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Viscous Incompressible Flows Under Stress-Free Boundary Conditions. The Smoothness Effect of Near Orthogonality or Near Parallelism Between Velocity and Vorticity

H. Beirão da Veiga

Dedicated to the Memory of Professor Enrico Magenes

Abstract. – *We consider the initial boundary value problem for the 3D Navier-Stokes equations under a slip type boundary condition. Roughly speaking, we are concerned with regularity results,* up to the boundary, *under suitable assumptions on the directions of velocity and vorticity. Our starting point is a recent, interesting, result obtained by Berselli and Córdoba concerning the "near orthogonal case". We also consider a "near parallel case".*

1. – Introduction and results

In this paper we consider the 3D Navier-Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)\, u - \nu \triangle u + \nabla p = 0 & \text{in } \Omega \times]0, T], \\ \nabla \cdot u = 0 & \text{in } \Omega \times]0, T], \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where the unknowns are the velocity u and the pressure p. For brevity we assume that external force vanishes. The symbol ν denotes the (positive) kinematic viscosity. The open, bounded, set $\Omega \subset R^3$ has a smooth boundary $\partial\Omega$, say of class $C^{2,\alpha}$, for some $\alpha > 0$.

We supplement the initial value problem with the "stress-free" boundary conditions

$$\begin{cases} u \cdot n = 0 & \text{on } \partial\Omega \times]0, T], \\ \omega \times n = 0 & \text{on } \partial\Omega \times]0, T], \end{cases} \tag{2}$$

where $\omega = \nabla \times u = \operatorname{curl} u$ is the vorticity field, while n denotes the exterior unit normal vector to the boundary. In the case of flat boundaries, the above conditions coincide with the classical Navier boundary conditions without friction. See the classical reference Serrin [8].

In the present paper we consider the problem of global existence of smooth solutions, under suitable hypotheses which imply, in particular, orthogonality or parallelism between velocity and vorticity.

We avoid here non strictly necessary references. We merely recall that the starting point for these kind of studies was Constantin and Fefferman's paper [5]. See also [3]. In these two references the vorticity direction alone was considered.

We denote by $L^p := L^p(\Omega),\ 1 \le p \le \infty$, the usual Lebesgue spaces equipped with norm $\|\,.\,\|_p$. $H^s := H^s(\Omega),\ s \ge 0$, are the classical Sobolev spaces. We use the same symbol for both scalar and vector function spaces, and set $\partial_i = \dfrac{\partial}{\partial x_i}$. Moreover,

$$L^p_T(X) \stackrel{def}{=} L^p(0,\,T;\,X(\Omega))\,,$$

where $X = X(\Omega)$ is a generical functional space, and $1 \le p \le \infty$.

In [4] the following result is proved.

THEOREM 1.1 ([4], Theorem 2.1). – *Let u be a weak solution of (1) in $(0,T)$ with $u_0 \in H^4(\mathcal{T})$ and $\nabla \cdot u_0 = 0$, where $\mathcal{T}$ is the three dimensional 2π periodic cube (a torus). If there exists a constant c_1 such that, for all $x \in \Omega$, and for $|\,y\,|$ small enough, it holds*

$$|\,u(x+y,\,t)\cdot\omega(x,\,t)\,| \le |y|\,|\,u(x+y,\,t)\,|\,|\omega(x,\,t)\,|\,, \tag{3}$$

for a.a. $t \in\,]0,T[$, then the u is regular.

As remarked by the authors, the above condition implies the orthogonality between u and ω.

The authors also state, without proof, the same result in a bounded, regular, domain Ω, under the boundary condition (2). See [4], theorem 3.1. In the sequel we give the following improvement of this last claim. Our approach looks easier, even if the ingredients are similar. In section 2 we prove the following result.

THEOREM 1.2. – *Let $\Omega \subset R^3$ be an open, bounded set with a smooth boundary $\partial\Omega$, say of class $C^{2,\gamma}$, for some $\gamma > 0$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla \cdot u_0 = 0$, and u is a weak solution to (1)-(2) in $[0,T]$. In addition, suppose that there is a constant c_1 and a positive $\delta(x,\,t)$, such that, for almost all $t \in\,]0,T[$, the following assumption holds.*

For a.a. $x \in \Omega$ one has

$$|\,u(x+y,\,t)\cdot\omega(x,\,t)\,| \le c_1\,|y|\,(\,1 + |\,u(x+y,\,t)\,|^{\frac{4}{3}})\,|\omega(x,\,t)|\,, \tag{4}$$

for a.a. y satisfying $|y - x| < \delta(x,\,t)$. Then u is a strong solution in $[0,T]$, hence is smooth.

