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# Functional Solutions for Fluid Flows Through Porous Media

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**Abstract.** – *The Levy-Caccioppoli global inversion theorem is applied to prove the existence and uniqueness of functional solutions for a problem of flow of a viscous incompressible fluid in a porous medium when the viscosity and the thermal conductivity depend on the temperature. A method based on the Abel integral equation, for determining the dependence of the viscosity from the temperature is also proposed.*

## 1. – Introduction

A porous homogeneous body is represented in $\boldsymbol{R}^3$ by the open and bounded subset $\Omega$; the body is filled with a viscous and incompressible fluid obeying the Darcy's law [9]

$$\boldsymbol{v} = -\frac{K}{\mu}\nabla p, \tag{1.1}$$

where $\boldsymbol{v}$ is the local average fluid velocity, $p$ the pressure, $K > 0$ the constant permeability and $\mu$ the viscosity which we suppose to be a given function of the temperature $u$. The regular boundary $\Gamma$ of $\Omega$ consists of three parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_0$. Between $\Gamma_1$ and $\Gamma_2$ a constant difference of pressure $P$ is maintained. Moreover, a constant temperature which in a suitable empirical scale we assume to be zero, is kept on $\Gamma_1$ and $\Gamma_2$. Besides, $\Gamma_0$ is thermally insulated. The condition of incompressibility implies

$$\nabla \cdot \left(\frac{\nabla p}{\mu(u)}\right) = 0 \text{ in } \Omega. \tag{1.2}$$

From the law of Fourier and the energy equation we have

$$\nabla \cdot (\tau(u)\nabla u) + \rho\alpha\frac{K}{\mu(u)}\nabla p \cdot \nabla u + \frac{\rho K}{\mu(u)}|\nabla p|^2 = 0 \text{ in } \Omega. \tag{1.3}$$

The third term in the left hand side of (1.3) reflects the energy dissipation in the fluid [3] whereas the second term on the left is related to the convective phenomena. $\rho$ is the (constant) mass density, $\alpha$ (also a constant) denotes the heat capacity at constant volume and $\tau$ the thermal conductivity, a given function of the temperature as $\mu(u)$. We set

$$\kappa(u) = \tau(u)(\rho K)^{-1}.$$

Taking into account (1.2) we have

$$\nabla \cdot \left[ p \frac{\nabla p}{\mu(u)} \right] = \frac{|\nabla p|^2}{\mu(u)}.$$

Thus we arrive, under stationary conditions, at the following boundary problem $(P)$ for the determination of $p(\boldsymbol{x})$ and $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3) \in \Omega$

$$\nabla \cdot \left[ \frac{\nabla p}{\mu(u)} \right] = 0 \text{ in } \Omega \tag{1.4}$$

$$\nabla \cdot \left[ \frac{1}{\mu(u)} (\eta(u) \nabla u + \alpha u \nabla p + p \nabla p) \right] = 0 \text{ in } \Omega \tag{1.5}$$

$$p = 0 \text{ on } \Gamma_1,\ p = P \text{ on } \Gamma_2,\ \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma_0 \tag{1.6}$$

$$u = 0 \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_2,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0, \tag{1.7}$$

where

$$\eta(u) = \kappa(u)\mu(u).$$

In the next section we study the functional solutions of problem $(P)$ according to the following:

DEFINITION 1.1. – *We say that a classical solution* $(p(\boldsymbol{x}), u(\boldsymbol{x}))$ *of problem* $(P)$ *is a "functional solution" if a function* $\mathcal{U}(p) \in C^2([0, P])$ *exists such that*

$$u(\boldsymbol{x}) = \mathcal{U}(p(\boldsymbol{x})).$$

Under quite general assumptions on the data we prove that a functional solution exists and is unique. Moreover, if $Q$ is the total mass of the fluid crossing $\Gamma_2$ in the unit time i.e.

$$Q = \rho \int_{\Gamma_2} \boldsymbol{v} \cdot \boldsymbol{n} \, d\Gamma,$$

where $\boldsymbol{n}$ is the unit vector normal to $\Gamma_2$, we show that the quantity

$$f(P) = \frac{Q(P)}{k\rho} \tag{1.8}$$

does not depend on $\Omega$. In (1.8) the constant $k$ depends only on the geometry of $\Omega$

and is given by

$$k = \int_{\Gamma_2} \frac{\partial z}{\partial n} d\Gamma, \tag{1.9}$$

where $z(\boldsymbol{x})$ is the solution of the problem

$$\Delta z = 0 \text{ in } \Omega,\ z = 0 \text{ on } \Gamma_1,\ z = 1 \text{ on } \Gamma_2,\ \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_0. \tag{1.10}$$

In the third section we discuss the following:

PROBLEM. – Suppose $f(P)$ and $\eta(u) = \mu(u)\kappa(u)$ are known, calculate $\mu(u)$.
We prove that this problem can be solved with the help of a generalized Abel integral equation.

