex and lower semicontinuous function

$$\begin{aligned} f_\alpha(v, v^*) &:= \langle v^*, v\rangle + \sup\left\{\langle v^* - v_0^*, v_0 - v\rangle : v_0^* \in \alpha(v_0)\right\} \\ &= \sup\left\{\langle v^*, v_0\rangle - \langle v_0^*, v_0 - v\rangle : v_0^* \in \alpha(v_0)\right\} \qquad \forall (v, v^*) \in V \times V', \end{aligned} \tag{1.1}$$

and proved that, whenever α is maximal monotone,

$$f_\alpha(v, v^*) \geq \langle v^*, v\rangle \qquad \forall (v, v^*) \in V \times V', \tag{1.2}$$

$$f_\alpha(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{1.3}$$

Extending Fitzpatrick's result, whenever a convex and lower semicontinuous function $f : V \times V' \to \boldsymbol{R} \cup \{+\infty\}$ fulfills the system (1.2) and (1.3), nowadays one says that f (variationally) *represents* the operator α. We shall denote by $\mathcal{F}(V)$ the class of *representative* functions and by $\mathcal{R}(V)$ the class of *representable* operators. The latter are necessarily monotone, but need not be either cyclically monotone or maximal monotone.

For instance, for any convex and lower semicontinuous function $\varphi : V \to \boldsymbol{R} \cup \{+\infty\}$, it is known that the *Fenchel function* $F(v, v^*) := \varphi(v) + \varphi^*(v^*)$ fulfills the system (1.2) and (1.3). Thus F represents the operator $\partial\varphi$. Other examples are provided e.g. in [13, 15, 16].

1.2 – *Extended B.E.N. Principle*

Let us now assume that we are given a Gelfand triplet of (real) Banach spaces

$$V \subset H = H' \subset V' \quad \text{with continuous and dense injections,} \tag{1.4}$$

Let a maximal monotone operator $\alpha : V \to \mathcal{P}(V')$ be represented by a function f_α, and $\Lambda : V \to V'$ linear, bounded and positive. The operator $\alpha + \Lambda$ may then be represented by the function

$$f_{\alpha+\Lambda}(u, v^*) = f_\alpha(u, v^* - \Lambda u) + \langle \Lambda u, u \rangle \qquad \forall (u, v^*) \in V \times V'. \tag{1.5}$$

This may be checked directly via the system (1.2) and (1.3).

We shall take $\Lambda = D_t$ (the time derivative). For any $h \in L^2(0, T; V')$, setting

$$\begin{gathered} J : \{v \in L^2(0,T;V) \cap H^1(0,T;V') : v(0) = u^0\} \to \boldsymbol{R} \cup \{+\infty\}, \\ J(v) := \int_0^T [f_\alpha(v, h - D_t v) - \langle h, v \rangle] \, dt + \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2, \end{gathered} \tag{1.6}$$

we may then reformulate the equation

$$D_t u + \alpha(u) \ni h \qquad \text{in } V', \text{ a.e. in }]0, T[\tag{1.7}$$

as a *null-minimization problem:* $J(v, h) = \inf J = 0$.

We shall refer to this representation as the *extended B.E.N. principle,* since it generalizes an approach that was pioneered by Brezis and Ekeland [3] and by Nayroles [9] for α cyclically monotone; see also the more recent works [10, 12] and references therein.

1.3 – *Variational Convergence and Structural Stability*

The above representation of maximal monotone operators offers the possibility to apply variational techniques to monotone flows like (1.7) as well as to other monotone problems, that so far were regarded as nonvariational. An important role is here played by De Giorgi's notion of Γ-convergence, see e.g. [4, 5]. The *structural stability* of the corresponding null-minimization problem may thus be addressed. By this we mean that the mapping that transforms the data (including the operator) into the solution is sequentially closed w.r.t. prescribed topologies. This extends more customary results on the closure of the dependence of the solution on data, by including variations of the operator.

In Sect. 2 we deal with the Γ-compactness of representative functions. In Sect. 3 we then apply these results to the compactness and structural stability of the monotone flow (1.7).