In section 3 we prove the Theorem 1.3 below. For convenience, we will assume that Ω is simply connected. We denote by $\theta(x,t)$ the angle between the velocity $u(t, x)$ and the vorticity $\omega(t, x)$ at the same point (x, t):

$$\theta(x,t) \stackrel{def}{=} \angle(\widehat{u}(x,t)\,, \widehat{\omega}(x,t)\,),$$

where, for each non-null vector v, we define $\widehat{v} \stackrel{def}{=} v/|v|$.

Further, we set

$$M(t) = \sup_{x\in \Omega} \sin \theta(x,t)\,.$$

THEOREM 1.3. – *Set* $\omega_0 = \operatorname{curl} u_0$, *where* u_0 *is the initial data to the above initial-boundary value problem. Assume that*

$$M^3(t) \leq \left(\frac{\nu}{\|\omega_0\|}\right)^4, \tag{5}$$

for all $t > 0$. *Then the solution* u *is strong and unique for all times.*

A well know, classical, argument, easily shows that there is a value t^* such that condition (5) is necessary only for $t \leq t^*$.

2. – The near orthogonal case. Proof of Theorem 1.2

For notation convenience we set

$$\alpha = \frac{4}{3}\,.$$

In the sequel the time variable t will be frozen. So, we often drop it from notation. Moreover, in the proof of theorem 1.2, we consider the more stringent case $\alpha = \frac{4}{3}$. Summation over repeated indices is assumed.

We denote by the same symbol c different positive constants, even in the same equation. A positive constant is labeled if this helps the reader to follow a particular manipulation.

We start by the following lemma proved in reference [3] (see [3], equation (23)). For the reader's convenience we recall here part of the proof.

PROPOSITION 2.1. – *Assume that u is a strong solution to* (1)-(2). *Then*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 dx + \frac{\nu}{2}\int_\Omega |\nabla\omega|^2 dx \leq c\,\nu \int_\Omega |\omega|^2 dx + \left|\int_\Omega (\omega\cdot\nabla)u\cdot\omega\, dx\right|. \tag{6}$$

PROOF. – By applying the curl operator to (1) we get the well-known equation

$$\text{(7)} \qquad \begin{cases} \omega_t + (u\cdot\nabla)\,\omega - \nu\,\Delta\omega &= (\omega\cdot\nabla)\,u \qquad \text{in } \Omega\times\,]0,T], \\ \nabla\cdot\omega &= 0 \qquad\qquad\;\; \text{in } \Omega\times\,]0,T]\,. \end{cases}$$

By taking the scalar product of both sides of the first equation (7) with ω, by integrating in Ω, and by appealing to Green's formula one gets

$$\text{(8)} \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 dx + \nu\int_\Omega |\nabla\omega|^2\,dx = \nu\int_{\partial\Omega} \frac{\partial\omega}{\partial n}\cdot\omega\,dS + \int_\Omega (\omega\cdot\nabla)u\cdot\omega\,dx\,.$$

Next we appeal to the estimate

$$\text{(9)} \qquad \left|\frac{\partial\omega(x)}{\partial n}\cdot\omega(x)\right| \le c\,|\omega(x)|^2, \qquad \forall\, x\in\partial\Omega\,,$$

proved in [3] (see the equation (14) in reference [3]) for divergence-free vector fields u such that $u\cdot n = 0$ and $\omega\times n = 0$ on $\partial\Omega$ (see also the lemma 2.2 in reference [1]). So (see [3], Lemma 2.6)

$$\text{(10)} \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 dx + \nu\int_\Omega |\nabla\omega|^2\,dx \le c_0\,\nu\int_{\partial\Omega} |\omega|^2 dx + \left|\int_\Omega (\omega\cdot\nabla)u\cdot\omega\,dx\right|.$$

Finally, by appealing to (16), and by tacking into account that

$$c_2\,\|\omega\|\,\|\nabla\,\omega\,\| \le \frac{c_0}{2}\,c_2^2\,\|\omega\|^2 + \frac{1}{2\,c_0}\|\,\nabla\,\omega\|^2\,,$$

the estimate (6) follows (for a proof of (16) see the end of this section). □

Our next goal is to estimate the last term in the right hand side of (6). The following result holds.

