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On the Internal Duality Between Topological Modules and Bornological Modules

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Abstract. – *In this paper the internal duality between topological modules and bornological modules is discussed, such a duality being expressed by an equivalence of suitable categories.*

The notion of a bounded subset of a topological vector space, introduced by Kolmogoroff [11] and von Neumann [15], plays a central role in functional analysis and its applications. In order to mention only two classical instances of the theory of locally convex spaces which are based on that notion, we may recall Kolmogoroff's characterization of normable spaces [11] and the concept of a bornological space [6,12]. The set of all bounded subsets of a topological vector space is a vector bornology, so that this special vector bornology is quite important. Since bornologies have shown to be a very useful tool in various aspects of functional analysis, they have been considered by several authors, in different contexts (see [1], [2], [3], [7], [16], [17], [19], [20] and [21], for example). The books [8] and [10] discuss many aspects of the theory of bornological vector spaces, including applications to differential equations, and the book [9] presents a treatment of the theory of distributions based on the concept of a bornology.

There is a natural internal duality between locally convex spaces and convex bornological vector spaces [1,10], established by an approach of an essentially geometric nature. On the one hand, to each locally convex space one associates a convex bornological vector space (by considering the von Neumann bornology) in such a way that, under such an association, continuous linear mappings are transformed into bounded linear mappings. And, on the other hand, to each convex bornological vector space one associates a locally convex space (by taking as fundamental system of neighborhoods of zero the absolutely convex sets which absorb the elements of the given convex bornology) in such a way that, under such an association, bounded linear mappings are transformed into continuous linear mappings.

In this paper we use a different approach, of a topological nature, to prove that there is an internal duality between topological modules and bornological modules, thereby showing that the above-mentioned property is also valid in the context of modules. In a nutshell, such an internal duality may be expressed by a functor which establishes an equivalence between the category of bornological topological R-modules and the category of topological bornological R-modules (for each given topological ring R), the notion of a bornological topological R-module being known and the one of a topological bornological R-module being introduced in the present paper.

Throughout this paper R shall denote a topological ring with a non-zero identity element and all R-modules under consideration shall be unitary left R-modules.

A bornology $\mathcal{B}$ on an R-module E is an R-module bornology, and $(E, \mathcal{B})$ is a bornological R-module [17], if the mappings

$$(x, y) \in (E \times E, \mathcal{B} \times \mathcal{B}) \mapsto x + y \in (E, \mathcal{B})$$

and

$$(\lambda, x) \in (R \times E, \mathcal{B}_R \times \mathcal{B}) \mapsto \lambda x \in (E, \mathcal{B})$$

are bounded, where $\mathcal{B} \times \mathcal{B}$ is the product bornology on $E \times E$, $\mathcal{B}_R$ is the bornology on R consisting of all bounded subsets of R ([22], Definition 16.1) and $\mathcal{B}_R \times \mathcal{B}$ is the product bornology on $R \times E$. For example, if $(E, \mathcal{T})$ is a topological R-module and $\boldsymbol{B}(\mathcal{T})$ is the set of all $\mathcal{T}$-bounded subsets of E ([22], Definition 15.1), it is easily seen that $\boldsymbol{B}(\mathcal{T})$ is an R-module bornology on E. Moreover, if $(E, \mathcal{T})$ and (F, θ) are two topological R-modules and

$$u : (E, \mathcal{T}) \to (F, \theta)$$

is a continuous R-linear mapping, then

$$u : (E, \boldsymbol{B}(\mathcal{T})) \to (F, \boldsymbol{B}(\theta))$$

is bounded by Theorem 15.2(2) of [22].

Now let us recall the notion of a bornological topological R-module [4], which has also been studied in [5]. Let $(E, \mathcal{T})$ be a topological R-module. Then the following assertion holds ([4], Proposition 1):

Among the R-module topologies $\mathcal{T}'$ on E for which $\boldsymbol{B}(\mathcal{T}') = \boldsymbol{B}(\mathcal{T})$ there is a finest one, denoted by $\mathcal{T}^b$, which is characterized as follows: for every R-module topology θ on E, we have that $\boldsymbol{B}(\mathcal{T})$ is finer that $\boldsymbol{B}(\theta)$ if and only if θ is coarser than $\mathcal{T}^b$.

Obviously, $\mathcal{T}$ is coarser than $\mathcal{T}^b$; $(E, \mathcal{T})$ is said to be bornological if $\mathcal{T} = \mathcal{T}^b$.

The following proposition has already been proved in the context of topological vector spaces ([14], p. 81 and p. 83; [18], pp. 67-69).