This note announces results that are expanded and proved in [16]. This is part of an ongoing research on the variational representation and structural stability of nonlinear P.D.E.s, along the lines that were indicated in [12]; see also [15]. A somehow comparable program, based on the use of the Fitzpatrick theory, has been accomplished for the homogenization of nonlinear flows, see e.g. [13, 14] and references therein. This theory opens the doors to further developments, that will be addressed apart. These include e.g. the onset of long memory in monotone flows; the formulation of a notion of convergence of representable functions, based on the Γ-convergence of the representative functions; the (variational) representation of pseudo-monotone operators; the identification of Γ-limits of sequences of representative operators, and others.

2. – Convergence and Compactness of Representable Operators

In this section we deal with the Γ-compactness of representative functions.

2.1 – *Γ-Compactness and Stability of Representative Functions*

Henceforth we shall assume that V' is separable. Let us define the following nonlinear notion of convergence: for any net $\{(v_a, v_a^*)\}$ in $V \times V'$, we set

$$\begin{aligned} &(v_a, v_a^*) \underset{\pi}{\to} (v, v^*) \quad \text{in } V \times V' \quad \Leftrightarrow \\ &v_a \rightharpoonup v \quad \text{in } V, \quad v_a^* \overset{*}{\rightharpoonup} v^* \quad \text{in } V', \quad \langle v_a^*, v_a \rangle \to \langle v^*, v \rangle. \end{aligned} \tag{2.1}$$

This convergence determines the π-topology, in which a set $B \subset V \times V'$ is closed if and only if it contains the limit of any π-convergent net of B. Let us also define the ws-topology as the product of the weak topology of V by the strong topology of V'. This topology is clearly finer than π. The "strong-weak star" topology of $V \times V'$ might be defined similarly.

THEOREM 2.1. – *Let a sequence $\{\psi_n\}$ of functions $V \times V' \to \boldsymbol{R} \cup \{+\infty\}$ be equicoercive in the sense that*

$$\forall C \in \boldsymbol{R}, \ \sup_{n \in \boldsymbol{N}} \left\{ \|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \psi_n(v, v^*) \le C \right\} < +\infty. \tag{2.2}$$

Then, up to extracting a subsequence, ψ_n sequentially Γ-converges to some function ψ w.r.t. the topology π. The same applies to the topology ws, and the Γ-limit may depend on the topology.

This statement rests upon the separability of $V \times V'$, and on the Γ-compactness of any subset of a topological space X with a countable basis, see e.g. [2; p. 152], [4; p. 90]. The role of equicoerciveness in providing the *sequential* Γ-convergence is illustrated in [1].

THEOREM 2.2. – *Let $\{\psi_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathcal{F}(V)$ that sequentially $\Gamma\pi$-converges to a function ψ. Then $\psi \in \mathcal{F}(V)$.*

Moreover, denoting by α_n (α, resp.) the operator $V \to \mathcal{P}(V')$ that is represented by ψ_n (ψ, resp.), for any sequence $\{(v_n, v_n^*)\}$ in $V \times V'$,

$$v_n^* \in \alpha_n(v_n) \quad \forall n, \quad (v_n, v_n^*) \underset{\pi}{\to} (v, v^*) \quad \Rightarrow \quad v^* \in \alpha(v). \tag{2.3}$$

The same results hold if π is replaced by the topology ws.

2.2 – *Representation of Time-Dependent Functions*

Let us fix any $T > 0$, any $p \in \,]1, +\infty[$ and set $\mathcal{V} := L^p(0, T; V)$. Let us define the topology π in $\mathcal{V} \times \mathcal{V}'$ as in (2.1), by replacing the space V by $\mathcal{V}$ and the associated duality pairing $\langle \cdot, \cdot \rangle$ by $\langle\langle v^*, v \rangle\rangle := \int_0^T \langle v^*(t), v(t) \rangle \, dt$ for any $(v, v^*) \in \mathcal{V} \times \mathcal{V}'$. The results above take over to time-dependent operators and to their time-integrated representative functions, simply by replacing the space V by $\mathcal{V}$.

