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New Periodic Solutions for N-Body Problems with Weak Force Potentials

PENGFEI YUAN - SHIQING ZHANG

Dedicated in gratitude to Zhang's teacher Professor Yang Wannian on the occasion of his 75th birthday

Abstract. – *In this paper, we apply a variant of the famous Mountain Pass Lemmas of Ambrosetti-Rabinowitz* ([5]) *and Ambrosetti-Coti Zelati* ([2]) *with* $(CPS)_c$ *type condition of Cerami-Palais-Smale* ([12]) *to study the existence of new periodic solutions with a prescribed energy for N-body problems with weak force type potentials.*

1. – Introduction and Main Results

In 1975 and 1977, Gordon ([26], [27]) firstly used variational methods to study periodic solutions of 2-body problems, later, many authors ([1]-[9], [11], [13]-[31], [33]-[40] etc. and the references there) used variational methods to study N-body ($N \geq 3$) type singular Hamiltonian systems. For Newtonian-type N-body problems with homogeneous potentials, the mathematicians can get some new non-collision symmetrical periodic solutions by using some priori estimates on the Lagrangian action or Marchal's theorem on fixed ends.

In [2], Ambrosetti-Coti Zelati used Mountain Pass Lemma with the $(PS)^+$ condition to study the existence of weak solutions for symmetrical N-body problems with any given masses $m_1, \dots, m_N > 0$ and a fixed energy $h < 0$:

$$(Ph) \qquad \begin{cases} m_i \ddot{x}_i + \nabla_{x_i} V(x_1, \dots, x_N) = 0, \quad (1 \leq i \leq N), & (Ph.1) \\ \dfrac{1}{2} \sum m_i |\dot{x}_i(t)|^2 + V(x_1(t), \dots, x_N(t)) = h. & (Ph.2) \end{cases}$$

They got:

THEOREM 1.1 ([2]). – *Suppose that* $V(x) = \dfrac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(x_i - x_j)$ *with* $V_{ij} \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ *satisfying:*

$(V1)$. $V_{ij}(\xi) = V_{ji}(\xi), \forall\, \xi \in \Omega = \mathbb{R}^n \backslash \{0\}$;
$(V2)$. $\exists\, \alpha \in [1, 2)$, *such that*

$$\nabla V_{ij}(\xi) \cdot \xi \geq -\alpha V_{ij}(\xi) > 0, \quad \forall\, \xi \in \Omega;$$

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$(V3)$. $\exists\,\delta\in(0,2)$ *and* $r>0$, *such that*

$$\nabla V_{ij}(\xi)\cdot\xi\leq-\delta V_{ij}(\xi),\quad \forall 0<|\xi|\leq r;$$

$(V4)$. $V_{ij}(\xi)\to 0$, *as* $|\xi|\to+\infty$.

Then $\forall\,h<0$, *the problem* (Ph) *has a periodic solution.*

THEOREM 1.2 ([2]). – *Suppose* V *satisfies* $(V1),(V3),(V4)$ *and*

$(V2')$. $\exists\,\alpha\in(0,2)$, *such that*

$$\nabla V_{ij}(\xi)\cdot\xi\geq-\alpha V_{ij}(\xi)>0,\quad \forall\,\xi\in\Omega;$$

$(V5)$. $V_{ij}\in C^2(\Omega,\mathbb{R})$ *and*

$$3\nabla V_{ij}(\xi)\cdot\xi+V''_{ij}(\xi)\xi\cdot\xi>0.$$

Then $\forall\,h<0,(Ph)$ *has a weak periodic solution.*

Motivated by Ambrosetti-Coti Zelati's work ,we have the following theorem

THEOREM 1.3. – *Suppose that* $V(q)=\sum_{1\leq i<j\leq N}V_{ij}(q_i-q_j)$ *with* $V_{ij}\in C^1(\mathbb{R}^n\backslash\{0\},\mathbb{R})$ *satisfying:*

(V_1). $V_{ij}(\xi)=V_{ji}(\xi)$;

(V_2). *There are constant* $0<\alpha<2$ *such that*

$$\langle V'_{ij}(\xi),\xi\rangle\geqslant-\alpha V_{ij}(\xi)>0,\quad \forall\,\xi\in\mathbb{R}^n\backslash\{0\};$$

(V_3). $\exists\,\delta\in(0,2),\delta\geqslant\alpha,r>0$, *such that*

$$\langle V'_{ij}(\xi),\xi\rangle\leqslant-\delta V_{ij}(\xi),\quad \forall\,0<|\xi|\leq r;$$

(V_4). $V_{ij}(\xi)\to 0$, *as* $|\xi|\to+\infty$.

Then for any given $h<0$, *the system (Ph) has at least a non-trivial weak periodic solution which can be obtained by Mountain Pass Lemma.*

REMARK. – Comparing Theorem 1.3 with Theorem 1.1-1.2, our Theorem 1.3 generalizes Theorem 1.2 since we don't assume $(V5)$, we also generalizers Theorem 1.1 since we relax α in (V_2).

COROLLARY 1.4. – *Suppose* $0<\alpha=\delta<2$ *and*

$$V(x)=\sum_{1\leq i<j\leq N}-|x_i-x_j|^{-\alpha}.$$

Then for any $h<0$, *(Ph) has at least one non-trivial weak periodic solution with the given energy h.*

2. – Some Lemmas

Let us introduce the following notation:

$$M = \sum_{i=1}^{N} m_i; \quad H^1 = W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n).$$
$$H^N = \{u = (u_1, \dots, u_N) \mid \ u_i \in H^1\}.$$
$$H_{\#} = \{u \in H^1 \mid \ u(t + 1/2) = -u(t)\}.$$
$$E = \{u = (u_1, \dots, u_N) \mid u_i \in H_{\#}(1 \leq i \leq N)\}.$$
$$\Lambda_0 = \{u \in E \mid u_i(t) \neq u_j(t), \forall\, t, \forall\, i \neq j\}.$$
$$\partial\Lambda_0 = \{u \in E \mid \exists\, t_0, 1 \leq i_0 \neq j_0 \leq N\, s.t.\, u_{i_0}(t_0) = u_{j_0}(t_0)\}.$$

LEMMA 2.1. – ([1]-[4]) *Let* $f(u) = \frac{1}{2}\int_0^1 \sum_{i=1}^{N} m_i|\dot{u}_i|^2\,\mathrm{d}t \int_0^1 (h - V(u))\,\mathrm{d}t$ *and* $\tilde{u} \in H^N$ *satisfy* $f'(\tilde{u}) = 0$ *and* $f(\tilde{u}) > 0$. *Set*

$$\frac{1}{T^2} = \frac{\int_0^1 (h - V(\tilde{u}))\,\mathrm{d}t}{\frac{1}{2}\int_0^1 \sum_{i=1}^{N} m_i|\dot{\tilde{u}}_i|^2\,\mathrm{d}t}. \tag{2.1}$$

Then $\tilde{q}(t) = \tilde{u}(t/T)$ *is a non-constant T-periodic solution for (Ph).*

LEMMA 2.2. – (Palais [32])
Let σ *be an orthogonal representation of a finite or compact group G in the real Hilbert space H such that for* $\forall \sigma \in G$,

$$f(\sigma \cdot x) = f(x),$$

where $f \in C^1(H, \mathbb{R})$.