PROPOSITION 1. – *Let $(E, \mathcal{B})$ be a bornological R-module. Then there exists a unique R-module topology $\boldsymbol{T}(\mathcal{B})$ on E satisfying the following properties:*

(a) *$\mathcal{B}$ is finer than $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$;*

(b) *for every R-module topology θ on E, we have that $\mathcal{B}$ is finer than $\boldsymbol{B}(\theta)$ if and only if θ is coarser than $\boldsymbol{T}(\mathcal{B})$.*

Moreover, $(E, \boldsymbol{T}(\mathcal{B}))$ is a bornological topological R-module.

PROOF. – Let $(\mathcal{T}_i)_{i\in I}$ be the family of all R-module topologies on E for which $\mathcal{B}$ is finer than $\boldsymbol{B}(\mathcal{T}_i)$ ($I \neq \emptyset$ since the chaotic topology on E belongs to this family). Then $\boldsymbol{T}(\mathcal{B}) = \sup_{i\in I} \mathcal{T}_i$ is an R-module topology on E ([22], Corollary 12.7) and (a) holds, because $B \in \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ if and only if $B \in \boldsymbol{B}(\mathcal{T}_i)$ for all $i \in I$. In order to prove (b), let θ be an R-module topology on E. If $\mathcal{B}$ is finer than $\boldsymbol{B}(\theta)$, then $\theta = \mathcal{T}_i$ for some $i \in I$, and so θ is coarser than $\boldsymbol{T}(\mathcal{B})$. Conversely, if θ is coarser than $\boldsymbol{T}(\mathcal{B})$, then $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ is finer than $\boldsymbol{B}(\theta)$, and so $\mathcal{B}$ is finer than $\boldsymbol{B}(\theta)$ by (a). Thus (b) is valid, and the uniqueness of $\boldsymbol{T}(\mathcal{B})$ follows immediately from (a) and (b). Finally, since $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B})^b) = \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ and $\mathcal{B}$ is finer than $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ (by (a)), then $\mathcal{B}$ is finer than $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B})^b)$, and (b) guarantees that $\boldsymbol{T}(\mathcal{B})^b$ is coarser than $\boldsymbol{T}(\mathcal{B})$. Therefore $\boldsymbol{T}(\mathcal{B})^b = \boldsymbol{T}(\mathcal{B})$, and $(E, \boldsymbol{T}(\mathcal{B}))$ is bornological.

PROPOSITION 2. – *Let $(E, \mathcal{T})$ be a topological R-module. Then $\mathcal{T} = \boldsymbol{T}(\boldsymbol{B}(\mathcal{T}))$ if and only if $(E, \mathcal{T})$ is bornological.*

PROOF. – If $\mathcal{T} = \boldsymbol{T}(\boldsymbol{B}(\mathcal{T}))$, then $(E, \mathcal{T})$ is bornological by the last assertion in Proposition 1. Conversely, assume that $(E, \mathcal{T})$ is bornological and put $\mathcal{B} = \boldsymbol{B}(\mathcal{T})$. Then, by Proposition 1(a), the identity mapping

$$(E, \mathcal{B}) \to (E, \boldsymbol{B}(\boldsymbol{T}(\mathcal{B})))$$

is bounded. But, since $(E, \mathcal{T})$ is bornological, the theorem obtained in [4] implies that the identity mapping

$$(E, \mathcal{T}) \to (E, \boldsymbol{T}(\mathcal{B}))$$

is continuous, and hence $\boldsymbol{T}(\mathcal{B})$ is coarser than $\mathcal{T}$. On the other hand, since $\mathcal{B}$ is finer than $\boldsymbol{B}(\mathcal{T})$, Proposition 1(b) shows that $\mathcal{T}$ is coarser than $\boldsymbol{T}(\mathcal{B})$. Therefore $\mathcal{T} = \boldsymbol{T}(\mathcal{B}) = \boldsymbol{T}(\boldsymbol{B}(\mathcal{T}))$.

COROLLARY 3. – *For every bornological R-module $(E, \mathcal{B})$, $\boldsymbol{T}(\mathcal{B}) = \boldsymbol{T}(\boldsymbol{B}(\boldsymbol{T}(\mathcal{B})))$.*

PROOF. – Since, by Proposition 1, $(E, \boldsymbol{T}(\mathcal{B}))$ is bornological, the result follows immediately from Proposition 2.

