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Topologies on Hyperspaces[1]

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Abstract. – *Let Y and Z be two arbitrary fixed topological spaces, $C(Y,Z)$ the set of all continuous maps from Y to Z, and $\mathcal{O}_Z(Y)$ the set consisting of all open subsets V of Y such that $V = f^{-1}(U)$, where $f \in C(Y,Z)$ and U is an open subset of Z. In this paper we continue the study of the $\mathcal{A}$-proper and $\mathcal{A}$-admissible topologies on $\mathcal{O}_Z(Y)$, where $\mathcal{A}$ is an arbitrary family of spaces, initiated in* [6] *and we offer new results concerning the finest X-proper topology $\tau(\{X\})$ on $\mathcal{O}_Z(Y)$ for several metrizable spaces X.*

1. – Preliminaries

We denote by Y and Z two arbitrary fixed topological spaces and by $C(Y,Z)$ the set of all continuous maps of Y into Z. If t is a topology on the set $C(Y,Z)$, then the corresponding topological space is denoted by $C_t(Y,Z)$.

By $\mathcal{O}(Y)$ we denote the family of all open subsets of Y and by $\mathcal{O}_Z(Y)$ the set

$$\{f^{-1}(U) : f \in C(Y,Z) \text{ and } U \in \mathcal{O}(Z)\}.$$

Let X be a space, $F : X \times Y \to Z$ a continuous map, and $x \in X$. Let F_x be the map of Y into Z, defined by $F_x(y) = F(x,y)$ for every $y \in Y$ and $\widehat{F}$ the map of X into the set $C(Y,Z)$, defined by $\widehat{F}(x) = F_x$ for every $x \in X$.

Let G be a map of X into $C(Y,Z)$. We denote by $\widetilde{G}$ the map of $X \times Y$ into Z, defined by $\widetilde{G}(x,y) = G(x)(y)$ for every $(x,y) \in X \times Y$.

A topology t on $C(Y,Z)$ is called *proper* if for every space X, the continuity of a map $F : X \times Y \to Z$ implies that of the map $\widehat{F} : X \to C_t(Y,Z)$. A topology t on $C(Y,Z)$ is called *admissible* if for every space X, the continuity of a map $G : X \to C_t(Y,Z)$ implies that of the map $\widetilde{G} : X \times Y \to Z$ (see [1], [2], [4], and [7]).

If in the above definitions the space X is assumed to belongs to a family $\mathcal{A}$ of spaces, then the topology τ is called $\mathcal{A}$-*proper* (respectively, $\mathcal{A}$-*admissible*) (see [5]). For $\mathcal{A} = \{X\}$, we write X-proper and X-admissible instead of $\mathcal{A}$-proper and $\mathcal{A}$-admissible, respectively.

Let $\mathbb{H} \subseteq \mathcal{O}_Z(Y)$, $\mathcal{H} \subseteq C(Y,Z)$, and $U \in \mathcal{O}(Z)$. We set

$$(\mathbb{H}, U) = \{f \in C(Y,Z) : f^{-1}(U) \in \mathbb{H}\}$$

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and

$$(\mathcal{H}, U) = \{f^{-1}(U) : f \in \mathcal{H}\}.$$

Let τ be a topology on $\mathcal{O}_Z(Y)$. The $t(\tau)$ topology on $C(Y,Z)$ is that having as subbasis all sets $(\mathbb{H}, U)$, where $\mathbb{H} \in \tau$ and $U \in \mathcal{O}(Z)$. The topology $t(\tau)$ is called *dual to* τ (see [6]).

Let t be a topology on $C(Y,Z)$. The $\tau(t)$ topology on $\mathcal{O}_Z(Y)$ is that having as subbasis all sets $(\mathcal{H}, U)$, where $\mathcal{H} \in t$ and $U \in \mathcal{O}(Z)\}$. The topology $\tau(t)$ is called *dual to* t (see [6]).

Let τ be a topology on $\mathcal{O}_Z(Y)$ and t a topology on $C(Y,Z)$. If $\tau = \tau(t)$ and $t = t(\tau)$, then the pair (τ, t) is called *a pair of mutually dual topologies* (see [6]).

Let X be a space and $F : X \times Y \to Z$ a continuous map. We denote by $\overline{F}$ the map of $X \times \mathcal{O}(Z)$ into $\mathcal{O}_Z(Y)$, defined by $\overline{F}(x, U) = F_x^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.

Let X be a space and $G : X \to C(Y,Z)$ a map. We denote by $\overline{G}$ the map of $X \times \mathcal{O}(Z)$ into $\mathcal{O}_Z(Y)$, defined by $\overline{G}(x, U) = (G(x))^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.