## 2. – Functional solutions

THEOREM 2.1. – *Assume*

$$\mu(u) > 0,\ \kappa(u) > 0 \textit{ for all } u \geq 0. \tag{2.1}$$

*Let* $\mathcal{U}(p)$ *be a solution of the following two-point problem* (*TPP*)

$$\eta(\mathcal{U}) \frac{d\mathcal{U}}{dp} + \alpha \mathcal{U} + p = \gamma,\ \gamma \in \boldsymbol{R}^1 \tag{2.2}$$

$$\mathcal{U}(0) = 0,\ \mathcal{U}(P) = 0. \tag{2.3}$$

*Let* $z(\boldsymbol{x})$ *be the solution of problem* (1.10). *Define*

$$G(p) = \int_0^p \frac{dt}{\mu(\mathcal{U}(t))}$$

*and*

$$p(\boldsymbol{x}) = G^{-1}(G(P)z(\boldsymbol{x})),\ u(\boldsymbol{x}) = \mathcal{U}(p(\boldsymbol{x})). \tag{2.4}$$

*Then* $(p(\boldsymbol{x}), u(\boldsymbol{x}))$ *is a functional solution of problem* (*P*).

PROOF. – $p(\boldsymbol{x})$ and $u(\boldsymbol{x})$ as given by (2.4), satisfy the boundary conditions (1.6) and (1.7). Equation (1.4) is also satisfied. For, we have

$$G(p(\boldsymbol{x})) = G(P)z(\boldsymbol{x}),\ \nabla G(p(\boldsymbol{x})) = \frac{\nabla p}{\mu(\mathcal{U}(p(\boldsymbol{x})))} = \frac{\nabla p}{\mu(u(\boldsymbol{x}))} = G(P)\nabla z.$$

Hence we obtain (1.4) by (1.10). It remains to verify that $(p(\boldsymbol{x}), u(\boldsymbol{x}))$ satisfies

(1.5). In fact, by (1.4) and (2.2) we have

$$\nabla\cdot\left[\frac{1}{\mu(u)}(\eta(u)\nabla u+\alpha u\nabla p+p\nabla p)\right]=\nabla\cdot\left[\frac{1}{\mu(u)}\left(\eta(u)\frac{d\mathcal{U}}{dp}+\alpha\mathcal{U}+p\right)\nabla p\right]$$
$$=\gamma\nabla\cdot\left[\frac{\nabla p}{\mu(u)}\right]=0.$$

□

REMARK 2.2. – The solutions of problem (*TPP*) give all the functional solutions of problem (*P*). For the proof we refer to [5].

We examine now the question of existence and uniqueness for the solutions of problem (*TPP*). Define

$$U=F(\mathcal{U})=\int_0^{\mathcal{U}}\eta(t)dt \tag{2.5}$$

and assume

$$\int_0^{\infty}\eta(t)dt=\infty. \tag{2.6}$$

By (2.6), $F(\mathcal{U})$ maps diffeomorphically $[0,\infty)$ onto $[0,\infty)$ and gives therefore also a new scale for the temperature equivalent to the original one. In terms of $U$ problem (*TPP*) becomes

$$\frac{dU}{dp}+\alpha B(U)+p-\gamma=0 \tag{2.7}$$

$$U(0)=0,\ \ U(P)=0, \tag{2.8}$$

where

$$B(U)=F^{-1}(U).$$

We quote, for later use, the following special case of the Levy-Caccioppoli global inversion theorem (see [8], [4] and [2]).

THEOREM 2.3. – *Let* $X=\{\mathcal{U}(p)\in C^1([0,P]),\ \mathcal{U}(0)=0,\ \mathcal{U}(P)=0\}\times\boldsymbol{R}^1$ *and* $Y=C^0([0,P])$. *Let* $\mathcal{F}(\mathcal{U},\gamma)$ *be a map of class* $C^1$ *from* $X$ *to* $Y$. *Assume: (i)* $\mathcal{F}$ *is locally invertible in* $X$, *(ii)* $\mathcal{F}$ *is proper. Then* $\mathcal{F}$ *is a diffeomorphism from* $X$ *into* $Y$.