PROPOSITION 4. – *Let $(E, \mathcal{B})$ be a bornological R-module. Then $\mathcal{B} = \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ if and only if there exists an R-module topology $\mathcal{T}$ on E such that $\mathcal{B} = \boldsymbol{B}(\mathcal{T})$.*

PROOF. – If $\mathcal{B} = \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$, take $\mathcal{T} = \boldsymbol{T}(\mathcal{B})$. Conversely, assume that $\mathcal{B} = \boldsymbol{B}(\mathcal{T})$ for some R-module topology $\mathcal{T}$ on E. By Proposition 1(a), $\mathcal{B}$ is finer than $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$. On the other hand, since $\mathcal{B}$ is finer than $\boldsymbol{B}(\mathcal{T})$, Proposition 1(b) implies that $\mathcal{T}$ is coarser than $\boldsymbol{T}(\mathcal{B})$, and so $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ is finer than $\boldsymbol{B}(\mathcal{T}) = \mathcal{B}$. Thus $\mathcal{B} = \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$.

An immediate consequence of Proposition 4 reads:

COROLLARY 5. – *For every topological R-module* $(E, \mathcal{T})$, $\boldsymbol{B}(\mathcal{T}) = \boldsymbol{B}(\boldsymbol{T}(\boldsymbol{B}(\mathcal{T})))$.

The next example shows that an R-module bornology is not necessarily induced by an R-module topology in the sense of Proposition 4. Consequently, by Proposition 4, the equality $\mathcal{B} = \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ is not valid for every R-module bornology $\mathcal{B}$.

EXAMPLE 6. – Consider the ring $\mathbb{Z}$ of integers endowed with the discrete topology and $E = \mathbb{R}$ regarded as a $\mathbb{Z}$-module. Then the set $\mathcal{B}$ of all at most countable subsets of E is a $\mathbb{Z}$-module bornology on E. Nevertheless, for every $\mathbb{Z}$-module topology $\mathcal{T}$ on E, $\boldsymbol{B}(\mathcal{T}) = 2^E \neq \mathcal{B}$.

The above considerations motivate the following

DEFINITION 7. – *A bornological R-module* $(E, \mathcal{B})$ *is said to be* topological *if there exists an R-module topology* $\mathcal{T}$ *on E such that* $\mathcal{B} = \boldsymbol{B}(\mathcal{T})$.

If R is endowed with the discrete topology and $(E, \mathcal{B})$ is a bornological R-module, then $(E, \mathcal{B})$ is topological if and only if $\mathcal{B} = 2^E$.

PROPOSITION 8. – *Let* $(E, \mathcal{B})$ *and* $(F, \mathcal{C})$ *be two bornological R-modules. If*

$$u : (E, \mathcal{B}) \to (F, \mathcal{C})$$

is a bounded R-linear mapping, then

$$u : (E, \boldsymbol{T}(\mathcal{B})) \to (F, \boldsymbol{T}(\mathcal{C}))$$

is continuous.

PROOF. – Let θ be the inverse image under u of the topology $\boldsymbol{T}(\mathcal{C})$, which is an R-module topology on E by Theorem 12.5 of [22]. We claim that θ is coarser than $\boldsymbol{T}(\mathcal{B})$ which, in view of Proposition 1(b), is equivalent to the assertion that $\mathcal{B}$ is finer than $\boldsymbol{B}(\theta)$. So let $B \in \mathcal{B}$ be arbitrary and let V be an arbitrary $\boldsymbol{T}(\mathcal{C})$-neighborhood of 0 in F. Since $u(B) \in \mathcal{C}$ and $\mathcal{C}$ is finer than $\boldsymbol{B}(\boldsymbol{T}(\mathcal{C}))$ by Proposition 1(a), there is a neighborhood W of 0 in R such that $Wu(B) \subset V$, and hence $WB \subset u^{-1}(V)$. This proves that $B \in \boldsymbol{B}(\theta)$. Therefore θ is coarser than $\boldsymbol{T}(\mathcal{B})$, and the continuity of $u : (E, \boldsymbol{T}(\mathcal{B})) \to (F, \boldsymbol{T}(\mathcal{C}))$ is established.

The converse of Proposition 8 is not necessarily true. In fact, we have the following

PROPOSITION 9. – *For a bornological R-module $(F,\mathcal{C})$, the following conditions are equivalent:*

(a) $(F,\mathcal{C})$ *is topological;*

(b) *for every bornological R-module $(E,\mathcal{B})$ and every R-linear mapping $u : E \to F$, we have that $u : (E,\mathcal{B}) \to (F,\mathcal{C})$ is bounded if and only if $u : (E,\boldsymbol{T}(\mathcal{B})) \to (F,\boldsymbol{T}(\mathcal{C}))$ is continuous.*

