BOLLETTINO
UNIONE MATEMATICA ITALIANA

ALESSIO RUSSO

# On Numbers which are Orders of Nilpotent Groups Only

*Bollettino dell'Unione Matematica Italiana, Serie 9*, Vol. **5** (2012), n.1, p. 121–124.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2012_9_5_1_121_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2012.

Bollettino U. M. I.
(9) **V** (2012), 121-124

# On Numbers which are Orders of Nilpotent Groups Only

ALESSIO RUSSO

**Abstract.** – *In [T. W. Müller, An arithmetic theorem related to groups of bounded nilpotency class, J. Algebra* 300 (2006), 10-15*] T. W. Müller characterizes the positive integers n satisfying the property that every group of order n is nilpotent of class bounded by a fixed positive integer c. In this article a different proof of the above result will be given.*

## 1. – Introduction

Let $\mathfrak{X}$ be a class of groups. A positive integer $n$ is said to be an $\mathfrak{X}$-*number* if every finite group of order $n$ is an $\mathfrak{X}$-group. Many authors have investigated the $\mathfrak{X}$-numbers for several choices of the class $\mathfrak{X}$. The case in which $\mathfrak{X}$ is the class of cyclic groups was attributed to Burnside and appeared in many articles (see, for instance, [1], [2], [4]). A necessary and sufficient condition that there is only one group of order $n$ is that $(n, \varphi(n)) = 1$, where $\varphi(n)$ is the totient function of Euler. Clearly, this condition is equivalent to saying that $n$ is square-free, and no two prime factors $p$ and $q$ satisfy the congruence $p \equiv 1 (mod\ q)$.

In what follows it is useful to consider the function $\psi(n)$ defined by extending multiplicatively the recursion for prime powers $\psi(1) = 1$, $\psi(p^s) = (p^s - 1)\psi(p^{s-1})$. Clearly, $\psi(n) = \varphi(n)$ if and only if $n$ is square-free. Moreover, if $n = p_1^{n_1} \cdots p_t^{n_t}$, $p_i$ distinct primes, is a positive integer, then the condition $(n, \psi(n)) = 1$ is equivalent to saying that $p_j$ does not divide $p_i^l - 1$, for all integers $i$, $j$ and $l$ with $1 \leq l \leq n_i$. Therefore $n$ is a cyclic number if and only if $(n, \psi(n)) = 1$ and $n$ is square-free.

The abelian case was considered by Dickson [1] and successively was rediscovered by Rédei [7, Satz 1] as an application of the theory of minimal non-abelian groups (a group $G$ is said to be *minimal non-*$\mathfrak{X}$, where $\mathfrak{X}$ is a class of groups, if $G$ does not belong to $\mathfrak{X}$, but all its proper subgroups are $\mathfrak{X}$-groups). They proved that $n$ is an abelian number if and only if $(n, \psi(n)) = 1$ and $n$ is cube-free. The nilpotent numbers were firstly characterized in 1959 by Pazderski [6, Satz 1] as the numbers $n$ such that $(n, \psi(n)) = 1$. Recently, Müller [5] has generalized latest results stating the following arithmetic condition for a number to be order only of nilpotent groups of bounded nilpotency class.

THEOREM 1. – *Fix $c \in \mathbb{N} \cup \{\infty\}$, and let $n$ be a positive integer. Then every group of order $n$ is nilpotent with class at most $c$ (briefly: $n$ is a $c$-nilpotent number) if and only if $n$ is $(c+2)$-power free and $(n, \psi(n)) = 1$.*

The proof of Müller involves P. Hall's bound on the automorphism group of a finite $p$-group (see, for instance, [9, 5.3.3]) and the Rédei's structure theorems for minimal non-abelian groups [7] and for minimal non-nilpotent groups [8]. The aim of this short article is to produce a different proof of the above theorem. In particular, concerning the structure of minimal non-nilpotent groups, only the non-simplicity of such groups will be used.

Most of our notation is standard and can be found in [9].

## 2. – Proof of the Theorem

Before we can prove the result we must establish a simple lemma which is well known to finite group theorists. We report the proof for convenience of the reader.

LEMMA 2. – *Let $G$ be a finite group whose maximal subgroups are self-normalizing. Then there exist distinct maximal subgroups whose intersection is non-trivial.*

PROOF. – Suppose for a contradiction that any two maximal subgroups have trivial intersection. Let $(M_1, M_2)$ be a pair of maximal subgroups of $G$ such that $M_2$ is not a conjugate of $M_1$. By hypothesis the conjugates of $M_i$ $(i = 1, 2)$ contain exactly $|G| - |G|/|M_i|$ elements $\neq 1$. It follows that

$$|G| - |G|/|M_1| + |G| - |G|/|M_2| < |G|,$$

and hence the contradiction $|M_1||M_2| < |M_1| + |M_2|$. □

Now we are in a position to prove the Theorem 1.

We divide the proof in two steps.