Let τ be a topology on $\mathcal{O}_Z(Y)$. We say that a map M of $X \times \mathcal{O}(Z)$ into $\mathcal{O}_Z(Y)$ is *continuous with respect to the first variable* if for every fixed element U of $\mathcal{O}(Z)$, the map $M_U : X \to (\mathcal{O}_Z(Y), \tau)$, defined by $M_U(x) = M(x, U)$ for every $x \in X$, is continuous. We denote by $CF(X \times \mathcal{O}(Z), \mathcal{O}_Z(Y))$ the set of all continuous maps with respect to the first variable from the set $X \times \mathcal{O}(Z)$ to $\mathcal{O}_Z(Y)$.

DEFINITION 1 (see [6]). – *A topology τ on $\mathcal{O}_Z(Y)$ is called $\mathcal{A}$-proper if for every space $X \in \mathcal{A}$ the continuity of a map $F : X \times Y \to Z$ implies the continuity with respect to the first variable of the map $\overline{F} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$. For $\mathcal{A} = \{X\}$, we write X-proper instead of $\mathcal{A}$-proper.*

In the set $\mathcal{O}_Z(Y)$ there exists the finest $\mathcal{A}$-proper topology which is denoted by $\tau(\mathcal{A})$ (see [6]).

DEFINITION 2 (see [6]). – *A topology τ on $\mathcal{O}_Z(Y)$ is called $\mathcal{A}$-admissible if for every space $X \in \mathcal{A}$ and for every map $G : X \to C(Y,Z)$ the continuity with respect to the first variable of the map $\overline{G} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$ implies the continuity of the map $\widetilde{G} : X \times Y \to Z$. For $\mathcal{A} = \{X\}$, we write X-admissible instead of $\mathcal{A}$-admissible.*

If $\mathcal{A}$ is the family of all spaces, then the $\mathcal{A}$-proper (respectively, $\mathcal{A}$-admissible) topology τ on $\mathcal{O}_Z(Y)$ is called *proper* (respectively, *admissible*).

In this paper we continue the study of the $\mathcal{A}$-proper and $\mathcal{A}$-admissible topologies on $\mathcal{O}_Z(Y)$, where $\mathcal{A}$ is an arbitrary family of spaces, initiated in [6] and we offer new results concerning the finest X-proper topology $\tau(\{X\})$ on $\mathcal{O}_Z(Y)$ for several metrizable spaces X.

2. – On $\mathcal{A}$-proper and $\mathcal{A}$-admissible topologies

In this section we describe some properties of $\mathcal{A}$-proper and $\mathcal{A}$-admissible topologies on $\mathcal{O}_Z(Y)$, where $\mathcal{A}$ is an arbitrary family of spaces.

THEOREM 2.1. – *Let $\mathcal{A}$ be an arbitrary family of spaces, τ a topology on $\mathcal{O}_Z(Y)$, and e the map from $\mathcal{O}_Z(Y) \times Y$ to Z, defined by $e(f^{-1}(U), y) = f(y)$ for every $y \in Y$ and $f^{-1}(U) \in \mathcal{O}_Z(Y)$. If the map e is continuous, then the topology τ is $\mathcal{A}$-admissible.*

PROOF. – Let $X \in \mathcal{A}$ and $G : X \to C(Y, Z)$ be a continuous map such that the corresponding

$$\overline{G} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$$

is continuous with respect to the first variable. For every $U \in \mathcal{O}(Z)$ the map $\overline{G}_U : X \to (\mathcal{O}_Z(Y), \tau)$ is continuous. Also, the identity map $id : Y \to Y$ is continuous. Thus, the map

$$\overline{G}_U \times id : X \times Y \to \mathcal{O}_Z(Y) \times Y$$

is continuous for every $U \in \mathcal{O}(Z)$ and, therefore, the map

$$e \circ (\overline{G}_U \times id) : X \times Y \to Z$$

is continuous for every $U \in \mathcal{O}(Z)$. We observe that

$$\begin{aligned} e \circ (\overline{G}_U \times id)(x, y) &= e((\overline{G}_U \times id)(x, y)) = e(\overline{G}_U(x), id(y)) \\ &= e(G(x)^{-1}(U), y) = G(x)(y) = \widetilde{G}(x, y), \end{aligned}$$

for every $(x, y) \in X \times Y$. Thus, the map $\widetilde{G}$ is continuous and, therefore, the topology τ is $\mathcal{A}$-admissible.