PROOF. – Let us prove that (a) implies (b). Indeed, let $(E,\mathcal{B})$ and u be as in (b). If $u : (E,\mathcal{B}) \to (F,\mathcal{C})$ is bounded, then $u : (E,\boldsymbol{T}(\mathcal{B})) \to (F,\boldsymbol{T}(\mathcal{C}))$ is continuous by Proposition 8. Conversely, if $u : (E,\boldsymbol{T}(\mathcal{B})) \to (F,\boldsymbol{T}(\mathcal{C}))$ is continuous, then

$$u : (E,\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))) \to (F,\boldsymbol{B}(\boldsymbol{T}(\mathcal{C})))$$

is bounded. Thus, since $\mathcal{B}$ is finer than $\boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ by Proposition 1(a) and $\mathcal{C} = \boldsymbol{B}(\boldsymbol{T}(\mathcal{C}))$ by Proposition 4 ($(F,\mathcal{C})$ is topological), we conclude that $u : (E,\mathcal{B}) \to (F,\mathcal{C})$ is bounded.

Now, let us assume that (a) is false. Then, in view of Propositions 1(a) and 4, the identity mapping $1_F : (F,\boldsymbol{B}(\boldsymbol{T}(\mathcal{C}))) \to (F,\mathcal{C})$ is not bounded. However, $\boldsymbol{T}(\boldsymbol{B}(\boldsymbol{T}(\mathcal{C}))) = \boldsymbol{T}(\mathcal{C})$ by Corollary 3, and hence $1_F : (F,\boldsymbol{T}(\boldsymbol{B}(\boldsymbol{T}(\mathcal{C}))) \to (F,\boldsymbol{T}(\mathcal{C}))$ is continuous. This shows that (b) is false.

Let Modtb_R be the category whose objects are bornological topological R-modules and whose morphisms are continuous R-linear mappings, and let Modbt_R be the category whose objects are topological bornological R-modules and whose morphisms are bounded R-linear mappings. If $(E,\mathcal{T})$ and (F,θ) are two arbitrary topological R-modules and

$$u : (E,\mathcal{T}) \to (F,\theta)$$

is a continuous R-linear mapping, then

$$u : (E,\boldsymbol{B}(\mathcal{T})) \to (F,\boldsymbol{B}(\theta))$$

is bounded. Moreover, $(E,\boldsymbol{B}(\mathcal{T}))$ is a topological bornological R-module by Corollary 5. Therefore we can consider the covariant functor

$$\mathcal{F} : \mathrm{Modtb}_R \to \mathrm{Modbt}_R$$

which, to each $(E,\mathcal{T}) \in \mathrm{Ob}(\mathrm{Modtb}_R)$, associates

$$\mathcal{F}((E,\mathcal{T})) = (E,\boldsymbol{B}(\mathcal{T})) \in \mathrm{Ob}(\mathrm{Modbt}_R).$$

The relationship between the categories Modtb_R and Modbt_R may be summarized in the following

THEOREM 10. – *The functor*

$$\mathcal{F} : \mathrm{Modtb}_R \to \mathrm{Modbt}_R$$

is an equivalence of categories.

PROOF. – First, we claim that $\mathcal{F}$ is full and faithful, that is, for all $(E, \mathcal{T}), (F, \theta) \in \mathrm{Ob}(\mathrm{Modtb}_R)$, the mapping

$$u \in \mathrm{Mor}_{\mathrm{Modtb}_R}\big((E, \mathcal{T}); (F, \theta)\big) \mapsto u \in \mathrm{Mor}_{\mathrm{Modbt}_R}\big((E, \boldsymbol{B}(\mathcal{T})); (F, \boldsymbol{B}(\theta))\big)$$

is bijective. In fact, the injectivity of this mapping is obvious and its surjectivity follows from the fact that if $(E, \mathcal{T}) \in \mathrm{Ob}(\mathrm{Modtb}_R)$, (F, θ) is an arbitrary topological R-module and $u : (E, \boldsymbol{B}(\mathcal{T})) \to (F, \boldsymbol{B}(\theta))$ is bounded, then $u : (E, \mathcal{T}) \to (F, \theta)$ is continuous.

Second, the mapping

$$(E, \mathcal{T}) \in \mathrm{Ob}(\mathrm{Modtb}_R) \mapsto (E, \boldsymbol{B}(\mathcal{T})) \in \mathrm{Ob}(\mathrm{Modbt}_R)$$

is surjective. In fact, let $(E, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modbt}_R)$ be arbitrary. Then, by Proposition 1, $(E, \boldsymbol{T}(\mathcal{B})) \in \mathrm{Ob}(\mathrm{Modtb}_R)$. And, since $(E, \mathcal{B})$ is topological, $\mathcal{B} = \boldsymbol{B}(\boldsymbol{T}(\mathcal{B}))$ by Proposition 4.

Therefore $\mathcal{F}$ is an equivalence of categories ([13], Chapter IV, § 4, Theorem 1).

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