**(1)** *A positive integer $n = p_1^{n_1} \cdots p_t^{n_t}$, $p_i$ distinct primes, is a nilpotent number if and only if $(n, \psi(n)) = 1$.*

First suppose that $n$ is a nilpotent number. For a contradiction let $p_i^l \equiv 1 (mod p_j)$, for some $i, j \in \{1, \ldots, t\}$ and $l \in \{1, \ldots, n_i\}$. Denote by $N$ an elementary abelian $p_i$-group of order $p_i^l$. It is well known that $AutN \cong GL(l, p_i)$, and hence $|AutN| = (p_i^l - 1)(p_i^l - p_i) \cdots (p_i^l - p_i^{l-1})$. It follows that there exists an automorphism $a$ of $N$ of order $p_j$, so that the non-trivial semi-direct product

$H = \langle a \rangle \ltimes N$ is not nilpotent (see, for example, [9, 5.2.4]). Let now $K$ be a group of order $n/p_i^l p_j$. Then the direct product $G = H \times K$ is a non-nilpotent group of order $n$. This contradiction shows that $(n, \psi(n)) = 1$.

Conversely, suppose that $(n, \psi(n)) = 1$ and for a contradiction, let $n$ be the least positive integer satisfying the latter condition which is not a nilpotent number. It follows that there exists a group $G$ which is minimal non-nilpotent. Suppose that every maximal subgroup of $G$ is self-normalizing. By Lemma 2 there exist distinct maximal subgroups $M_1$ and $M_2$ such that $X = M_1 \cap M_2$ is not trivial. Choose $M_1$ and $M_2$ so that $X$ has largest possible order. If $N_G(X) \neq G$, we may consider a maximal subgroup $M$ of $G$ such that $N_G(X) \leq M$ and $M \neq M_1$. Clearly, $X < N_G(X) \cap M_1$ since $M_1$ is nilpotent. It follows the contradiction $X < M \cap M_1$. Thus $G$ is not simple, and let $N_1$ be a proper non-trivial normal subgroup of $G$. Then $Z(N_1)$ contains a minimal $G$-invariant subgroup $N$ which is an elementary abelian $p_i$-group, for some $p_i \in \{p_1, \ldots, p_t\}$. Moreover, as the proper factors of $G$ are nilpotent too, then $N$ is the unique minimal normal subgroup of $G$. Let $\Phi(G)$ be the Frattini subgroup of $G$. By a result of Wielandt (see, for instance, [9, 5.2.16]) $G/\Phi(G)$ is not nilpotent. Thus $\Phi(G)$ is trivial, and hence there exists a maximal subgroup $M$ of $G$ such that $G = M \ltimes N$. Clearly, $C_M(N) = \{1\}$ by the uniqueness of $N$. It follows that $M$ is isomorphic to a subgroup of $AutN$, so that $p_j$ divides $p_i^l - 1$ for some $j \neq i$ and $l \in \{1, \ldots n_i\}$. This last contradiction completes the proof of (1).

**(2)** *Let $n = p_1^{n_1} \cdots p_t^{n_t}$, $p_i$ distinct primes, be a nilpotent number. Then $n$ is c-nilpotent if and only if $n$ is $(c+2)$-power free.*

Let $n$ be a $c$-nilpotent number, and assume for a contradiction that there exists a prime $p$ such that $p^{c+2}$ divides $n$. Let $P$ be a group of order $p^{c+2}$ and class $c+1$ (such a group is said to be of *maximal class* – see, for instance, [3, Kapitel III] for a general reference on this topic). If $Q$ is an abelian group, then the direct product $G = P \times Q$ is nilpotent with class $c+1$, a contradiction.

Conversely, suppose that $n$ is $(c+2)$-power free. Let $G$ be a finite group of order $n$. Since $G$ is nilpotent, then $G = G_{p_1} \times \cdots \times G_{p_t}$, where each $G_{p_i}$ is the unique $p_i$-Sylow subgroup of $G$. As the nilpotency class of $G_{p_i}$ is at most $n_i - 1$ (see, for instance, [9, 5.3.1]), then by hypothesis the nilpotency class of $G_{p_i}$ is at most $c$. It follows that $G$ is nilpotent with class at most $c$ as required. □

## REFERENCES

[1] L. E. DICKSON, *Definitions of a group and a field by independent postulates*, Trans. Math. Soc., **6** (1905), 198-204.

[2] J. A. GALLIAN - D. MOULTON, *When $\mathbb{Z}_n$ is the only group of order n?*, Elem. Math., **48** (1993), 117-119.

[3] B. HUPPERT, *Endliche Gruppen I*, 2nd edition, Springer, Berlin (1967).

[4] D. Jungnickel, *On the uniqueness of the cyclic group of order n*, Amer. Math. Monthly., **99** (1992), 545-547.
[5] T. W. Müller, *An arithmetic theorem related to groups of bounded nilpotency class*, J. Algebra, **300** (2006), 10-15.
[6] G. Pazderski, *Die Ordnungen, zu denen nur Gruppen mit gegebener Eigenschaft gehören*, Arch. Math., **10** (1959), 331-343.
[7] L. Rédei, *Das schiefe Produkt in der Gruppentheorie*, Comm. Math. Helv., **20** (1947), 225-264.
[8] L. Rédei, *Die endlichen einstufig nicht nilpotenten Gruppen*, Publ. Math. Debrecen, **4** (1956), 303-324.
[9] D. J. S. Robinson, *A course in the theory of groups*, 2nd edition, Springer, New York (1996).

Dipartimento di Matematica, Seconda Università di Napoli,
Via Vivaldi 43, I - 81100 Caserta (Italy).
E-mail: alessio.russo@unina2.it

*Received May 20, 2011 and in revised form August 8, 2011*