THEOREM 2.2. – *Let τ be a topology on $\mathcal{O}_Z(Y)$.*

(1) *If τ is larger than an $\mathcal{A}$-admissible topology, then τ is also $\mathcal{A}$-admissible.*
(2) *If τ is smaller than an $\mathcal{A}$-proper topology , then τ is also $\mathcal{A}$-proper.*

PROOF. – (1) Let τ' be an $\mathcal{A}$-admissible topology on $\mathcal{O}_Z(Y)$ such that $\tau' \subseteq \tau$. By Lemma 4.2 of [6] we have $t(\tau') \subseteq t(\tau)$. Also, by Theorem 3.9 of [6], the topology $t(\tau')$ is $\mathcal{A}$-admissible. Since $t(\tau') \subseteq t(\tau)$, the topology $t(\tau)$ is $\mathcal{A}$-admissible (see [5]). Therefore, by Theorem 3.9 of [6], the topology τ is $\mathcal{A}$-admissible.

(2) Let τ' be an $\mathcal{A}$-proper topology and τ be a topology on $\mathcal{O}_Z(Y)$ such that $\tau \subseteq \tau'$. By Lemma 4.2 of [6] we have $t(\tau) \subseteq t(\tau')$. Also, by Theorem 3.5 of [6], the topology $t(\tau')$ is $\mathcal{A}$-proper. Since $t(\tau) \subseteq t(\tau')$, the topology $t(\tau)$ is $\mathcal{A}$-proper (see [5]). Therefore, by Theorem 3.5 of [6], the topology τ is $\mathcal{A}$-proper.

COROLLARY 2.3. – *Let τ be a topology on $\mathcal{O}_Z(Y)$.*

(1) *If τ is larger than an admissible topology, then τ is also admissible.*
(2) *If τ is smaller than a proper topology, then τ is also proper.*

THEOREM 2.4. – *Let (t, τ) and (t_1, τ_1) two pairs of mutual dual topologies such that $C_{t(\tau)}(Y, Z) \in \mathcal{A}$. If τ is $\mathcal{A}$-admissible and τ_1 $\mathcal{A}$-proper topology, then $\tau_1 \subseteq \tau$.*

PROOF. – By Theorems 3.5 and 3.9 of [6] the topologies $t(\tau) = t$ and $t(\tau_1) = t_1$ are $\mathcal{A}$-admissible and $\mathcal{A}$-proper, respectively. Since $C_{t(\tau)}(Y, Z) \in \mathcal{A}$ we have $t(\tau_1) = t_1 \subseteq t(\tau) = t$. Indeed, let

$$G : C_t(Y, Z) \to C_t(Y, Z)$$

be the identity map. Since t is an $\mathcal{A}$-admissible topology on $C(Y, Z)$ and $C_t(Y, Z) \in \mathcal{A}$ we have that the map

$$\widetilde{G} : C_t(Y, Z) \times Y \to Z$$

is continuous. Also, since the topology t_1 is $\mathcal{A}$-proper, the map

$$\widehat{\widetilde{G}} : C_t(Y, Z) \to C_{t_1}(Y, Z)$$

is continuous. We observe that $\widehat{\widetilde{G}}(f) = f$ for every $f \in C(Y, Z)$. Thus, $t_1 \subseteq t$. Now, by Lemma 4.3 of [6],

$$\tau(t_1) = \tau_1 \subseteq \tau(t) = \tau.$$

COROLLARY 2.5. – *Let (t, τ) and (t_1, τ_1) two pairs of mutual dual topologies. If the topologies τ and τ_1 on $\mathcal{O}_Z(Y)$ are admissible and proper, respectively, then $\tau_1 \subseteq \tau$.*

THEOREM 2.6. – *Let $\mathcal{A}_i$, $i \in I$, be a family of spaces. Then, the following propositions are true:*

(1) *If $\mathcal{A} = \bigcup\{\mathcal{A}_i : i \in I\}$, then*

$$\tau(\mathcal{A}) = \bigcap\{\tau(\mathcal{A}_i) : i \in I\}.$$

(2) *If $\mathcal{A} = \bigcap\{\mathcal{A}_i : i \in I\} \neq \emptyset$, then*

$$\vee\{\tau(\mathcal{A}_i) : i \in I\} \subseteq \tau(\mathcal{A}).$$

(3)

$$\tau(\mathcal{A}) = \bigcap\{\tau(\{X\}) : X \in \mathcal{A}\}.$$

PROOF. – (1) Since $\mathcal{A} = \bigcup\{\mathcal{A}_i : i \in I\}$ we have that every topology which is $\mathcal{A}$-proper is also $\mathcal{A}_i$-proper, for every $i \in I$. Thus, the finest $\mathcal{A}$-proper topology