It is promptly seen that, whenever a function $\psi \in \mathcal{F}(V)$ is coercive in the sense that

$$\forall C \in \mathbf{R}, \ \sup\big\{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \psi(v, v^*) \le C\big\} < +\infty, \tag{2.4}$$

ψ represents an operator $\alpha : V \to \mathcal{P}(V')$ if and only if the functional

$$\Psi(v, v^*) := \int_0^T \psi(v(t), v^*(t)) \, dt \qquad \forall (v, v^*) \in \mathcal{V} \times \mathcal{V}' \tag{2.5}$$

represents the operator $\widehat{\alpha} : \mathcal{V} \to \mathcal{P}(\mathcal{V}') : v \mapsto \alpha(v(\,\cdot\,))$.

Next we relate the π-convergence in $V \times V'$ a.e. in $]0, T[$ with the π-convergence in $\mathcal{V} \times \mathcal{V}'$.

THEOREM 2.3. – *Let $p \in \,]1, +\infty[$, and $\{(v_n, v_n^*)\}$ be a bounded sequence in $W^{\varepsilon,p}(0, T; V) \times W^{\varepsilon,p'}(0, T; V')$ for some $\varepsilon > 0$. If*

$$(v_n, v_n^*) \underset{\pi}{\to} (v, v^*) \qquad \text{in } V \times V' \text{ a.e. in }]0, T[, \tag{2.6}$$

then

$$(v_n, v_n^*) \underset{\pi}{\to} (v, v^*) \qquad in\ \mathcal{V}\times\mathcal{V}'. \tag{2.7}$$

On the other hand, (2.7) *does not entail* (2.6), *not even for a subsequence.*

3. – Compactness and Structural Stability of Monotone Flows

On the basis of the previous results, in this section we study the compactness and the structural stability of the variational formulation of the equation (1.7) w.r.t. variations of the operator α and of the source h.

More specifically, for any n we prescribe a pair of admissible α_n and h_n, and denote by u_n the solution of (1.7) coupled e.g. with the condition of time periodicity. We fix a representative function ψ_n of α_n, and reformulate (1.7) variationally. Under the hypothesis of equicoercivity, by Theorems 2.1 and 2.2 the ψ_n's accumulate at some $\psi \in \mathcal{F}(V)$ w.r.t. the π-topology. As $h_n \to h$ in a suitable topology, we then show that the solutions accumulate at some u, which solves the corresponding limit equation (1.7), where α is the operator that is represented by ψ.

3.1 – *Abstract Quasilinear Parabolic Operators*

Let $V \subset H = H' \subset V'$ be as in (1.4), here with V and H Hilbert spaces and V' separable. For any $n \in \boldsymbol{N}$, let us assume that

$$\alpha_n : V \to \mathcal{P}(V') \text{ is maximal monotone,} \tag{3.1}$$

$$\exists a_n > 0 : \forall (v, w) \in \text{graph}(\alpha_n), \quad \langle w, v\rangle \geq a_n \|v\|_V^2, \tag{3.2}$$

$$\exists C_{1n}, C_{2n} > 0 : \forall (v, w) \in \text{graph}(\alpha_n), \quad \|w\|_{V'} \leq C_{1n}\|v\|_V + C_{2n}, \tag{3.3}$$

$$h_n \in L^2(0, T; V'). \tag{3.4}$$

Let us fix any $T \in\ ]0, +\infty]$ ($T = +\infty$ included), and set

$$\begin{gathered} X := L^2(0, T; V) \cap H^1(0, T; V'), \qquad Y := H^1(0, T; V) \cap H^2(0, T; V'), \\ H^1_\sharp(0, T; V') := \{v \in H^1(0, T; V') : v(0) = v(T)\}, \\ X_\sharp := L^2(0, T; V) \cap H^1_\sharp(0, T; V'), \qquad Y_\sharp := Y \cap X_\sharp; \end{gathered} \tag{3.5}$$

here we set $v(+\infty) := \lim_{t\to+\infty} v(t)$. Next we shall deal with the sequence of flows

$$u_n \in X_\sharp, \qquad D_t u_n + \alpha_n(u_n) \ni h_n \qquad \text{in } V',\ \text{a.e. in }]0, T[. \tag{3.6}$$

Note that $X_\sharp = \{v \in X : v(0) = 0\}$ if $T = +\infty$. The Cauchy problem for all $t \geq 0$ with vanishing initial datum may thus be regarded as a periodic problem with infinite period. The condition $u_n(0) = 0$ is not restrictive, since it may be retrieved by shifting the unknown function u. Next we review a result of existence, uniqueness and boundedness.