We recall that $\mathcal{F}$ is *proper* if, for every compact subset $\mathcal{K}$ of $Y$, $\mathcal{F}^{-1}(\mathcal{K})$ is compact in $X$. The above result is used in the following Theorem to show that problem (2.7), (2.8) has only and only one solution.

THEOREM 2.4. – *Let* $B(U) \in C^1([0,\infty))$ *and*

$$0 \le B(U) \le (m_1 + m_2 U), \tag{2.9}$$

*where* $\alpha$, $m_1$ *and* $m_2$ *are positive constants. Then the two-point problem* (2.7), (2.8) *has one and only one solution* $U(p) \in C^2([0,P])$.

PROOF. – Define $X$ and $Y$ as in Theorem 2.3. By (2.9), $L(U) = \alpha B(U) - \alpha m_2 U$ is a bounded function. Let $\mathcal{F} : X \to Y$ be given by

$$\mathcal{F}(U,\gamma) = \frac{dU}{dp} + p + \alpha m_2 U + L(U) - \gamma.$$

$\mathcal{F}$ is locally invertible. For, the linearized two-point problem corresponding to (2.7), (2.8) reads

$$\frac{dH}{dp} - \Gamma + \alpha m_2 H + L'(U)H = g(p) \tag{2.10}$$

$$H(0) = 0 \tag{2.11}$$

$$H(P) = 0 \tag{2.12}$$

where $\Gamma$ is a real number, $(U(p),\Gamma) \in X$ and $g(p) \in Y$. This linear two-point problem has a unique solution. To prove that we solve the Cauchy problem (2.10) and (2.11) and find

$$H(p) = e^{-\int_0^p (\alpha m_2 + L'(U(t))dt} \int_0^p e^{\int_0^\tau \alpha m_2 + L'(U(t))dt} (g(\tau) + \Gamma) d\tau. \tag{2.13}$$

It easily seen that there exists one and only one value of $\Gamma$ which permits to satisfy the condition (2.12). We claim that $\mathcal{F}$ is a proper map. For, let $\mathcal{K}$ be a compact subset of $Y$ and $g(p) \in \mathcal{K}$. If $(U(p),\gamma) \in \mathcal{F}^{-1}(\mathcal{K})$ we have

$$\frac{dU}{dp} - \gamma + p + \alpha m_2 U + L(U) = g(p) \tag{2.14}$$

$$U(0) = 0, \;\; U(P) = 0. \tag{2.15}$$

Applying the variation of constant formula to (2.14) and taking into account of the condition $U(0) = 0$ we have

$$U(p) = e^{-\alpha m_2 p} \int_0^p e^{\alpha m_2 \tau} (\gamma - \tau - L(U(\tau)) + g(\tau)) d\tau. \tag{2.16}$$

Setting $p = P$ in (2.16) we obtain, after an easy calculation,

$$\gamma \left( \frac{1 - e^{-\alpha m_2 P}}{\alpha m_2} \right) = e^{-\alpha m_2 P} \int_0^P (\tau + L(U(\tau)) - g(\tau)) d\tau. \tag{2.17}$$

Since the functions $g(p)$ belong to $\mathcal{K}$ they are equibonded. Hence, from (2.17) we draw the conclusion that there exists a constant $C_1(\mathcal{K})$ depending only on $\mathcal{K}$ such that for all $g \in \mathcal{K}$ we have

$$|\gamma| \leq C_1(\mathcal{K}). \tag{2.18}$$

Using again (2.16) and (2.18) and recalling that $L(U)$ is a bounded function we conclude that there exists a constant $C_2(\mathcal{K})$ such that, for all $g \in \mathcal{K}$,

$$|U(p)| \leq C_2(\mathcal{K}). \tag{2.19}$$

Moreover from (2.14) we have, for all $g \in \mathcal{K}$,

$$\left|\frac{dU}{dp}\right| \leq C_3(\mathcal{K}). \tag{2.20}$$

To apply Arzela's theorem to the set $\mathcal{F}^{-1}(\mathcal{K})$ it remains to prove the equicontinuity of the functions $\frac{dU}{dp}$. Let $p_1,\ p_2 \in [0,P]$ and $M = \sup\{|L'(u)|;\ |u| \leq C_2(\mathcal{K})\}$ From (2.14) we have, by difference,

$$\left|\frac{dU}{dp}(p_2) - \frac{dU}{dp}(p_1)\right| \leq (\alpha m_2 + M)|U(p_2) - U(p_1)| + |p_2 - p_1| + |g(p_2) - g(p_1)|. \tag{2.21}$$