Let $S = \{x \in H \mid \sigma x = x, \forall \sigma \text{ in } G\}$, *then the critical point of f in S is also a critical point of f in H.*

By Lemma 2.1-2.2 *and* (V_1), *we have*

LEMMA 2.3. – ([1] – [4])
If $\bar{u} \in \Lambda_0$ *is a critical point of* $f(u)$ *and* $f(\bar{u}) > 0$, *then* $\bar{q}(t) = \bar{u}(t/T)$ *is a non-constant T-periodic solution of (Ph).*

Cerami [12] *introduced the following* $(CPS)_c$ *condition:*

DEFINITION 2.4 ([12]). – *Let X be a Banach space,* $\{q_n\} \subset X$ *satisfies*

$$f(q_n) \to c, \ (1 + \| q_n \|)f'(q_n) \to 0, \tag{2.2}$$

then $\{q_n\}$ has a strongly convergent subsequence, then we call that $\{q_n\}$ satisfies Cerami-Palais-Smale condition at level c, we denote it as $(CPS)_c$. If for all c, $(CPS)_c$ holds, then we call $f(q)$ satisfies (CPS) condition.

Combining the different forms of the Mountain Pass Lemmas in ([2], [5], [12], [19], [23], [25]), it's not difficult to get:

LEMMA 2.5. – *Suppose $f \in C^1(\Lambda_0, \mathbb{R})$ and*

(AR_1). $\exists\, \mathrm{r}, \beta > 0$, *s.t.* $f(u) \geq \beta, \quad \forall\, u \in \Lambda_0,\ \| u \|_{H^N} = \mathrm{r}$,

(AR_2). $\exists u_0 \in \Lambda_0$ *with* $\|u_0\| = \rho < r$ *and* $f(u_0) < \beta$.

(AR_3). $\forall \mathrm{M} > 0, \exists\, 0 < \rho = \rho(\mathrm{M}) < r$, *s.t.* $\forall u \in \Sigma_{\mathrm{M},\rho}$, $\langle df(u), u\rangle > 0$,

where $\Sigma_{\mathrm{M},\rho} = \{u \in \Lambda_0 \mid \|u\| = \rho, f(u) \leq \mathrm{M}\}$.

(AR_4). $\exists u_1 \in \Lambda_0$, *s.t.* $\|u_1\| \geq \mathrm{r}$, $f(u_1) \leq 0$.

Let

$$C = \inf_{P \in \Gamma_\rho} \max_{0 \leq \xi \leq 1} f(P(\xi)),$$

where

$$\Gamma_\rho = \{P \in C([0,1], \Sigma_\rho) \mid \| P(0) \|_{H^N} = \rho, P(1) = u_1\},$$

$$\Sigma_\rho = \{u \in \Lambda_0 \mid \| u \|_{H^N} \geq \rho\}.$$

Then there exists $\{u_n\} \subset \Lambda_0$ such that

$$f(u_n) \to C, \quad (1 + \| u_n \|) f'(u_n) \to 0.$$

Furthermore, if f satisfies $(CPS)_C$ condition, that is, $\{u_n\}$ has a convergent subsequence, furthermore if

$$f(u_n) \to +\infty, \quad \forall\, u_n \rightharpoonup u \in \partial \Lambda_0,$$

then C is a critical value of f, so there exists $u \in \Lambda_0$ such that $f'(u) = 0$, and $f(u) = C \geq \beta > 0$.

LEMMA 2.6. – *(Gordon[27])*
Suppose that V_{ij} satisfies so called Gordon's Strong Force condition: There exists a neighborhood $\mathcal{N}_{ij}$ of 0 and a function $U_{ij} \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ such that:

(i). $\lim_{\xi \to 0} U_{ij}(\xi) = -\infty$;

(ii). $-V_{ij}(\xi) \geq |U'_{ij}(\xi)|^2$ *for every* $\xi \in \mathcal{N}_{ij} \setminus \{0\}$.

Then we have

$$\int_0^1 V(u_n)\,\mathrm{d}t \to -\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda_0.$$

LEMMA 2.7. – *(Sobolev-Rellich-Kondrachov, Compact Imbedding Theorem* [10], [41]*)*

$$W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n) \subset C(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$$

and the embedding is compact.

LEMMA 2.8. – *(Eberlein-Shmulyan*[10]*)*
A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.

LEMMA 2.9. – ([41])
Let $q \in W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ *and* $\int_0^T q(t)\,\mathrm{d}t = 0$, *then we have*

(i). *Poincaré-Wirtinger's inequality:*

$$\int_0^T |\dot q(t)|^2\,\mathrm{d}t \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2\,\mathrm{d}t.$$

(ii). *Sobolev's inequality:*

$$\max_{0\le t\le T} |q(t)| = \| q \|_\infty \le \sqrt{\frac{T}{12}}\left(\int_0^T |\dot q(t)|^2 \mathrm{d}t\right)^{1/2}.$$

It's not difficult to prove:

LEMMA 2.10. – *For* $\forall\, u \in \Lambda_0$, *we have*

$$\int_0^1 u(t)\,\mathrm{d}t = 0.$$

By Lemma 2.9 *and Lemma* 2.10, *for* $\forall\, u \in \Lambda_0$, $\|u\| = \left(\int_0^1 \sum_{i=1}^N m_i |\dot u_i|^2\,\mathrm{d}t\right)^{1/2}$ *is equivalent to the* H^N *norm:*

$$\| u \|_{H^N} = \left(\int_0^1 |\dot u|^2\,\mathrm{d}t\right)^{1/2} + \left(|\int_0^1 u\,\mathrm{d}t|\right)^{1/2}.$$

LEMMA 2.11. – *(Coti Zelati*[20]*)*
Let $X = (x_1, \ldots, x_N) \in R^n \times \cdots R^n$. *Then*

$$\sum_{1 \le i < j \le N} \frac{m_i m_i}{|x_i - x_j|^\alpha} \ge C_\alpha(m_1, \ldots, m_N)\Big(\sum_{i=1}^{N} m_i |x_i|^2\Big)^{-\alpha/2},$$

where $C_\alpha(m_1, \ldots, m_N) \triangleq C_\alpha = M^{-\alpha/2}\Big(\sum_{1 \le i < j \le N} m_i m_j\Big)^{\frac{2+\alpha}{2}}$.