$\tau(\mathcal{A})$ is $\mathcal{A}_i$-proper and, therefore,

$$\tau(\mathcal{A}) \subseteq \tau(\mathcal{A}_i),$$

for every $i \in I$. So, we have

$$\tau(\mathcal{A}) \subseteq \bigcap\{\tau(\mathcal{A}_i) : i \in I\}.$$

For the converse relation it suffices to prove that the topology

$$\bigcap\{\tau(\mathcal{A}_i) : i \in I\}$$

is $\mathcal{A}$-proper. Let $X \in \mathcal{A}$ and let $F : X \times Y \to Z$ be a continuous map. We prove that the map

$$\overline{F} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \bigcap\{\tau(\mathcal{A}_i) : i \in I\})$$

is continuous with respect to the first variable. Since $X \in \mathcal{A}$, there exists $i \in I$ such that $X \in \mathcal{A}_i$. This means that the map

$$\overline{F} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau(\mathcal{A}_i))$$

is continuous with respect to the first variable. Since

$$\bigcap\{\tau(\mathcal{A}_i) : i \in I\} \subseteq \tau(\mathcal{A}_i),$$

the identity map

$$id : (\mathcal{O}_Z(Y), \tau(\mathcal{A}_i)) \to (\mathcal{O}_Z(Y), \bigcap\{\tau(\mathcal{A}_i) : i \in I\})$$

is continuous. Clearly, by the above fact, the map $\overline{F}$ is continuous with respect to the first variable. Thus, the topology

$$\bigcap\{\tau(\mathcal{A}_i) : i \in I\}$$

is $\mathcal{A}$-proper.

(2) The proof of this follows by the fact that the topology

$$\vee\{\tau(\mathcal{A}_i) : i \in I\}$$

is $\mathcal{A}$-proper.

(3) It follows from (1).

DEFINITION 3. – *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two classes of spaces. We say that these families are equivalent and write $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if:*

(α) *a topology τ on $\mathcal{O}_Z(Y)$ is $\mathcal{A}_1$–proper if and only if τ is $\mathcal{A}_2$–proper and*

(β) *a topology τ on $\mathcal{O}_Z(Y)$ is $\mathcal{A}_1$–admissible if and only if τ is $\mathcal{A}_2$–admissible.*

THEOREM 2.7. – *Let $\mathcal{A}$ be a family of spaces. Then, there exists a space $X(\mathcal{A})$ such that*

$$\mathcal{A} \sim \{X(\mathcal{A})\}$$

PROOF. – Let T_p^c be the set of all topologies on $\mathcal{O}_Z(Y)$ which are not $\mathcal{A}$-proper and let T_{ad}^c be the set of all topologies on $\mathcal{O}_Z(Y)$ which are not $\mathcal{A}$-admissible. For each topology $\tau \in T_p^c$ there exists in $\mathcal{A}$ a space X_τ^p such that τ is not X_τ^p-proper. Similarly, for each $\tau \in T_{ad}^c$ there exists in $\mathcal{A}$ a space X_τ^a such that τ is not X_τ^a-admissible. Let

$$\mathcal{A}_0 = \{X_\tau^p : \tau \in T_p^c\} \cup \{X_\tau^a : \tau \in T_{ad}^c\}.$$

We can suppose that the spaces in $\mathcal{A}_0$ are pairwise disjoint. Let $X(\mathcal{A})$ be the free union of all spaces in $\mathcal{A}_0$. We prove that

$$\mathcal{A} \sim \{X(\mathcal{A})\}$$

Let τ be an $\mathcal{A}$-proper topology on $\mathcal{O}_Z(Y)$, $F : X(\mathcal{A}) \times Y \to Z$ a continuous map, and $X \in \mathcal{A}$. We prove that the topology τ is $X(\mathcal{A})$-proper. In order to show that the map

$$\overline{F} : X(\mathcal{A}) \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$$

is continuous with respect to the first variable, let F_X be the restriction of F on $X \times Y \subseteq X(\mathcal{A}) \times Y$. By continuity of F_X, it follows that

$$\overline{F}_X : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$$

is continuous with respect to the first variable. Since $X(\mathcal{A})$ is a free union of $X \in \mathcal{A}_0$, we have that

$$\overline{F} : X(\mathcal{A}) \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$$

is continuous with respect to the first variable. Thus, the topology τ on $\mathcal{O}(Z)$ is $X(\mathcal{A})$-proper.