Since the functions $U(p)$ and the $g(p)$ are equicontinuous we conclude that the functions $\frac{dU}{dp}$ are also equicontinuous. Therefore $\mathcal{F}^{-1}(\mathcal{K})$ is compact in $X$. □

REMARK 2.5. – Assumption (2.9) is crucial for the validity of Lemma 2.4. As a counterexample we can take the problem

$$\frac{dU}{dp} + U^2 + p = \gamma,\ \ U(0) = 0,\ \ U(P) = 0 \tag{2.22}$$

which can be solved explicitly in terms of the Airy's functions.

## 3. – The inverse problem and the integral equation of Abel

In problem $(P)$ the mass of fluid crossing $\Gamma_2$ in the unit time is given by

$$Q = \rho \int_{\Gamma_2} \boldsymbol{v} \cdot \boldsymbol{n}\ d\Gamma,$$

where $\boldsymbol{n}$ is the unit vector normal to $\Gamma_2$. From (1.1) we obtain, recalling (2.4),

$$\int_0^{p(\boldsymbol{x})} \frac{dt}{\mu(\mathcal{U}(t))} = z(\boldsymbol{x}) \int_0^P \frac{dt}{\mu(\mathcal{U}(t))},$$

where $z(\boldsymbol{x})$ is given by the solution of problem (1.10). Hence

$$\frac{\nabla p}{\mu(\mathcal{U}(p(\boldsymbol{x})))} = \nabla z \int_0^P \frac{dt}{\mu(\mathcal{U}(t))}.$$

On the other hand, on $\Gamma_2$ we have

$$\frac{1}{\mu(\mathcal{U}(P))}\frac{\partial p}{\partial n} = \int_0^P \frac{dt}{\mu(\mathcal{U}(t))}\,\frac{\partial z}{\partial n}.$$

Therefore

$$Q = \rho K k \int_0^P \frac{dt}{\mu(\mathcal{U}(t))},$$

where $k$ is given by

$$k = \int_{\Gamma_2} \frac{\partial z}{\partial n}.$$

By the maximum principle in Hopf's form [10], we have $k > 0$. Moreover $k$ depends only on $\Omega$, $\Gamma_1$, $\Gamma_2$, $\Gamma_0$ . Thus we obtain the following

LEMMA 3.1. – *The function*

$$f(P) = \frac{Q}{\rho K k} = \int_0^P \frac{dt}{\mu(\mathcal{U}(t,P))}$$

*does not depend on* $\Omega$, $\Gamma_0$, $\Gamma_1$, $\Gamma_2$.

The inverse problem, stated in the Introduction, of finding $\mu(u)$ if $f(P)$ and $\eta(u)$ are known may be solved if we can solve the integral equation

$$f(P) = \int_0^P \frac{dt}{\mu(\mathcal{U}(t,P)} \tag{3.1}$$

in the unknown $\mu(u)$. We first treat the case in which we have in equation (1.5) $\alpha = 0$. This means to neglect in problem $(P)$ the convective phenomena. In this case the two-point problem (2.2) and (2.3) becomes

$$\eta(\mathcal{U})\frac{d\mathcal{U}}{dp} + p = \gamma,\ \mathcal{U}(0) = 0,\ \mathcal{U}(P) = 0$$

which can easily be solved by separation of variables. If $U = F(\mathcal{U})$ is given by

(2.5) we have

$$\mathcal{U}(p) = F^{-1}\left(\frac{Pp}{2} - \frac{p^2}{2}\right)$$

and the integral equation (3.1) becomes

$$f(P) = \int_0^P \frac{dt}{\mu\left(F^{-1}\left(\frac{Pt}{2} - \frac{t^2}{2}\right)\right)}. \tag{3.2}$$