LEMMA 2.2. – *For for a.a.* $x\in\Omega$,

$$\text{(11)} \qquad |\,((\omega(x)\cdot\,\nabla)\,u(x))\cdot\,\omega(x)\,| \le c_1\,(1+\,|\,u(x)\,|^\alpha\,)\,|\omega(x)\,|^2\,.$$

In particular,

$$\text{(12)} \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 dx + \frac{\nu}{2}\int_\Omega |\nabla\omega|^2\,dx \le c\,(1+\,\nu\,)\int_\Omega |\omega|^2 dx + c_1\int_\Omega |u|^\alpha\,|\omega|^2\,dx\,.$$

PROOF. – By letting in equation (4) $y\to 0$, $y+x\in\Omega$, it follows that $u(x)\cdot\omega(x) = 0$. Hence,

$$\text{(13)} \qquad |\,(u(x+\,y) -\,u(x))\cdot\,\omega(x)\,| \le c_1\,|y|\,(1+\,|\,u(x+\,y)\,|^\alpha\,)\,|\omega(x)|\,,$$

where y satisfies $|y - x| < \delta(x)$. In particular, for each fixed index j,

$$\left| \frac{u(x + h_j e_j) - u(x)}{h_j} \cdot \omega(x) \right| \leq c_1 (1 + |u(x + h_j e_j)|^\alpha) |\omega(x)|, \tag{14}$$

where e_j denotes the unit vector in the positive j-direction. So, by letting $h_j \to 0$, we show that

$$\left| \omega_i \frac{\partial u_i}{\partial x_j} \right| \leq c_1 (1 + |u|^\alpha) |\omega|.$$

for each index j. Equation (11) follows. This last equation, together with (6), shows (12).

Finally we prove the theorem 1.2. By appealing to Hölder's inequality, to interpolation, and to a well known Sobolev's embedding theorem, it follows that

$$\int_\Omega |u|^{\frac{4}{3}} |\omega|^2 \, dx \leq \|u\|_4^{\frac{4}{3}} \|\omega\|_3^2 \leq \|u\|_4^{\frac{4}{3}} \|\omega\| \, \|\omega\|_6 \leq c \, \|u\|_4^{\frac{4}{3}} \|\omega\| (\|\omega\| + \|\nabla \omega\|).$$

So,

$$c_1 \int_\Omega |u|^{\frac{4}{3}} |\omega|^2 \, dx \leq c \, \|u\|_4^{\frac{4}{3}} \|\omega\|^2 + c \, \nu^{-1} \|u\|_4^{\frac{8}{3}} \|\omega\|^2 + \frac{\nu}{4} \|\nabla \omega\|^2,$$

for suitable positive constants c. Hence, from (12) written for $\alpha = \frac{4}{3}$, one obtains

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\omega|^2 dx + \frac{\nu}{4} \int_\Omega |\nabla \omega|^2 \, dx \leq c (1 + \nu + \|u\|_4^{\frac{4}{3}} + \nu^{-1} \|u\|_4^{\frac{8}{3}}) \|\omega\|^2. \tag{15}$$

By interpolation it readily follows that

$$\|u\|_4^{\frac{8}{3}} \leq \|u\|^{\frac{2}{3}} \|u\|_6^2.$$

Since weak solutions u satisfy

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^6(\Omega)),$$

it follows that $\|u(t)\|_4^{\frac{8}{3}}$ is integrable in $[0, T]$. Hence, from (15), we show that $\omega(t)$ is bounded in $(0, T)$ with values in $L^2(\Omega)$, and square integrable in $(0, T)$ with values in $H^1(\Omega)$. So

$$u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

is a strong solution. A well know argument shows that $u(t)$ is weakly continuous in $[0, T]$ with values in $H^1(\Omega)$.

As already remarked, we end this section with the proof of equation (16) below.

LEMMA 2.3. – *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, regular set. There is a constant $c = c(\Omega)$ such that*

$$\| v \|^2_{2,\Gamma} \leq c_2 (\|v\|^2 + \|v\| \, \|\nabla v \|) , \tag{16}$$

for each $v \in H^1(\Omega)$.

PROOF. – We merely show the basic argument in the proof, by working in the framework of an unit cube $Q = \{x : 0 < x_j < 1 , j = 1, 2, ..., n \}$. The extension of the argument to a regular Ω is easy, and left to the reader.