Now, let τ be a $X(\mathcal{A})$-proper topology on $\mathcal{O}_Z(Y)$. We assume that τ is not an $\mathcal{A}$-proper topology. Then $\tau \in T_p^c$ and so τ is not X_τ^p-proper. Thus there exists a continuous map

$$F_{X_\tau^p} : X_\tau^p \times Y \to Z$$

such that the map

$$\overline{F}_{X_\tau^p} : X_\tau^p \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$$

is not continuous with respect to the first variable. The map $F_{X_\tau^p}$ can be extended to a continuous map $F : X(\mathcal{A}) \times Y \to Z$. Since the restriction of $\overline{F}$ to $X_\tau^p \times \mathcal{O}(Z)$ is not continuous, it follows that $\overline{F}$ also is not continuous, which contradicts our assumption that τ is a $X(\mathcal{A})$-proper topology.

In a similar way we can prove that a topology τ on $\mathcal{O}_Z(Y)$ is $\mathcal{A}$-admissible if and only if τ is $X(\mathcal{A})$-admissible.

COROLLARY 2.8. – *There exists a space X such that:*

(α) *a topology on $\mathcal{O}_Z(Y)$ is proper if and only if this topology is $\{X\}$–proper and*

(β) *a topology on $\mathcal{O}_Z(Y)$ is admissible if and only if this topology is $\{X\}$-admissible.*

DEFINITION 4. – *Let τ be a proper topology on $\mathcal{O}_Z(Y)$. The exponential function*

$$E^\tau : C(X \times Y, Z) \to CF(X \times \mathcal{O}(Z), \mathcal{O}_Z(Y))$$

is defined by $E^\tau(F) = \overline{F}$, for every $F \in C(X \times Y, Z)$.

We note that since τ is proper this function is well defined.
It is easy to verify the following theorem:

THEOREM 2.9. – *If for every space X the mapping E^τ is onto, then τ is an admissible topology.*

THEOREM 2.10. – *The following propositions are true:*

(1) *A topology τ on $\mathcal{O}_Z(Y)$ is proper if and only if it is $\mathcal{A}$-proper, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.*

(2) *A topology τ on $\mathcal{O}_Z(Y)$ is admissible if and only if is $\mathcal{A}$-admissible, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.*

PROOF. – To prove (1), let τ be a proper topology on $\mathcal{O}_Z(Y)$. By Theorem 3.9 of [6] the topology τ on $\mathcal{O}_Z(Y)$ is proper if and only if the topology $t(\tau)$ on $C(Y, Z)$ is proper. Also, by Theorem II.2 of [5], $t(\tau)$ is proper if and only $t(\tau)$ is $\mathcal{A}$-proper, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point. Thus, the topology τ on $\mathcal{O}_Z(Y)$ is proper if and only if it is $\mathcal{A}$-proper, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.
In a similar way (2) can be shown.

3. – The finest X-proper topology on $\mathcal{O}_Z(Y)$

In this section we study the finest X-proper topology on $\mathcal{O}_Z(Y)$ for several metrizable spaces X.

THEOREM 3.1. – *Let H be a quotient map (see* [3], *page* 125*) of a space X_1 onto a space X_2. Then, we have*

$$\tau(\{X_1\}) \subseteq \tau(\{X_2\}).$$

PROOF. – We prove that the topology $\tau(\{X_1\})$ on $\mathcal{O}_Z(Y)$ is X_2-proper. Let $F : X_2 \times Y \to Z$ be a continuous map. We prove that the corresponding map

$$\overline{F} : X_2 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau(\{X_1\}))$$

is continuous with respect to the first variable.

Let $U \in \mathcal{O}(Z)$. We prove that the map

$$\overline{F}_U : X_2 \to (\mathcal{O}_Z(Y), \tau(\{X_1\}))$$

is continuous.

We consider the map $F^1 : X_1 \times Y \to Z$ defined by

$$F^1(x, y) = F(H(x), y),$$

for every $(x, y) \in X_1 \times Y$. The map F^1 is continuous. Indeed, let

$$F^1(x, y) = F(H(x), y) = z \in Z$$

and U_z be an open neighborhood of z in Z. Since F is continuous at the point $(H(x), y) \in X_2 \times Y$, there are open neighborhoods $U_{H(x)}$ and U_y of $H(x)$ and y, respectively such that

$$F(U_{H(x)} \times U_y) \subseteq U_z.$$

Since $H : X_1 \to X_2$ is a quotient map, we have that H is continuous. Thus, there exists an open neighborhood U_x of x in X_1 such that $H(U_x) \subseteq U_{H(x)}$.