Since $U = F(\mathcal{U})$ is simply a new scale for the temperature, we may take as unknown in the integral equation $\nu(U) = \mu(F^{-1}(U))$. This permits to rewrite (3.2) as follows

$$f(P) = \int_0^P \frac{dt}{\nu\left(\frac{Pt}{2} - \frac{t^2}{2}\right)}.$$

With the change of variable of integration $\zeta = t - \dfrac{P}{2}$ we obtain

$$f(P) = 2\int_0^{\frac{P}{2}} \frac{d\zeta}{\nu\left(\frac{P^2}{8} - \frac{\zeta^2}{2}\right)}$$

and with the further substitution $z = \dfrac{P^2}{8} - \dfrac{\zeta^2}{2}$ we have

$$f(P) = \sqrt{2}\int_0^{\frac{P^2}{8}} \frac{dz}{\nu(z)\sqrt{\frac{P^2}{8} - z}}. \tag{3.3}$$

Setting $x = \dfrac{P^2}{8}$ and $\varphi(x) = \dfrac{1}{\sqrt{2}} f(2\sqrt{2x})$ equation (3.3) becomes

$$\varphi(x) = \int_0^x \frac{(\nu(z))^{-1}dz}{\sqrt{x - z}}.$$

This is the classical Abel integral equation [1] which has the simple solution [12]

$$\nu(x) = \frac{1}{\pi}\int_0^x \frac{\varphi'(t)dt}{\sqrt{x - t}}. \tag{3.4}$$

REMARK – We note that if $\eta(u) \in C^0([0,\infty))$ we have $Q(P) \in C^1([0,\infty))$ moreover $\varphi(x)$ is absolutely continuous by (2.1) and its definition. In addition $\varphi(0) = 0$. Thus the solution (3.4) makes sense according to the work of Tonelli [11].

If $\alpha > 0$ the two-point problem $(TPP)$

$$\eta(\mathcal{U})\frac{d\mathcal{U}}{dp} + p + \alpha\mathcal{U} = \gamma,\ \mathcal{U}(0) = 0,\ \mathcal{U}(P) = 0$$

or alternatively, in terms of $U = F(\mathcal{U}) = \int_0^{\mathcal{U}} \eta(t)dt$ and with $B(U) = F^{-1}(U)$,

$$\frac{dU}{dp} + p + \alpha B(U) = \gamma,\ U(0) = 0,\ U(P) = 0 \tag{3.5}$$

cannot be solved explicitly. However, we shall prove that if $\alpha > 0$ is sufficiently small the integral equation

$$f(P) = \int_0^P \frac{dt}{v(U(t,P,\alpha))}$$

with $v(u)$ as unknown, can be restated as a generalized Abel integral equation. To this end, we present below two lemmas on the properties of the unique solution $U(p,P,\alpha)$ of problem (3.5).

LEMMA 3.2. – *The solution $U(p,P,\alpha)$ of problem* (3.5) *has only one point of maximum $p_M(P,\alpha) \in (0,P)$.*

PROOF. – In view of the regularity of the data problem (3.5) can equivalently be rewritten as

$$\frac{d^2U}{dp^2} + 1 + \alpha B'(U)\frac{dU}{dp} = 0,\ U(0) = 0,\ U(P) = 0. \tag{3.6}$$

Therefore, in every point $p^* \in (0,P)$ in which $\frac{dU}{dp}(p^*) = 0$, we have $\frac{d^2U}{dp^2}(p^*) < 0$. This excludes the presence in $(0,P)$ of points of minimum. On the other hand, the existence of two points of maximum is also excluded since this would imply the existence of a point of minimum between them. □

LEMMA 3.3. – *Let $p_M(P,\alpha)$ be the point of maximum of $U(p,P,\alpha)$. There exists a positive number $\tilde{\alpha}$ such that, if $0 \le \alpha < \tilde{\alpha}$ then $M(P,\alpha) = U(p_M(P,\alpha),\alpha,P)$ is strictly increasing as a function of $P > 0$ for every fixed $0 \le \alpha < \tilde{\alpha}$.*

PROOF. – Since $\frac{\partial U}{\partial p}(p_M(P,\alpha),P,\alpha) = 0$, we have

$$\frac{\partial M}{\partial P}(P,\alpha) = \frac{\partial U}{\partial P}(p_M(P,\alpha),P,\alpha).$$