3. – The Proof of Theorem 1.3

In order to apply Mountain Pass Lemma for the variational functional defined on Λ_0 (an open set of Banach space), we need a complete condition:

$$f(u_n) \to +\infty, \quad u_n \rightharpoonup \partial\Lambda_0, \tag{3.0}$$

which can guarantee that the critical point is in Λ_0, not on it's boundary. But in the assumptions of Theorem 1.3, we don't have the strong force condition, so we need to revise the potential V as V_ε

$$\begin{aligned} V_\varepsilon(u) &= V(u) + W_\varepsilon(u) \\ W_\varepsilon(u) &= -\sum_{1 \le i < j \le N} \frac{\varepsilon m_i m_j}{|u_i - u_j|^\gamma}, \quad \gamma > 2 \\ V_{\varepsilon ij}(u_i - u_j) &= V_{ij}(u_i - u_j) - \frac{\varepsilon m_i m_j}{|u_i - u_j|^\gamma}. \end{aligned} \tag{3.1}$$

We also need to revise the functional $f(u)$ as

$$\begin{aligned} f_\varepsilon(u) =& \frac{1}{2}\int_0^1 \sum_{i=1}^{N} m_i |\dot{u}_i|^2 \, dt \int_0^1 (h - V_\varepsilon(u)) \, dt \\ =& \frac{1}{2} \| u \|^2 \int_0^1 (h - V_\varepsilon(u)) \, dt. \end{aligned} \tag{3.2}$$

REMARK. – Different from earlier papers, here we use $W_\varepsilon(u)$ with $\gamma > 2$ not $\gamma = 2$ to perturb V in order that f_ε satisfies (3.0) and we can verify all conditions of Mountain Pass Lemma.

After we apply Mountain Pass Lemma to the variational functional f_ε to get critical point u_ε, then let $\varepsilon \to 0$ to get the limit point, which is a weak solution which satisfies (Ph) except on a Lebegue's zero-measurable set.

In order to find critical point of f_ε in Λ_0, we need to verify all conditions of Mountain Pass Lemma, let's begin to prove:

LEMMA 3.1. – *If* $(V_1)-(V_2)$ *hold, then for all* $C>0$ *and any given* $\varepsilon>0$, *if* $\{u_n\}\subset\Lambda_0$ *satisfies*

$$f_\varepsilon(u_n)\to C>0,\quad (1+\|u_n\|)f'_\varepsilon(u_n)\to 0. \tag{3.3}$$

Then $\{u_n\}\subset\Lambda_0$ *has a strongly convergent subsequence, the limit must be in* Λ_0, *that is,* f_ε *satisfies the* $(CPS)_C$ *condition in* Λ_0.

PROOF. – The proof will be divided into three steps:

STEP1. – We show that $\{u_n\}$ is bounded.
In fact, by $f_\varepsilon(u_n)\to C$, we have

$$-\frac{1}{2}\|u_n\|^2\int_0^1 V_\varepsilon(u_n)\,\mathrm{d}t\to C-\frac{h}{2}\|u_n\|^2\,. \tag{3.4}$$

So when n is large enough, it follows that

$$-\frac{1}{2}\|u_n\|^2\int_0^1 V_\varepsilon(u_n)\,\mathrm{d}t\le C+1-\frac{h}{2}\|u_n\|^2\,. \tag{3.5}$$

By simple calculation, we can get

$$\langle V'_\varepsilon(u_n),u_n\rangle=\langle V'(u_n),u_n\rangle-\gamma W_\varepsilon(u_n). \tag{3.6}$$

Noting that

$$-\gamma W_\varepsilon\ge-\alpha W_\varepsilon. \tag{3.7}$$

From (V_2), (3.6) and (3.7) we have

$$\langle V'_\varepsilon(u_n),u_n\rangle\ge-\alpha V_\varepsilon(u_n)>0. \tag{3.8}$$

So

$$\begin{aligned}\langle f'_\varepsilon(u_n),u_n\rangle&=\|u_n\|^2\int_0^1\left(h-V_\varepsilon(u_n)-\frac{1}{2}\langle V'_\varepsilon(u_n),u_n\rangle\right)\mathrm{d}t\\&\le\|u_n\|^2\int_0^1\left(h-V_\varepsilon(u_n)+\frac{\alpha}{2}V_\varepsilon(u_n)\right)\mathrm{d}t\\&=\|u_n\|^2\int_0^1\left(h-(1-\frac{\alpha}{2})V_\varepsilon(u_n)\right)\mathrm{d}t.\end{aligned} \tag{3.9}$$

Since $0<\alpha<2$, using (3.5) and (3.9) we have

$$\begin{aligned}\langle f_\varepsilon'(u_n), u_n\rangle &\leq h \parallel u_n \parallel^2 + (1-\frac{\alpha}{2})\big[2(C+1) - h \parallel u_n \parallel^2\big] \\ &= \frac{\alpha}{2} h \parallel u_n \parallel^2 + C_1,\end{aligned} \tag{3.10}$$

where $C_1 = 2\big(1-\frac{\alpha}{2}\big)(C+1) > 0$.

By (3.3) we have

$$\langle f_\varepsilon'(u_n), u_n\rangle \leq \parallel u_n \parallel \parallel f_\varepsilon'(u_n) \parallel \to 0. \tag{3.11}$$

(3.10), (3.11) and $h<0$ imply

$$\parallel u_n \parallel \leq C_2. \tag{3.12}$$

STEP 2. – We prove $u_n \rightharpoonup u \in \Lambda_0$.

Since H^N is a reflexive Banach space, by Lemma 2.8 and (3.12), $\{u_n\}$ has a weakly convergent subsequence still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$.

To prove $u \in \Lambda_0$, we need two Lemmas.