We start by assuming that $w(x', 1) = 0$, for $x' \in \Gamma$, where now $\Gamma = \{x : x_n = 0 \}$ represents the "effective boundary", in which we are interested. One has

$$w^2(x', 0) = -2 \int_0^1 w(x', \xi) \, \partial_n w(x', \xi) \, d\xi .$$

Integration of both sides of the above equation with respect to the x' variables easily leads to the estimate

$$\| w \|^2_{2,\Gamma} \leq 2 \, \|w\|_{2,Q} \, \|\nabla w\|_{2,Q} . \tag{17}$$

Next define $\phi(x', x_n) = 1$ if $0 < x_n \leq \frac{1}{2}$, and $\phi(x', x_n) = 2 - 2x_n$ if $\frac{1}{2} \leq x_n \leq 1$. By setting $w = \phi v$ in equation (18) one shows that

$$\| v \|^2_{2,\Gamma} \leq \| v \|^2 + 2 \, \|v\|_{2,Q} \, \|\nabla v\|_{2,Q} . \tag{18}$$

□

3. – The near parallel case. Proof of Theorem 1.3

LEMMA 3.1. – *Assume that u is a strong solution to* (1)-(2). *Then*

$$\frac{1}{2} \frac{d}{dt} \| \omega\|^2 + \nu \| \operatorname{curl} \underline{\omega} \|^2 = \int_\Omega u \times \omega \cdot \operatorname{curl} \omega \, dx , \tag{19}$$

PROOF. – We start by replacing the first equation (1) by the equivalent, well known, equation

$$u_t - \nu \triangle u + \omega \times u + \nabla \left(\frac{1}{2} | u |^2 + p \right) = 0 .$$

By applying the curl operator to both sides of the above equation we get

$$\omega_t - \nu \Delta \omega + \operatorname{curl} (\omega \times u) = 0 . \tag{20}$$

Next we recall the identity

$$\int_\Omega (\operatorname{curl} \underline{f}) \cdot \underline{g}\, dx = \int_\Omega \underline{f} \cdot (\operatorname{curl} \underline{g})\, dx + \int_\Gamma (\underline{n} \times \underline{f}) \cdot \underline{g}\, d\Gamma . \tag{21}$$

Scalar multiplication of both sides of equation (20) by $\underline{\omega}$, integration over Ω, plus a suitable integration by parts lead to

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \nu \|\operatorname{curl} \underline{\omega}\|^2 = -\int_\Omega \operatorname{curl}(\omega \times u) \cdot \underline{\omega}\, dx . \tag{22}$$

We have appealed to (21), to the identity

$$-\Delta \underline{\omega} = \operatorname{curl} \operatorname{curl} \underline{\omega} - \nabla \operatorname{div} \underline{\omega},$$

and to $(\underline{n} \times \operatorname{curl} \underline{\omega}) \cdot \underline{\omega} = 0$ on Γ. From (22), and by appealing once more to (21), we show that (19) holds, since $n \times (\omega \times u) \cdot \omega = 0$. □

Next we estimate the right hand side of (19). Note that

$$|\, u \times \omega \cdot \operatorname{curl} \omega\,| \le M(t)\, |u|\, |\omega|\, |\operatorname{curl} \omega\,| .$$

By Hölder's inequality with exponents 6, 3 and 2, followed by interpolation, we show that

$$\left| \int_\Omega u \times \omega \cdot \operatorname{curl} \omega\, dx \right| \le M(t)\, \|u\|_6\, \|\omega\|^{\frac{1}{2}}\, \|\omega\|_6^{\frac{1}{2}}\, \|\operatorname{curl} \omega\| . \tag{23}$$

Since Ω is simply-connected, one has

$$\|u\|_6 \le c\, \|\omega\| , \tag{24}$$

and

$$\|\omega\|_6 \le c\, \|\operatorname{curl} \omega\| . \tag{25}$$

See, for instance, [6] equation (1.19), and [7] equations (2.27) and (2.29). It follows that the right hand side of (23) is bounded by $c\, M(t)\, \|\omega\|^{\frac{3}{2}}\, \|\operatorname{curl}\, \omega\|^{\frac{3}{2}}$. So, it is bounded by

$$\frac{c}{\nu^3}\, M^4(t)\, \|\omega\|^6 + \frac{\nu}{2}\, \|\operatorname{curl}\, \omega\|^2 .$$

So, by (19),

$$\frac{1}{2}\frac{d}{dt}\|\, \omega\|^2 + \frac{\nu}{2}\, \|\operatorname{curl} \underline{\omega}\|^2 \le \frac{c}{\nu^3}\, M^4(t)\, \|\omega\|^6 . \tag{26}$$

In particular, since $\|\operatorname{curl} \underline{\omega}\| \geq c \|\underline{\omega}\|^2$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 \leq -c\,(\nu - \nu^{-3} M^3(t)\,\|\omega\|^4)\,\|\omega\|^2 .$$

Since we assume that (5) holds for all $t > 0$, a classical, straightforward, argument shows that the solution is strong and global in time.

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