For the open neighborhood $U_x \times U_y$ of (x, y) in $X_1 \times Y$ we have

$$F^1(U_x \times U_y) = F(H(U_x) \times U_y) \subseteq F(U_{H(x)} \times U_y) \subseteq U_z.$$

Thus, the map F^1 is continuous.

Since the topology $\tau(\{X_1\})$ is X_1-proper, the map

$$\overline{F^1} : X_1 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau(\{X_1\}))$$

is continuous with respect to the first variable. Thus, for every $U \in \mathcal{O}(Z)$ the map

$$\overline{F^1}_U : X_1 \to (\mathcal{O}_Z(Y), \tau(\{X_1\}))$$

is continuous.

Let $U \in \mathcal{O}(Z)$. Then for every $x \in X_1$ we have

$$(\overline{F}_U \circ H)(x) = \overline{F}_U(H(x) = \overline{F}(H(x), U) = \overline{F^1}_U(x).$$

So we have $\overline{F^1}_U = \overline{F}_U \circ H$, for every $U \in \mathcal{O}(Z)$.

Since the map H is quotient, $\overline{F^1}_U$ is continuous, and $\overline{F^1}_U = \overline{F}_U \circ H$, we have that the map $\overline{F}_U$ is continuous. Thus, the topology $\tau(\{X_1\})$ is X_2-proper and, therefore,

$$\tau(\{X_1\}) \subseteq \tau(\{X_2\}).$$

COROLLARY 3.2. – *Let X_1 and X_2 be two topological spaces. If there exists a quotient map H_1 of X_1 onto X_2 and a quotient map H_2 of X_2 onto X_1, then*

$$\tau(\{X_1\}) = \tau(\{X_2\}).$$

COROLLARY 3.3. – *Let X be a connected locally connected compact metrizable infinite space X. Then, we have*

$$\tau(\{X\}) = \tau(\{[0,1]\}).$$

COROLLARY 3.4. – *Let C be the Cantor set. Then, we have*

$$\tau(\{C\}) \subseteq \tau(\{[0,1]\}),$$

COROLLARY 3.5. – *Let X be a sequential space (see* [3], *page* 134*) and β the family of all sequences $x_0, x_1, x_2, \ldots$ of points of X such that $x_0 \in \lim x_i$. For every $c = \{x_i\} \in \beta$ let $X_c = \{c\} \times \{0, 1, \frac{1}{2}, \ldots\}$, where $\{c\}$ is the one-point discrete space and $\{0, \frac{1}{2}, \ldots\}$ has the topology of subspace of R, where R is the set of all real numbers with the usual topology. Then, we have*

$$\tau(\{\oplus_{c\in\beta} X_c\}) \subseteq \tau(\{X\}).$$

PROOF. – Let $f_c : X_c \to X$ be the map definened by

$$f_c((c,0)) = x_0 \text{ and } f_c((c,\frac{1}{i})) = x_i, \text{ for every } i = 1, 2, \ldots.$$

The map

$$f_X = \bigtriangledown_{c\in\beta} f_c : \oplus_{c\in\beta} X_c \to X$$

(see [3], page 134) is a quotient map. Thus, by Theorem 3.1, we have

$$\tau(\{\oplus_{c\in\beta} X_c\}) \subseteq \tau(\{X\}).$$

THEOREM 3.6. – *Let X be a locally compact space and $f : Y \to Z$ a quotient map. Then, we have*

$$\tau(\{X \times Y\}) \subseteq \tau(\{X \times Z\}).$$

PROOF. – The map

$$id_X \times f : X \times Y \to X \times Z,$$

where $id_X : X \to X$ is the identity map, is a quotient map (see [3], page 200). Thus, by Theorem 3.1, we have

$$\tau(\{X \times Y\}) \subseteq \tau(\{X \times Z\}).$$

DEFINITION 5. – *Let R be the set of real numbers with the usual topology. The subspace of R^{n+1}, where n is a positive integer, consisting of all points $(x_1, x_2, ..., x_{n+1})$ such that $x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1$ is called the unit n-sphere and is denoted by S^n. The 1-sphere is a circle and the cartesian product $S^1 \times S^1$ is a torus.*

COROLLARY 3.7. – *Let X be an arbitrary discrete space. Then, we have*

(1) $\tau(\{X \times R\}) \subseteq \tau(\{X \times S^1\})$.
(2) $\tau(\{X \times [0,1]\}) \subseteq \tau(\{X \times S^1\})$.
(3) $\tau(\{X \times [0,1] \times [0,1]\}) \subseteq \tau(\{X \times S^1 \times S^1\})$.