LEMMA 3.2. – *$V_{\varepsilon ij}$ satisfies Gordon's Strong Force condition.*

PROOF. – Let

$$\overline{V_{ij}}(\xi) = \frac{-m_i m_j}{\lambda|\xi|^\lambda}, \quad \Big(0<\lambda<\frac{\gamma-2}{2}\Big).$$

Then

$$\lim_{|\xi|\to 0} \overline{V_{ij}} = -\infty. \tag{3.13}$$

By simple calculation, we obtain

$$|\overline{V_{ij}}'(\xi)|^2 = \frac{m_i m_j}{|\xi|^{2\lambda+2}}.$$

Since

$$\frac{\varepsilon m_i m_j}{|\xi|^\gamma} \geq \frac{m_i m_j}{|\xi|^{2\lambda+2}}, \quad \forall \varepsilon > 0, \tag{3.14}$$

when $|\xi|$ is small enough, so there exists a neighborhood $\mathcal{N}_{ij}$ of 0 such that $-V_{\varepsilon ij} \geq |\overline{V_{ij}}'|^2, \forall \xi \in \mathcal{N}_{ij} \setminus \{0\}$. Therefore, $V_{\varepsilon ij}$ satisfies Gordon's Strong Force condition. □

LEMMA 3.3. – *For any weakly convergent sequence* $u_n \rightharpoonup u \in \partial\Lambda_0$, *where* $u_n = (u_n^1, \dots, u_n^N)$, *there holds:*

$$f_\varepsilon(u_n) \to +\infty.$$

PROOF. – First of all, we recall that

$$f_\varepsilon(u_n) = \frac{1}{2}\int_0^1 \sum_{i=1}^N m_i|\dot{u}_n^i|^2\,\mathrm{d}t \int_0^1 (h - V_\varepsilon(u_n))\,\mathrm{d}t.$$

(1). If $u \equiv$ constant, we deduce that $u \equiv 0$ by $u_i(t+1/2) = -u_i(t)$. By Sobolev's embedding theorem, we have

$$\| u_n \|_\infty \to 0, \quad n \to \infty. \tag{3.15}$$

Using (V_2) we have $C_3 > 0$, such that

$$V_{ij}(\xi) \le -\frac{C_3 m_i m_j}{|\xi|^\alpha}, \quad \forall |\xi| > 0. \tag{3.16}$$

Therefore, $h - V(u_n) > 0$ when n is large enough, then by Lemma 2.11 we have

$$\begin{aligned} f_\varepsilon(u_n) &= \frac{1}{2}\int_0^1 \sum_{i=1}^N m_i|\dot{u}_n^i|^2\,\mathrm{d}t \int_0^1 \left(h - \sum_{i<j} V(u_n^i - u_n^j) + \sum_{i<j}\frac{\varepsilon m_i m_j}{|u_n^i - u_n^j|^\gamma}\right)\mathrm{d}t \\ &\ge \frac{1}{2}\int_0^1 \sum_{i=1}^N m_i|\dot{u}_n^i|^2\,\mathrm{d}t \int_0^1 \sum_{i<j}\frac{\varepsilon m_i m_j}{|u_n^i - u_n^j|^\gamma}\,\mathrm{d}t \\ &\ge \frac{\varepsilon}{2} C_\alpha \int_0^1 \sum_{i=1}^N m_i|\dot{u}_n^i|^2\,\mathrm{d}t \,\| u_n \|_\infty^{-\gamma}, \end{aligned} \tag{3.17}$$

where $\| u_n \|_\infty \triangleq \sum_{i=1}^N m_i|u_n^i|_\infty^2$.

Then by Sobolev's inequality, (3.15) and $\gamma > 2$ we have

$$f_\varepsilon(u_n) \ge 6\varepsilon C_\alpha \| u_n \|_\infty^{2-\gamma} \to +\infty, \quad n \to \infty.$$

(2). If $u \not\equiv$ constant, by the weakly lower-semi-continuity property for norm, we have

$$\liminf_{n\to\infty} \int_0^1 \sum_{i=1}^N m_i|\dot{u}_n^i|^2\,\mathrm{d}t \ge \int_0^1 \sum_{i=1}^N m_i|\dot{u}_i|^2\,\mathrm{d}t > 0. \tag{3.18}$$

Since $u \in \partial \Lambda_0$, there exist $t_0, 1 \leq i_0 \neq j_0 \leq N$ s.t. $u_{i_0}(t_0) = u_{j_0}(t_0)$ Set

$$\xi_n(t) = u_n^{i_0}(t) - u_n^{j_0}(t)$$

$$\xi(t) = u_{i_0}(t) - u_{j_0}(t)$$

By $u_n \rightharpoonup u$, we have $\xi_n(t) \rightharpoonup \xi(t)$. Then by Lemma 2.6 and Lemma 3.2, $\forall\, \varepsilon > 0$, we have

$$\int_0^1 V_{\varepsilon i_0 j_0}(u_n^{i_0} - u_n^{j_0})\,\mathrm{d}t \to -\infty.$$

Recalling that

$$V_\varepsilon(u_n) = \sum_{i<j} V_{\varepsilon ij}(u_n^i - u_n^j).$$

So we have

$$f_\varepsilon(u_n) \to +\infty, \quad n \to \infty. \tag{3.19}$$

□

Combining (3.3) and Lemma 3.3, we deduce that $u_n \rightharpoonup u \in \Lambda_0$.

STEP 3. – We prove that $u_n \to u$ strongly.
By $u_n \rightharpoonup u \in \Lambda_0$ and compact embedding theorem we have

$$\max_{0 \leq t \leq 1} |u_n(t) - u(t)| \to 0.$$

By the continuity of V_ε, V'_ε and the inner product $\langle \cdot\,, \cdot \rangle$, we have the uniformly convergent for $0 \leq t \leq 1$

$$\begin{aligned} V_\varepsilon(u_n) &\to V_\varepsilon(u), \\ W_\varepsilon(u_n) &\to W_\varepsilon(u), \\ \langle V'_\varepsilon(u_n), u_n \rangle &\to \langle V'_\varepsilon(u), u \rangle. \end{aligned} \tag{3.20}$$

From STEP 2, we know $u \in \Lambda_0$, so $\| u \| = \int_0^1 \sum_{i=1}^N m_i |\dot{u}_i|^2\,\mathrm{d}t > 0$, otherwise $u \equiv 0 \in \partial \Lambda_0$ by $u_i(t + 1/2) = -u_i(t)$. Then by $u_n \rightharpoonup u$ and the weakly lower-semi-continuous property of the norm ,we have

$$\liminf_{n \to \infty} \| u_n \| \geq \| u \| > 0. \tag{3.21}$$

By (3.11) we have

$$\langle f'_\varepsilon(u_n), u_n \rangle = \| u_n \|^2 \int_0^1 [h - V_\varepsilon(u_n) - \frac{1}{2}\langle V'_\varepsilon(u_n), u_n \rangle]\,\mathrm{d}t \to 0. \tag{3.22}$$