PROPOSITION 3.1. – *Let* (3.1)–(3.4) *be fulfilled, for* $0 < T \leq +\infty$. *Then:*

(i) *Problem* (3.6) *has a solution. This is unique if either* $T = +\infty$ *or* α_n *is strictly monotone. The sequence* $\{u_n\}$ *is bounded in* $X_\sharp$ *whenever*

$$\sup_n \left\{|a_n^{-1}| + |C_{1n}| + |C_{2n}| + \|h_n\|_{L^2(0,T;V')}\right\} < +\infty, \tag{3.7}$$

(ii) *If* $h_n \in H^1_\sharp(0,T;V')$ *and*

$$\begin{aligned} &\exists b_n > 0 : \forall (v_1, w_1), (v_2, w_2) \in \mathrm{graph}(\alpha_n),\\ &\langle w_1 - w_2, v_1 - v_2 \rangle \geq b_n \|v_1 - v_2\|_V^2 \ , \end{aligned} \tag{3.8}$$

then $u_n \in Y_\sharp$. *Moreover* $\{u_n\}$ *is bounded in* $Y_\sharp$ *whenever*

$$\sup_n \left\{|a_n^{-1}| + |b_n^{-1}| + |C_{1n}| + |C_{2n}| + \|h_n\|_{H^1(0,T;V')}\right\} < +\infty. \tag{3.9}$$

For any n, let the operator α_n be represented by a proper function $\psi_n \in \mathcal{F}(V)$, so that $\widehat{\alpha}_n$ (defined as in Sect. 2) is represented by the time-integrated functional $\Psi_n \in \mathcal{F}(\mathcal{V})$, see (2.5). Let us assume that

$$h_n \to h \qquad \text{in } \mathcal{V}'. \tag{3.10}$$

THEOREM 3.2. – *(Global Formulation)*

(i) *(Compactness) Let* (3.1)–(3.4) *be fulfilled, and let each operator* α_n *be represented by a function* $\psi_n : V \to \mathcal{P}(V')$. *Let us assume that*

$$\forall C \in \boldsymbol{R}, \ \sup_{n \in \boldsymbol{N}} \ \left\{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \psi_n(v, v^*) \leq C\right\} < +\infty, \tag{3.11}$$

and define Ψ_n *as in* (2.5). *Then there exists* $\Psi \in \mathcal{F}(\mathcal{V})$ *such that, up to extracting a subsequence,*

$$\Psi_n \text{ sequentially } \Gamma\text{-converges to } \Psi \text{ w.r.t. the topology } \pi \text{ of } \mathcal{V} \times \mathcal{V}'. \tag{3.12}$$

(ii) *(Structural Stability) Let* $\widehat{\alpha} : \mathcal{V} \to \mathcal{P}(\mathcal{V}')$ *be the operator that is represented by* Ψ. *For any* n, *let* $h_n \in \mathcal{V}'$, *and* u_n *be a solution of problem* (3.6). *If* (3.7) *is fulfilled, then there exists* $u \in X_\sharp$ *such that* $u_n \rightharpoonup u$ *in* $X_\sharp$, *up to extracting a subsequence. This entails*

$$D_t u + \widehat{\alpha}(u) \ni h \qquad \text{in } \mathcal{V}'. \tag{3.13}$$

3.2 – *Outline of the Proof*

The Γ-compactness stems from Theorems 2.1 and 2.2. By part (i) of Proposition 3.1, the sequence $\{u_n\}$ is bounded in $X_\sharp$. Hence there exists $u \in X_\sharp$ such that $u_n \rightharpoonup u$ in $X_\sharp$, up to a subsequence. It is easily seen that

$$(u_n, h_n - D_t u_n) \underset{\pi}{\rightarrow} (u, h - D_t u) \qquad \text{in } \mathcal{V} \times \mathcal{V}'. \tag{3.14}$$

The flow (3.6) may be reformulated in global form as

$$u_n \in X_\sharp, \qquad \Psi_n(u_n, h_n - D_t u_n) \le \langle\langle u_n, h_n \rangle\rangle. \tag{3.15}$$