PROOF. – (1) Let $f : R \to S^1$ be a map defined by

$$f(x) = (\cos 2\pi x, \sin 2\pi x),$$

for every $x \in R$. The map f is a quotient map (see [3], page 127). Also, the space X is locally compact. Thus, by Theorem 3.6,

$$\tau(\{X \times R\}) \subseteq \tau(\{X \times S^1\}).$$

(2) We consider the map $f : R \to S$ of (1). Then, the map

$$g = f|_{[0,1]} : [0,1] \to S^1$$

is a quotient map (see [3], page 127). Since the space X is locally compact, by Theorem 3.6, we have that

$$\tau(\{X \times [0,1]\}) \subseteq \tau(\{X \times S^1\}).$$

(3) We consider the map $g = f|_{[0,1]} : [0,1] \to S^1$ of (2). The map

$$g \times g : [0,1] \times [0,1] \to S^1 \times S^1$$

is a quotient map (see [3], page 127)). Thus, by Theorem 3.6,

$$\tau(\{X \times [0,1] \times [0,1]\}) \subseteq \tau(\{X \times S^1 \times S^1\}).$$

In a similar way the following corollary can be shown.

COROLLARY 3.8. – *The following relations are true:*

(1) $\tau(\{R^{n+1}\}) \subseteq \tau(\{R^n \times S^1\})$, *and*
(2) $\tau(\{R^n \times [0,1] \times [0,1]\}) \subseteq \tau(\{R^n \times S^1 \times S^1\})$.

THEOREM 3.9. – *Let X be a Hausdorff space, $C(X)$ the family of all non-empty compact subspaces of X, and $X^* = \oplus_{K \in C(X)} K$ (see* [3]*). If a set $A \subseteq X$ is closed provided that the intersections $A \cap K$ are closed in K for all $K \in C(X)$, then*

$$\tau(\{X^*\}) \subseteq \tau(\{X\}).$$

PROOF. – We consider the map

$$f = \nabla_{K \in C(X)} i_K : X^* \to X,$$

where i_K is the embedding of the space K into the space X (see [3], page 201). The map f is a quotient map. Thus, by Theorem 3.1, we have

$$\tau(\{X^*\}) \subseteq \tau(\{X\}).$$

THEOREM 3.10. – *Let $f_i : X_i \to Y_i$ be quotient maps for $i = 1, 2$ and X_1, $X_1 \times Y_2$ k-spaces. Then, we have*

$$\tau(\{X_1 \times X_2\}) \subseteq \tau(\{Y_1 \times Y_2\}).$$

PROOF. – We consider the map

$$f = f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2.$$

By [3] (Theorem 3.3.28, page 203) f is a quotient map. Thus, by Theorem 3.1, we have

$$\tau(\{X_1 \times X_2\}) \subseteq \tau(\{Y_1 \times Y_2\}).$$

THEOREM 3.11. – *Let $f_i : X_i \to Y_i$ be quotient maps for $i = 1, 2$, X_1 a locally compact space, and Y_2 a k-space. Then, we have*

(1) $\tau(\{X_1 \times X_2\}) \subseteq \tau(\{X_1 \times Y_2\})$.
(2) $\tau(\{X_1 \times X_1\}) \subseteq \tau(\{X_1 \times Y_1\})$.
(3) $\tau(\{X_1 \times Y_2\}) \subseteq \tau(\{Y_1 \times Y_2\})$.
(4) $\tau(\{X_1 \times X_2\}) \subseteq \tau(\{Y_1 \times Y_2\})$.

PROOF. – (1) We consider the map

$$id_{X_1} \times f_2 : X_1 \times X_2 \to X_1 \times Y_2,$$

where $id_{X_1} : X_1 \to X_1$ be the identity map. Then, by Theorem 3.3.17 of [3], the map $id_{X_1} \times f_2$ is a quotien map. Thus, by Theorem 3.1, we have

$$\tau(\{X_1 \times X_2\}) \subseteq \tau(\{X_1 \times Y_2\}).$$

(2) In a similar way (2) can be shown.
(3) We consider the map

$$f_1 \times id_{Y_2} : X_1 \times Y_2 \to Y_1 \times Y_2,$$

where $id_{Y_2} : Y_2 \to Y_2$ be the identity map. Then, $f_1 \times id_{Y_2}$ is a quotien map (see [3], page 204). Thus, by Theorem 3.1, we have

$$\tau(\{X_1 \times Y_2\}) \subseteq \tau(\{Y_1 \times Y_2\}).$$

(4) We consider the map

$$f = (f_1 \times id_{Y_2}) \circ (id_{X_1} \times f_2).$$

Since the maps $f_1 \times id_{Y_2}$ and $id_{X_1} \times f_2$ are quotient, the map f is quotient (see [3]). Thus, by Theorem 3.1, the relation (4) of the theorem is true.