Let $n \to \infty$ in (3.22), by (3.20) and (3.21) we have

$$\int_0^1 (h - V_\varepsilon(u))\,\mathrm{d}t = \frac{1}{2}\int_0^1 \langle V'_\varepsilon(u), u\rangle\,\mathrm{d}t > 0. \tag{3.23}$$

From (3.3), we deduce that $f'_\varepsilon(u_n) \to 0$, then $\langle f'_\varepsilon(u_n),\ v\rangle \to 0, \forall\, v \in H^N$, that is

$$\int_0^1 \sum_{i=1}^N m_i\langle \dot{u}_n^i, \dot{v}_i\rangle \mathrm{d}t \int_0^1 (h - V_\varepsilon(u_n))\,\mathrm{d}t - \frac{1}{2}\parallel u_n \parallel^2 \int_0^1 \langle V'_\varepsilon(u_n), v\rangle\,\mathrm{d}t \to 0,\ \ \forall v \in H^N. \tag{3.24}$$

Taking $v = u$ in (3.24), we get

$$\lim_{n\to\infty}\int_0^1 \sum_{i=1}^N m_i\langle \dot{u}_n^i, \dot{u}_i\rangle\,\mathrm{d}t = \lim_{n\to\infty} \parallel u_n \parallel^2 . \tag{3.25}$$

By $u_n \rightharpoonup u$, we have

$$\lim_{n\to\infty}\int_0^1 \sum_{i=1}^N m_i\langle \dot{u}_n^i, \dot{u}_i\rangle\,\mathrm{d}t = \int_0^1 \sum_{i=1}^N m_i|\dot{u}_i|^2\,\mathrm{d}t = \parallel u \parallel^2 . \tag{3.26}$$

From (3.25) and (3.26), it follows that

$$\begin{aligned}\parallel u_n - u \parallel^2 = \int_0^1 \sum_{i=1}^N m_i|\dot{u}_n^i - \dot{u}_i|^2 \mathrm{d}t &= \int_0^1 \Big(\sum_{i=1}^N m_i|\dot{u}_n^i|^2 - 2\sum_{i=1}^N m_i\langle \dot{u}_n^i, \dot{u}_i\rangle + \sum_{i=1}^N m_i|\dot{u}_i|^2 \Big)\,\mathrm{d}t \\ &\to \parallel u \parallel^2 - 2 \parallel u \parallel^2 + \parallel u \parallel^2 \\ &= 0.\end{aligned} \tag{3.27}$$

That is $u_n \to u$ strongly in H^1. □

LEMMA 3.4. – *f_ε satisfies the condition (AR1) in the Mountain Pass Lemma.*

PROOF. – By (3.8) we have $C_4 > 0$, such that $-V_\varepsilon(u) \geq \sum_{i<j} \dfrac{C_4 m_i m_j}{|u_i - u_j|^\alpha}$, so by Coti Zelati's inequality [20], we have

$$\begin{aligned} f_\varepsilon(u) &= \frac{1}{2} \parallel u \parallel^2 \int_0^1 (h - V_\varepsilon(u))\,\mathrm{d}t \\ &= \frac{h}{2} \parallel u \parallel^2 - \frac{1}{2} \parallel u \parallel^2 \int_0^1 V_\varepsilon(u)\,\mathrm{d}t \\ &\geq \frac{h}{2} \parallel u \parallel^2 + \frac{C_\alpha C_4}{2} \parallel u \parallel^2 \parallel u \parallel_\infty^{-\alpha} . \end{aligned}$$

Then by Sobolev's inequality, we have $C_5 > 0$ s.t.

$$f_\varepsilon(u) \geq \frac{h}{2} \| u \|^2 + \frac{C_5}{2} \| u \|^{2-\alpha} .$$

Since $0<\alpha<2$, we can choose $\| u \| = \mathrm{r}$ small enough such that $\frac{h}{2}\mathrm{r}^2 + \frac{C_5}{2}\mathrm{r}^{2-\alpha} = \beta > 0$. Hence

$$f_\varepsilon(u) \geq \beta > 0, \quad \forall \| u \| = \mathrm{r}.$$

□

LEMMA 3.5. – *f_ε satisfies the condition (AR2) in the Mountain Pass Lemma,that is, $\exists u_0 \in \Lambda_0$ and $\varepsilon_0 > 0$, with $\| u_0 \| = \rho < \mathrm{r}$ s.t. $f_\varepsilon(u_0) < \beta$, $\forall 0 < \varepsilon < \varepsilon_0$.*

PROOF. – For $\tilde{R} > 0$, we consider

$$f_\varepsilon(\tilde{R}u) = \frac{1}{2} \| \tilde{R}u \|^2 \int_0^1 (h - V_\varepsilon(\tilde{R}u))\,\mathrm{d}t.$$

Using (V_3) we have $C_6 > 0$, such that

$$V_{ij}(\xi) \geq -C_6 m_i m_j |\xi|^{-\delta}, \quad \forall\ 0 < |\xi| \leq \mathrm{r}.$$

Then we have

$$\begin{aligned} f_\varepsilon(\tilde{R}u) \leq & \frac{h}{2}\tilde{R}^2 \| u \|^2 + C_6 \tilde{R}^{2-\delta} \| u \|^2 \sum_{i<j} \int_0^1 m_i m_j |u_i - u_j|^{-\delta}\,\mathrm{d}t \\ & + \varepsilon C_7 \tilde{R}^{2-\gamma} \| u \|^2 \sum_{i<j} \int_0^1 m_i m_j |u_i - u_j|^{-\gamma}\,\mathrm{d}t. \end{aligned} \tag{3.28}$$

Take $u_i(t) = \xi \cos[2\pi(t + \frac{i}{N})] + \eta \sin[2\pi(t + \frac{i}{N})]$, where $|\xi| = 1$, $|\eta| = 1$, $\langle \xi, \eta \rangle = 0$, $\xi, \eta \in \mathbb{R}^n$, then

$$\sum_{i<j} m_i m_j |u_i(t) - u_j(t)|^{-\delta} = \sum_{i<j} m_i m_j \left\{ 2 - 2\cos[\frac{2\pi(i-j)}{N}] \right\}^{-\delta/2} \triangleq a_\delta.$$

$$\sum_{i<j} m_i m_j |u_i(t) - u_j(t)|^{-\gamma} = \sum_{i<j} m_i m_j \left\{ 2 - 2\cos[\frac{2\pi(i-j)}{N}] \right\}^{-\gamma/2} \triangleq a_\gamma.$$

$$\| u \|^2 = 4\pi^2 M.$$

Hence

$$\begin{aligned} f_\varepsilon(\tilde{R}u) & \leq 4\pi^2 M \Big(\frac{h}{2}\tilde{R}^2 + C_6 a_\delta \tilde{R}^{2-\delta} + \varepsilon C_7 a_\gamma \tilde{R}^{2-\gamma} \Big) \\ & \leq 4\pi^2 M (C_6 a_\delta \tilde{R}^{2-\delta} + \varepsilon C_7 a_\gamma \tilde{R}^{2-\gamma}). \end{aligned} \tag{3.29}$$

Since $0<\delta<2$, so we can take R_0 small enough such that $4\pi^2 MC_6 a_\delta R_0^{2-\delta}<\beta$.