We claim that

$$\lim_{\alpha\to 0^+} p_M(P,\alpha) = \frac{P}{2}. \tag{3.7}$$

Integrating (3.5) from 0 to $P$ we have

$$\gamma = \frac{P}{2} + \frac{\alpha}{P}\int_0^P B(U(t,P,\alpha))dt.$$

If we define

$$\mathcal{F}(p_M,P,\alpha) = \frac{P}{2} + \frac{\alpha}{P}\int_0^P B(U(t,P,\alpha))dt - p_M - \alpha B(U(p_M,P,\alpha),P,\alpha))$$

we have

$$\mathcal{F}\Big(\frac{P}{2},P,0\Big) = 0$$

and

$$\frac{\partial \mathcal{F}}{\partial p_M}(p_M,P,0) = -1.$$

Therefore we may solve locally the equation $\mathcal{F}(p_M,P,\alpha) = 0$ with respect to $p_M$ i.e., there exists $\tilde{\alpha}_1 > 0$ and a function $p_M = \tilde{p}_M(P,\alpha) \in C^1$ such that $\tilde{p}_M(P,0) = \frac{P}{2}$ and $\mathcal{F}(\tilde{p}_M(P,\alpha),P,\alpha) = 0$. Thus (3.7) follows. It remains to show that $\frac{\partial U}{\partial P}(p,P,\alpha)$ is continuous for $P > 0$ and in particular that

$$\lim_{\alpha\to 0^+} \frac{\partial U}{\partial P}(p,P,\alpha) = \frac{p}{2}. \tag{3.8}$$

With the substitution $p = P\xi$ problem (3.6) can be restated in the fixed interval $[0,1]$ as follows

$$\frac{d^2W}{d\xi^2} + P^2 + \alpha P B'(W)\frac{dW}{d\xi} = 0,\ \ W(0) = 0,\ \ W(1) = 0,$$

where $W(\xi,P,\alpha) = U(P\xi,P,\alpha)$. To prove the needed regularity of $W(\xi,P,\alpha)$ with respect to $P$, we apply the implicit function in Banach spaces. Define in the space $\{W(\xi) \in C^2([0,1]);\ W(0) = 0,\ W(1) = 0\} \times \boldsymbol{R}^2$ the operator

$$\mathcal{G}(W,(P,\alpha)) = \frac{d^2W}{d\xi^2} + P^2 + \alpha P B'(W)\frac{dW}{d\xi},\ \ (P,\alpha) \in \boldsymbol{R}^2$$

with values in $C^0([0,1])$. We have for $P > 0$ and $\bar{W}(\xi) = \dfrac{P^2}{2}(\xi - \xi^2)$

$$\mathcal{G}(\bar{W}, (P, 0)) = 0. \tag{3.9}$$

Moreover, $\mathcal{G}$ is of class $C^2$ and for the the partial derivative we have

$$\mathcal{G}_W(\bar{W}, (P, 0))[H] = \frac{d^2H}{d\xi^2}.$$

Since the problem $\dfrac{d^2H}{d\xi^2} = 0,\ H(0) = 0,\ H(1) = 0$ has a unique solution, there exists $\tilde{\alpha}_2 > 0$ such that $\mathcal{G}(W, (P, \alpha)) = 0$ is locally solvable with respect to $W(\xi)$ when $0 < \alpha < \tilde{\alpha}_2$ and $P > 0$ and, as a function of $(P, \alpha)$ is of class $C^1$. Thus we can write

$$\frac{\partial U}{\partial P}(p, P, \alpha) = -\frac{\partial W}{\partial \xi}\left(\frac{p}{P}, P, \alpha\right)\frac{1}{P^2} + \frac{\partial W}{\partial P}\left(\frac{p}{P}, P, \alpha\right).$$

Therefore (3.8) holds. In particular we have, by (3.7),

$$\lim_{\alpha \to 0} \frac{\partial U}{\partial P}(p_M(P, \alpha), P, \alpha) = \frac{P}{4} > 0.$$

Hence there exists $\tilde{\alpha} > 0$ such that the conclusion of the Lemma holds. □

REMARK 3.4. – When $B(U) = U$ (which corresponds to the linear case) we find with a direct computation

$$\frac{\partial U}{\partial P}(p_M(P, \alpha), P, \alpha) > 0$$

not only when $\alpha$ is small, but for all $\alpha \geq 0$ and for all $P > 0$. This is probably true also for the nonlinear problem (3.5) under the sole hypothesis

$$B(U) \leq m_1 + m_2 U.$$

For future use we quote the following elementary Lemma, referring for the proof to [6].