By Theorem 2.2 then

$$u \in X_\sharp, \qquad \Psi(u, h - D_t u) \le \langle\langle u, h \rangle\rangle, \tag{3.16}$$

and this is tantamount to (3.13). □

THEOREM 3.3. – *(Pointwise Formulation)*

(i) *(Compactness) Let* (3.1)–(3.4) *be fulfilled. For any n, let the operator α_n be represented by a function $\psi_n : V \to \mathcal{P}(V')$, and* (3.11) *be fulfilled. Then there exists $\psi \in \mathcal{F}(V)$ such that, up to extracting a subsequence,*

$$\psi_n \text{ sequentially } \Gamma\text{-converges to } \psi \text{ w.r.t. the topology } \pi \text{ of } V \times V'. \tag{3.17}$$

(ii) *(Structural Stability) Let $\alpha : V \to \mathcal{P}(V')$ be the operator that is represented by ψ. For any n, let $h_n \in H^1_\sharp(0, T; V')$, and u_n be a solution of problem* (3.6). *If* (3.8) *and* (3.9) *are fulfilled, then there exists $u \in Y_\sharp$ such that $u_n \rightharpoonup u$ in $Y_\sharp$, up to extracting a subsequence. If the canonic injection $V \to H$ is compact, this entails*

$$D_t u + \alpha(u) \ni h \qquad \text{in } V', \text{ a.e. in }]0, T[. \tag{3.18}$$

The proof of this result follows the lines of Theorem 3.2. Part (ii) of Proposition 3.1 provides the boundedness in $Y_\sharp$, and this allows one to derive the pointwise formulation.

The global and a pointwise formulations and the two latter theorems may be extended to the Cauchy problem for the equation (1.7) on a bounded time interval, see [16].

By Theorem 2.3, the pointwise formulation (3.18) entails the global problem (3.13). A priori the converse might fail: for any instant t, $[\widehat{\alpha}(u)](\cdot, t)$ might depend not only on $u(\cdot, t)$ but also on $u|_{\Omega \times]0,t[}$. This would correspond to the occurrence of *long memory* in the equation. Even the proof of the causality of the operator $\widehat{\alpha}$ does not seem obvious.

3.3 – *Applications to P.D.E.s.*

Let Ω be a bounded Lipschitz domain of $\boldsymbol{R}^N$ ($N \geq 1$) and $\{\vec{\gamma}_n(x,\cdot)\}$ be a sequence of maximal monotone mappings $\boldsymbol{R}^N \to \mathcal{P}(\boldsymbol{R}^N)$, that are Lebesgue-measurable w.r.t. $x \in \Omega$. One may select

$$V = H^1_0(\Omega), \quad H = L^2(\Omega), \quad \alpha_n(v) = -\nabla\cdot\vec{\gamma}_n(x, \nabla v) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.19}$$

For $N = 3$, one may also take

$$\begin{aligned} &V = \{\vec{v} \in L^2(\Omega)^3 : \nabla\times\vec{v} \in L^2(\Omega)^3,\ \vec{\nu}\times\vec{v} = \vec{0} \text{ in } H^{-1/2}(\partial\Omega)^3\},\\ &H = L^2(\Omega)^3, \qquad \vec{\alpha}_n(\vec{v}) = \nabla\times\vec{\gamma}_n(x, \nabla\times\vec{v}) \quad \text{in } \mathcal{D}'(\Omega)^3, \forall\vec{v}\in V, \end{aligned} \tag{3.20}$$

where by $\vec{\nu}$ we denote the outward-oriented unit normal vector-field on $\partial\Omega$.

Theorem 3.2 applies to both sequences of operators (3.19) and (3.20), whereas Theorem 3.3 only applies to (3.19), since the canonic injection $V \to H$ is compact just in this case. For (3.20) onset of long memory may thus be expected in the limit.

This may be compared with an example due to Tartar. The equation $D_t u + a_n(x)u = 0$ is obviously associated to the linear semigroup $S_n(t) : v(x) \mapsto \exp\{-a_n(x)t\}\, v(x)$. This converges to a semigroup if the sequence a_n *strongly* converges in $L^1_{\text{loc}}(\Omega)$. If instead a_n converges only weakly, the exponential form is lost in the limit, and a long memory effect does occur; see [11; Chap. 23] and references therein.

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