For the above fixed R_0, we choose $\varepsilon>0$ small enough such that

$$4\pi^2 MC_7 a_\gamma R_0^{2-\gamma}\varepsilon<\beta-4\pi^2 MC_6 a_\delta R_0^{2-\delta}. \tag{3.30}$$

In fact, we can choose

$$0<\varepsilon_0<\frac{\beta-4\pi^2 MC_6 a_\delta R_0^{2-\delta}}{4\pi^2 MC_7 a_\gamma R_0^{2-\gamma}}. \tag{3.31}$$

Choose R_1 small enough such that $\| R_1 u \|=\rho<\mathrm{r}$, take $R=\min\{R_0,R_1\}$, let $u_0=Ru$, then we have

$$f_\varepsilon(u_0)<\beta, \| u_0 \|=\rho<\mathrm{r}, \forall\, 0<\varepsilon\le\varepsilon_0. \tag{3.32}$$

□

LEMMA 3.6. – *f_ε satisfies the condition (AR3) in the Mountain Pass Lemma, that is, $\forall\,\mathrm{M}>0, \exists\rho(\mathrm{M})>0, \exists\varepsilon_0>0$, s.t. $\langle df_\varepsilon(u),u\rangle>0$, $\forall u\in\Sigma_{\mathrm{M},\rho}$, $\forall\,0<\varepsilon<\varepsilon_0$, where $\Sigma_{M,\rho}=\{u\in\Lambda_0 \mid f_\varepsilon(u)\le\mathrm{M}, \|u\|=\rho\}$.*

PROOF. –

$$\begin{aligned}\langle df_\varepsilon(u),u\rangle &= \|u\|^2\int_0^1\Big(h-V_\varepsilon(u)-\frac{1}{2}\langle V_\varepsilon'(u),u\rangle\Big)\,\mathrm{d}t\\ &\ge \|u\|^2\bigg[h+\Big(1-\frac{\delta}{2}\Big)C_3C_\alpha\|u\|^{-\alpha}+\varepsilon\Big(1-\frac{\gamma}{2}\Big)\int_0^1\sum_{i<j}\frac{m_im_i}{|u_i-u_j|^\gamma}\,\mathrm{d}t\bigg].\end{aligned}$$

Choose ρ small enough s.t. $h+\Big(1-\dfrac{2}{\delta}\Big)C_3C_\alpha\rho^{-\alpha}>0$.

We claim $\int_0^1\sum_{i<j}\dfrac{m_im_i}{|u_i-u_j|^\gamma}\mathrm{d}t$ is bounded for $\forall u\in\Sigma_{\mathrm{M},\rho}$, that is, there exists $A>0$, s.t. $\int_0^1\sum_{i<j}\dfrac{m_im_i}{|u_i-u_j|^\gamma}\mathrm{d}t\le A$ for $\forall u\in\Sigma_{\mathrm{M},\rho}$.

In fact, if not, then $\exists\{u^n\}, \|u^n\|=\rho, \exists i_0\ne j_0, t_0\in[0,1]$ such that $u^n_{i_0}(t_0)-u^n_{j_0}(j_0)\to 0$, that is, there is a subsequence of u^n ,we still denote it as u_n, and $u_n\rightharpoonup u\in\partial\Lambda_0$ as $n\to+\infty$, furthermore by Lemma 3.3, $f_\varepsilon(u^n)\to+\infty$, which is a contradiction with $f_\varepsilon(u)\le\mathrm{M}$.

Thus, if we choose $\varepsilon_0=\inf\limits_{u\in\Sigma_{M,\rho}}\dfrac{h+(1-\frac{\delta}{2})C_3C_\alpha\rho^{-\alpha}}{(\frac{\gamma}{2}-1)\int_0^1\sum_{i<j}\frac{m_im_j}{|u_i-u_j|^\gamma}\mathrm{d}t}$, we have $\varepsilon_0>0$ and $\langle df_\varepsilon(u),u\rangle>0, \forall u\in\Sigma_{\mathrm{M},\rho}, \forall 0<\varepsilon<\varepsilon_0$.

□

LEMMA 3.7. – *f_ε satisfies the condition (AR4) in the Mountain Pass Lemma, that is, $\exists\, u_1\in\Lambda_0$ with $\| u_1 \|>\mathrm{r}$ s.t. $f_\varepsilon(u_1)<0$.*

PROOF. – Let $R > 0$, we consider

$$f_\varepsilon(Ru) = \frac{1}{2} \| Ru \|^2 \int_0^1 (h - V_\varepsilon(Ru))\,dt.$$

Take $u = (u_1, \ldots, u_N), u_i = \xi \cos[2\pi(t + \frac{i}{N})] + \eta \sin[2\pi(t + \frac{i}{N})], \ |u_i| = 1, |u| = \left(\sum_{i=1}^N |u_i|^2\right)^{1/2} = N, |Ru| = RN, \| u \|^2 = 4\pi^2 M$, by (V_4) it follows that

$$\int_0^1 V_\varepsilon(Ru)\,dt \to 0, \qquad R \to +\infty.$$

So $f_\varepsilon(R_0 u) < 0$, when R_0 is large enough. Choose R_1 large enough such that $\| R_1 u \| > \mathrm{r}$. Take $R = \max\{R_0, R_1\}$, let $u_1 = Ru$, then

$$f_\varepsilon(u_1) < 0 < \beta, \ \| u_1 \| > \mathrm{r}.$$

□

From Lemma 3.1-3.7, $\forall 0 < \varepsilon \le \varepsilon_0, f_\varepsilon$ satisfies (AR_1), (AR_2), (AR_3), (AR_4), $(CPS)_C$ with $C > 0$, and $f_\varepsilon(u_{\{n,\varepsilon\}}) \to +\infty, \forall u_{\{n,\varepsilon\}} \rightharpoonup u_\varepsilon \in \partial\Lambda_0$. Let

$$C_\varepsilon = \inf_{P \in \Gamma_\rho} \max_{0 \le \xi \le 1} f_\varepsilon(P(\xi)).$$