LEMMA 3.5. – *Let* $f(x) \in C^3((-\delta, \delta))$ *satisfy*

$$f(0) = 0,\ f'(0) = 0,\ f''(0) > 0.$$

*Let* $x = \mathcal{H}_1(y)$ *be the branch of the local inverse of* $f(x)$ *with* $0 \leq y \leq \mu_1$, $0 \leq x \leq \eta$ *and* $x = \mathcal{H}_2(y)$ *the branch of the local inverse of* $f(x)$ *with* $0 \leq y \leq \mu_2$ *and* $-\eta \leq x \leq 0$ ($\mu_1 > 0$, $\mu_2 > 0$, $\eta > 0$). *Then*

$$\lim_{y \to 0^+} \mathcal{H}_1'(y)\sqrt{y} = -\lim_{y \to 0^+} \mathcal{H}_2'(y)\sqrt{y} = \frac{1}{\sqrt{2f''(0)}}. \tag{3.10}$$

We use in the sequel Lemma 3.3 and Lemma 3.5 to restate the integral equation

$$f(P)=\int_0^P \frac{dt}{\nu(U(t,P,\alpha))},$$

where $\nu(U)$ is the unknown, as an Abel integral equation.

THEOREM 3.6. – *Let $U(p,P,\alpha)$ be the solution of the problem*

$$\frac{dU}{dp}=\gamma-p-\alpha B(U),\ \ U(0)=0,\ \ U(P)=0$$

*and*

$$x=M(P),\ \ M(P)=U(p_M(P,\alpha),P,\alpha)$$

*the value of the maximum of $U(p,P,\alpha)$ in $(0,P)$. If*

$$0\le\alpha<\tilde{\alpha}$$

*there exists a continuous function $G(x,z)$ defined in $\{(x,z);0\le z\le x\}$ such that*

$$G(z,z)\ne 0$$

*and*

$$\int_0^P \frac{dt}{\nu(U(t,P,\alpha)}=\int_0^{M(P)}\frac{G(M(P),z)dz}{\nu(z)\sqrt{M(P)-z}}=\int_0^x \frac{G(x,z)dz}{\nu(z)\sqrt{x-z}}.$$

PROOF. – We can write

$$\int_0^P \frac{dt}{\nu(U(t,P,\alpha)}=\int_0^{p_M}\frac{dt}{\nu(U(t,P,\alpha)}+\int_{p_M}^P \frac{dt}{\nu(U(t,P,\alpha)}=I_1+I_2.$$

We make in $I_1$ the substitution of variable of integration $t=\mathcal{H}_1(z,x)$ ($\mathcal{H}_1$ is global inverse of $U(t,P,\alpha)$ for $0\le z\le x$ and $0\le t\le p_M$) and in $I_2$ the substitution $t=\mathcal{H}_2(z,x)$ (inverse of $U(t,P,\alpha)$ for $0\le z\le x$ and $p_M\le t\le P$). We find

$$\int_0^P \frac{dt}{\nu(U(t,P,\alpha))}=\int_0^x \frac{\mathcal{H}_1'(z,x)-\mathcal{H}_2'(z,x)}{\nu(z)}dz,\quad x=M(P). \tag{3.11}$$

By Lemma 3.5

$$G(z,x)=\sqrt{x-z}\left[\mathcal{H}_1'(z,x)-\mathcal{H}_2'(z,x)\right]$$

is a continuous function for $0 \leq x \leq z$. Moreover, by (3.10) we have $G(z,z) \neq 0$. Hence, from (3.11) we obtain

$$\int_0^P \frac{dt}{\nu(U(t,P,\alpha))} = \int_0^x \frac{G(z,x)dz}{\nu(z)\sqrt{x-z}}. \tag{3.12}$$

□

Therefore the integral equation

$$f(P) = \int_0^P \frac{dt}{\nu(U(t,P,\alpha))}$$

can be reformulated, by (3.12), as follows

$$f(P) = \int_0^x \frac{G(x,z)dz}{\nu(z)\sqrt{x-z}}.$$

On the other hand, if $0 \leq \alpha \leq \tilde{\alpha}$ the function $x = M(P)$ is invertible by Lemma 3.3. Defining

$$g(x) = f(M^{-1}(x))$$

we obtain, for the determination of $\nu(u)$, the Abel integral equation

$$g(x) = \int_0^x \frac{G(x,z)dz}{\nu(z)\sqrt{x-z}}$$

which can be solved using standard methods see [7] and [12].

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