DEFINITION 6 (see [3], page 178). – *A space X is called a sequential space if a set $A \subseteq X$ is closed if and only if together with any sequence it contains all limits.*

THEOREM 3.12. – *Let X be a sequential space and let Seq the subspace of the real line (with the usual topology) consisting of the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Then,*

$$\tau(\{Seq\}) \subseteq \tau(\{X\}).$$

PROOF. – We prove that the topology $\tau(\{Seq\})$ on $\mathcal{O}_Z(Y)$ is X-proper.

Let $F : X \times Y \to Z$ be a continuous map. We prove that the map

$$\overline{F} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau(\{Seq\}))$$

is continuous with respect to the first variable. Let $U \in \mathcal{O}(Z)$.

Since the space X is sequential, the map

$$\overline{F}_U : X \to (\mathcal{O}_Z(Y), \tau(\{Seq\}))$$

is continuous if and only if for every net $\phi : N \to X$, where N is the set of all positive integers, we have

$$\overline{F}_U(\lim(\phi(i))) \subseteq \lim \overline{F}_U(\phi(i)).$$

(see [3], Proposition 1.6.15).

Let $\phi : N \to X$ be a net in X and $x \in \lim \phi(i)$. We prove that

$$\overline{F}_U(x) \in \lim \overline{F}_U(\phi(i)).$$

Let

$$\phi_{Seq} : Seq \to X,$$

be the map defined by $\phi_{Seq}(i/i) = \phi(i)$, for every $i = 1, 2, \ldots$, and $\phi_{Seq}(0) = x$. By Proposition 1.6.6 of [3] and by the fact that X is a sequential space, we have that the map ϕ_{Seq} is continuous.

Let

$$F_{Seq} : Seq \times Y \to Z$$

be the map defined by

$$F_{Seq}(x_1, y_1) = F(\phi_{Seq}(x_1), y_1),$$

for every $(x_1, y_1) \in Seq \times Y$. Since the maps F and ϕ_{Seq} are continuous, the map F_{Seq} is also continuous.

Since the topology $\tau(\{Seq\})$ is Seq-proper we have that the map

$$\overline{F_{Seq}} : Seq \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau(\{Seq\})$$

is continuous with respect to the first variable.

Thus, for every $U \in \mathcal{O}(Z)$, the map

$$\overline{F_{Seq}}_U : Seq \to (\mathcal{O}_Z(Y), \tau(\{Seq\})$$

is continuous. For every $x_1 \in Seq$, we have

$$(\overline{F}_U \circ \phi_{Seq})(x_1) = \overline{F}_U(\phi_{Seq}(x_1)) = \overline{F}(\phi_{Seq}(x_1), U) = \overline{F_{Seq}}_U(x_1).$$

Thus,

$$\overline{F_{Seq}}_U = \overline{F}_U \circ \phi_{Seq}.$$

So, we have

$$\begin{aligned}
\overline{F_{Seq}}_U(0) &= (\overline{F}_U \circ \phi_{Seq})(0) = \overline{F}_U(\phi_{Seq}(0)) \\
&= \overline{F}_U(x) \in \lim \overline{F_{Seq}}_U(1/i) = \lim\,(\overline{F}_U \circ \phi_{Seq})(1/i) \\
&= \lim \overline{F}_U(\phi_{Seq}(1/i)) = \lim \overline{F}_U(\phi(i))
\end{aligned}$$

and, therefore,

$$\overline{F}_U(\lim\,(\phi(i))) \subseteq \lim \overline{F}_U(\phi(i)).$$

This means that the map

$$\overline{F}_U : X \to (\mathcal{O}_Z(Y), \tau(\{Seq\}))$$

is continuous. Thus, the topology $\tau(\{Seq\})$ on $\mathcal{O}_Z(Y)$ is X-proper and

$$\tau(\{Seq\} \subseteq \tau(\{X\}).$$

COROLLARY 3.13. – *Let X be a compact metrizable space having infinitely components. Then, we have then*

$$\tau(\{Seq\}) \subseteq \tau(\{X\}).$$

COROLLARY 3.14. – *Let C be the Cantor set. Then, we have*

$$\tau(\{Seq\}) \subseteq \tau(\{C\}).$$

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