By Lemma 2.5, we know that $\forall\, 0 < \varepsilon \le \varepsilon_0$, there exists $u_\varepsilon \in \Lambda_0$ such that

$$f'_\varepsilon(u_\varepsilon) = 0, f_\varepsilon(u_\varepsilon) = C_\varepsilon \ge \beta > 0. \tag{3.33}$$

Let

$$\omega_\varepsilon^2 = \frac{\int_0^1 (h - V_\varepsilon(u_\varepsilon))\,dt}{\frac{1}{2}\int_0^1 \sum_{i=1}^N m_i |\dot{u}_\varepsilon^i|^2\,dt}.$$

Then by Lemma 2.3, $y_\varepsilon = u_\varepsilon(\omega_\varepsilon t)$ satisfies

$$m_i \ddot{y}_\varepsilon^i + \frac{\partial V_\varepsilon(y_\varepsilon)}{\partial y_\varepsilon^i} = 0. \tag{3.34}$$

$$\frac{1}{2}\omega_\varepsilon^2 \sum_{i=1}^N m_i |\dot{u}_\varepsilon^i(t)|^2 + V_\varepsilon(u_\varepsilon(t)) = h, \tag{3.35}$$

where $y_\varepsilon = (y_\varepsilon^1, \ldots, y_\varepsilon^N), u_\varepsilon = (u_\varepsilon^1, \ldots, u_\varepsilon^N)$.

Next, we show that u_ε converges to some $\tilde{u}$ which gives rise to a solution $\tilde{y}$ of (Ph).

LEMMA 3.8. – $\exists\, C_8, C_9 > 0$ *s.t.* $C_8 \le \| u_\varepsilon \| \le C_9$.

PROOF. – Since $u_\varepsilon \in \Lambda_0$, so $\| u_\varepsilon \|^2 = \int_0^1 \sum_{i=1}^N m_i |\dot{u}_\varepsilon^i|^2 \, dt \neq 0$, otherwise $u_\varepsilon(t) \equiv 0 \in \partial\Lambda_0$ by $u_\varepsilon^i(t+1/2) = -u_\varepsilon^i(t)$. By $\langle f_\varepsilon'(u_\varepsilon), u_\varepsilon \rangle = 0$, we have

$$\| u_\varepsilon \|^2 \int_0^1 \left[h - V_\varepsilon(u_\varepsilon) - \frac{1}{2} \langle V_\varepsilon'(u_\varepsilon), u_\varepsilon \rangle \right] dt = 0.$$

Then

$$h = \int_0^1 (V_\varepsilon(u_\varepsilon) + \frac{1}{2} \langle V_\varepsilon'(u_\varepsilon), u_\varepsilon \rangle) \, dt. \tag{3.36}$$

Letting $\gamma \to 2$, we have

$$h = \int_0^1 (V(u_\varepsilon) + \frac{1}{2} \langle V'(u_\varepsilon), u_\varepsilon \rangle) \, dt.$$

By (V_3), we get

$$h \leq (1 - \frac{\delta}{2}) \int_0^1 V(u_\varepsilon) \, dt. \tag{3.37}$$

If $\| u_\varepsilon \| \to 0$, as $\varepsilon \to 0$; then $\| u_\varepsilon \|_\infty \to 0$, from (3.16) we deduce

$$\int_0^1 V(u_\varepsilon) \, dt \to -\infty,$$

which is a contradiction with (3.37). So we have $C_8 > 0$ such that

$$\| u_\varepsilon \| \geq C_8 > 0. \tag{3.38}$$

On the other hand, from (3.33) we know

$$f_\varepsilon(u_\varepsilon) = \inf_{P \in \Gamma_\rho} \max_{0 \leq \xi \leq 1} f_\varepsilon(P(\xi)), \quad \forall 0 < \varepsilon \leq \varepsilon_0.$$

So we have

$$\begin{aligned} f_\varepsilon(u_\varepsilon) &\leq \inf_{P \in \Gamma_\rho} \max_{0 \leq \xi \leq 1} f_{\varepsilon_0}(P(\xi)) \\ &\leq \max_{0 \leq \xi \leq 1} f_{\varepsilon_0}(P(\xi)) = C_{10}, \quad \forall 0 < \varepsilon \leq \varepsilon_0. \end{aligned}$$

That is

$$f_\varepsilon(u_\varepsilon) = \frac{1}{2} \| u_\varepsilon \|^2 \int_0^1 (h - V_\varepsilon(u_\varepsilon)) \, dt \leq C_{10}, \forall 0 < \varepsilon \leq \varepsilon_0. \tag{3.39}$$

By (3.8) we have

$$h = \int_0^1 (V_\varepsilon(u_\varepsilon) + \frac{1}{2}\langle V'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle)\,\mathrm{d}t$$
$$\geq (\frac{1}{2} - \frac{1}{\alpha})\int_0^1 \langle V'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle\,\mathrm{d}t.$$

So

$$\int_0^1 \langle V'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle\,\mathrm{d}t \geq \frac{h}{\frac{1}{2} - \frac{1}{\alpha}} > 0. \tag{3.40}$$

Then by (3.36) we obtain

$$\int_0^1 (h - V_\varepsilon(u_\varepsilon))\,\mathrm{d}t \geq \frac{h}{1 - \frac{2}{\alpha}}. \tag{3.41}$$

(3.39) and (3.41) imply

$$\| u_\varepsilon \| \leq C_9. \tag{3.42}$$

Since E is a reflexive Banach space, by (3.42) and Lemma 2.8, there is a subsequence, still denoted by $\{u_\varepsilon\}$ such that $u_\varepsilon \rightharpoonup \tilde{u}$, then by compact embedding theorem, $u_\varepsilon \to \tilde{u}$ uniformly.

In the following, we can use almost the same proofs of Ambrosetti-Coti Zelati ([1], [2]) to get Lemma 3.9-3.11, but we should remember $\gamma > 2$, so in order to get our result, we need to let $\gamma \to 2$. For the convenience of the readers, we write the complete proofs.

LEMMA 3.9. –

(3.43) (1). $V(\tilde{u}(t)) \not\equiv h$.

(3.44) (2). $\exists \wp \subset [0,1]\, s.t.\, mes(\wp) = 1$ *and* $\tilde{u}_i(t) \neq \tilde{u}_j(t), \forall\, i \neq j,\ \forall\, t \in \wp$.

PROOF. – (1). if not, $V(\tilde{u}(t)) \equiv h$, then

$$V(u_\varepsilon(t)) \to V(\tilde{u}(t)) \equiv h.$$
$$\langle V'(u_\varepsilon(t)), u_\varepsilon(t)\rangle \to \langle V'(\tilde{u}(t)), \tilde{u}\rangle.$$

Since

$$h = \int_0^1 (V_\varepsilon(u_\varepsilon) + \frac{1}{2}\langle V'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle)\,\mathrm{d}t.$$

Letting $\gamma \to 2$, we can get

$$h = \int_0^1 (V(u_\varepsilon) + \frac{1}{2}\langle V'(u_\varepsilon), u_\varepsilon\rangle)\,\mathrm{d}t.$$

Then letting $\varepsilon \to 0$, we have

$$h = h + \frac{1}{2}\int_0^1 \langle V'(\tilde{u}), \tilde{u}\rangle\,\mathrm{d}t.$$

Hence $\langle V'(\tilde{u}), \tilde{u}\rangle = 0$, this is a contradiction with (V_2).

(2). Set $\ell_{ij} = \{t \in [0,1] \mid \tilde{u}_i(t) = \tilde{u}_j(t)\}(i \neq j)$, then each ℓ_{ij} is a closed set, and

$$u_\varepsilon^i - u_\varepsilon^j \to 0 \quad on\ \ell_{ij}$$

If $\mathrm{mes}(\ell_{ij}) > 0$, then

$$\lim_{n\to\infty} C_\varepsilon = \lim_{n\to\infty} f_\varepsilon(u_\varepsilon) \to +\infty.$$

This is a contradiction with (3.39), so we obtain

$$mes(\ell_{ij}) = 0(\forall\, i \neq j).$$

Let $\ell = \bigcup_{i<j} \ell_{ij}$, then $\mathrm{mes}(\ell) = 0$, we set $\wp = [0,1]\backslash\ell$, then

$$mes(\wp) = 1, \tilde{u}_i(t) \neq \tilde{u}_j(t), \forall\, i \neq j,\ \forall\, t \in \wp.$$

LEMMA 3.10. – *There are numbers* $\delta, \varDelta > 0$, *such that*

$$\delta \leq \omega_\varepsilon \leq \varDelta. \tag{3.45}$$

PROOF. – Integrating (3.35) on $\wp$, we have

$$\frac{1}{2}\omega_\varepsilon^2 \int_\wp \sum_{i=1}^N m_i|\dot{u}_\varepsilon^i|^2\,\mathrm{d}t + \int_\wp V_\varepsilon(u_\varepsilon)\,\mathrm{d}t = h\, mes(\wp). \tag{3.46}$$

From (3.42), we deduce

$$\int_\wp \sum_{i=1}^N m_i|\dot{u}_\varepsilon^i|^2\,\mathrm{d}t \leq \int_0^1 \sum_{i=1}^N m_i|\dot{u}_\varepsilon^i|^2\,\mathrm{d}t \leq C_9^2.$$

From (3.33), $h - V_\varepsilon(u_\varepsilon) > 0$, then by Lemma 3.9, $V_\varepsilon(u_\varepsilon) \to V(\tilde{u})$ uniformly on $\wp$ and $\int_\wp (h - V(\tilde{u})\,\mathrm{d}t > 0$, it follows that

$$\omega_\varepsilon^2 \geq \frac{2\int_\wp (h - V_\varepsilon(u_\varepsilon))\,\mathrm{d}t}{C_9^2} \to \frac{2\ \int_\wp (h - V(\tilde{u}))\,\mathrm{d}t}{C_9^2} > 0. \tag{3.47}$$

Integrating (3.35) on [0, 1], we have

$$\frac{1}{2}\omega_\varepsilon^2\int_0^1\sum_{i=1}^N m_i|\dot{u}_\varepsilon^i|^2\,\mathrm{d}t+\int_0^1 V_\varepsilon(u_\varepsilon)\,\mathrm{d}t=h.$$

Then by (3.2), (3.36), (3.38) and (3.39) we have

$$\omega_\varepsilon^2=\frac{4f_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|^4}\le\frac{4C_{10}}{C_8^4}. \tag{3.48}$$

LEMMA 3.11. – *Suppose that* $(V_1)-(V_4)$ *hold, then for any* $h<0$, $\tilde{u}$ *is a weak solution of (Ph).*

PROOF. – Let $K_n\subset\wp$ be an increasing sequence of compact sets with

$$\bigcup_{n\ge1}K_n=\wp,$$

and set

$$K_n^*=\{\tilde{u}(t)\mid t\in K_n\}.$$

Each $K_n^*\subset\hbar=\{x=(x_1,\dots,x_N)\mid x_i\in\mathbb{R}^n,\,x_i\ne x_j,\forall\,i\ne j\}$ is compact and has a neighborhood $\mathcal{N}_n$ such that $\overline{\mathcal{N}}_n\subset\hbar$. Then $V_\varepsilon\to V$ in $C^1(\mathcal{N}_n,\mathbb{R})$, and therefore $V_\varepsilon'(u_\varepsilon(t))\to V'(\tilde{u}(t))$ uniformly on K_n.

Since $u_\varepsilon=(u_\varepsilon^1,\dots,u_\varepsilon^N)$ satisfies

$$m_i\omega_\varepsilon^2\ddot{u}_\varepsilon^i+\frac{\partial V_\varepsilon(u_\varepsilon)}{\partial u_\varepsilon^i}=0.$$

By Lemma 3.10, ω_ε has subsequence, still denoted by ω_ε, and we have

$$\omega_\varepsilon\to\tilde{\omega}\ne0.$$

It follows that

$$u_\varepsilon\to\tilde{u}=(\tilde{u}_1,\dots,\tilde{u}_2)\quad in\ C^2(K_n,\mathbb{R}).$$

$$\tilde{\omega}^2m_i\frac{\mathrm{d}^2\tilde{u}_i}{\mathrm{d}t^2}+\frac{\partial V(\tilde{u})}{\partial\tilde{u}_i}=0\quad on\ K_n.$$

Since $\bigcup K_n=\wp$, it follows that

$$\tilde{\omega}^2m_i\frac{\mathrm{d}^2\tilde{u}_i}{\mathrm{d}t^2}+\frac{\partial V(\tilde{u})}{\partial\tilde{u}_i}=0\quad on\ K_n\quad\forall\,t\in\wp,$$

and $\tilde{y}(t)=\tilde{u}(\tilde{\omega}t)$ satisfies

$$m_i\ddot{\tilde{y}}_i+\frac{\partial V(\tilde{y})}{\partial\tilde{y}_i}=0,\quad\forall\,t\in\wp.$$

The energy conservation (Ph. 2) on $\wp$ follows directly from (3.35). □

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